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Torsional Impulsive Response of an Elastic Half-Space

Due to a Rigid Circular Cylinder*

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The rigid circular cylinder cemented on an elastic half-space is rotated impulsively through a small angle.

The problem is solved by setting up an integral equation for the Laplace transform of the coefficient of the stress function.

The inversion of the Laplace transform is carried out by using the numerical method.

The numerical results are shown for the impulsive response of the elastic half-space such as the stress intensity factor and the torque.

1. Introduction

The study of the dynamical mixed boundary value problem was firstly tried by Reissner-Sagoci(1) concerning the analysis of the rotational vibration of a rigid disk cemented on an elastic half-space.

Thereafter, the analytical method of the Reissner-Sagoci problem has been investigated by many people. Gladwell-Low(2) obtained numerical results of the transient motion of the disk which was twisted by static torque and then released from the load instantaneously. Jain-Kanwal(3) tried to extend the above problem to that of an annular rigid plate. The above authors paid attention mainly to the rotational motion of the disk.

Recently, in the field of the dynamic mixed boundary value problem, the investigation of the transient stress and displacement in elastic body is desired in order to apply the results to a dynamic contact problem and fracture mechanics and so on.

In the present paper, the method of analysis is described for the impulsive response of a semi-infinite elastic body which is twisted by a rigid circular cylinder. The numerical results are obtained for the time histories of displacement, stress, stress intensity factor and torque by using the method of the numerical inversion of the Laplace transforms.

2. Method of solution

2.1 Dual integral equations

In Fig.1, a rigid cylinder (diameter is $2a$) completely cemented on a semi-infinite elastic body is impulsively rotated by a small angle θ_0 .

Fig.1 Coordinate system

Denote the displacement components in the direction of cylindrical coordinates (r, θ, z) as (u, v, w) then, in the axisymmetrical torsional problem, we have non zero components of stress and displacement $\tau_{r\theta}, \tau_{\theta z}$; v and the others are all zero.

Now, we put nondimensional set as $\rho = r/a, \xi = z/a, \eta = C_2 t/a; U_2 = U/a, \tau_{r\theta} = \tau_{r\theta}/G, \tau_{\theta z} = \tau_{\theta z}/G$ where t is time, G is the modulus of rigidity, $C_2 = \sqrt{3G/\rho}$ is the distortional wave velocity, ρ is weight per unit volume and g is the acceleration of gravity.

The boundary conditions are as follows.

- (i) $(v_2)_{\xi=0} = \rho \theta_0 f(\eta) \quad (0 \leq \rho \leq 1)$
- (ii) $(\tau_{\theta z})_{\xi=0} = 0 \quad (1 < \rho < \infty)$
- (iii) Far from the origin, all the components of stress and displacement tend to zero.

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In condition (i), $f(\eta)$ means a time history function of the rotational angle of the cylinder.

Initial conditions are taken as follows, assuming that the elastic body is at rest for $\eta \leq 0$,

$$(v_2)_{\tau=0} = \left(\frac{\partial v_2}{\partial \eta} \right)_{\tau=0} = 0$$

Introducing the non-dimensional form $\lambda = \lambda_3 / \alpha^2$ of the stress function (4) λ_3 , we have the Laplace transform $\bar{\lambda}$ of λ considering the boundary condition (iii) as follows,

$$\bar{\lambda} = \int_0^\infty \frac{A(\beta, \alpha)}{\sqrt{\beta^2 + \alpha^2}} J_0(\beta \rho) \exp[-\sqrt{\beta^2 + \alpha^2} \xi] d\beta$$

where barred quantity shows the Laplace transform, and α is the transformation parameter with respect to η . Then, we have

$$\left. \begin{aligned} \bar{v}_2 &= \int_0^\infty \frac{\beta}{\sqrt{\beta^2 + \alpha^2}} A(\beta, \alpha) J_1(\beta \rho) \\ &\quad \times \exp[-\sqrt{\beta^2 + \alpha^2} \xi] d\beta \\ \bar{\tau}_{\theta\theta} &= - \int_0^\infty \beta A(\beta, \alpha) J_1(\beta \rho) \exp[-\sqrt{\beta^2 + \alpha^2} \xi] d\beta \\ \bar{\tau}_{\rho\theta} &= - \int_0^\infty \frac{\beta^2}{\sqrt{\beta^2 + \alpha^2}} A(\beta, \alpha) J_2(\beta \rho) \\ &\quad \times \exp[-\sqrt{\beta^2 + \alpha^2} \xi] d\beta \end{aligned} \right\} \dots\dots\dots (1)$$

where $J_\nu(x)$ is the Bessel function of order ν of the first kind.

Laplace transforming the boundary conditions (i), (ii) together with Eqs. (1), we have the following dual integral equations for unknown function $A(\beta, \alpha)$.

$$\left. \begin{aligned} \int_0^\infty \frac{\beta}{\sqrt{\beta^2 + \alpha^2}} A(\beta, \alpha) J_1(\beta \rho) d\beta &= \rho \theta_0 f(\alpha) \\ &\quad (0 \leq \rho \leq 1) \\ \int_0^\infty \beta A(\beta, \alpha) J_1(\beta \rho) d\beta &= 0, \quad (1 < \rho < \infty) \end{aligned} \right\} \dots\dots\dots (2)$$

2.2 Transformation to Fredholm integral equation of second kind

In order to satisfy the second formula of Eqs. (2), the unknown $A(\beta, \alpha)$ is represented by (2)

$$A(\beta, \alpha) = \int_0^1 \phi(x, \alpha) \sin \beta x dx, \quad (0 \leq x \leq 1) \dots\dots\dots (3)$$

Substituting Eq. (3) into the first of Eqs. (2), we get

$$\int_0^\infty \frac{\beta}{\sqrt{\beta^2 + \alpha^2}} J_1(\beta \rho) \left\{ \int_0^1 \phi(x, \alpha) \sin \beta x dx \right\} d\beta = \rho \theta_0 f(\alpha), \quad (0 \leq \rho \leq 1)$$

The order of integrals is changed and a new variable is used as follows,

$$g(\beta, \alpha) = 1 - \frac{\beta}{\sqrt{\beta^2 + \alpha^2}} \dots\dots\dots (4)$$

Then, we have

$$\begin{aligned} &\int_0^1 \phi(x, \alpha) \left\{ \int_0^\infty \sin \beta x J_1(\beta \rho) d\beta \right\} dx \\ &- \int_0^1 \phi(x, \alpha) \left\{ \int_0^\infty g(\beta, \alpha) \sin \beta x J_1(\beta \rho) d\beta \right\} dx \\ &= \rho \theta_0 f(\alpha) \end{aligned}$$

Using the following identities (5) in the first and the second terms of the left hand side of the above equation

$$\int_0^\infty \sin \beta x J_1(\beta \rho) d\beta = \begin{cases} \frac{x}{\rho(\rho^2 - x^2)^{1/2}}, & (x < \rho) \\ 0, & (\rho < x) \end{cases} \dots\dots\dots (5)$$

$$J_1(\beta \rho) = \frac{2}{\pi} \frac{1}{\rho} \int_0^\rho \frac{y \sin \beta y}{(\rho^2 - y^2)^{1/2}} dy \dots\dots\dots (6)$$

we have

$$\begin{aligned} &\int_0^\rho \frac{x \phi(x, \alpha)}{\rho(\rho^2 - x^2)^{1/2}} dx - \int_0^1 \phi(x, \alpha) \left[\int_0^\infty g(\beta, \alpha) \sin \beta x \right. \\ &\quad \times \left. \left\{ \frac{2}{\pi} \frac{1}{\rho} \int_0^\rho \frac{y \sin \beta y}{(\rho^2 - y^2)^{1/2}} dy \right\} d\beta \right] dx = \rho \theta_0 f(\alpha) \end{aligned}$$

This leads to an Abel type integral equation

$$\begin{aligned} &\int_0^\rho \frac{y}{(\rho^2 - y^2)^{1/2}} \left\{ \phi(y, \alpha) \right. \\ &\quad \left. + \int_0^1 \phi(x, \alpha) K(x, y, \alpha) dx \right\} dy = \rho^2 \theta_0 f(\alpha) \end{aligned}$$

with the kernel

$$K(x, y, \alpha) = -\frac{2}{\pi} \int_0^\infty g(\beta, \alpha) \sin \beta x \sin \beta y d\beta \dots\dots\dots (7)$$

The Abel type integral equation is converted to a Fredholm type integral equation

$$\phi(x, \alpha) + \int_0^1 \phi(y, \alpha) K(x, y, \alpha) dy = x \quad (0 \leq x \leq 1, 0 \leq y \leq 1) \dots\dots\dots (8)$$

where

$$\phi(x, \alpha) = \frac{\pi}{4\theta_0} \frac{\phi(x, \alpha)}{f(\alpha)}$$

The unknown $A(\beta, \alpha)$ can be represented by $\phi(x, \alpha)$ as

$$A(\beta, \alpha) = \frac{4\theta_0}{\pi} f(\alpha) \int_0^1 \phi(x, \alpha) \sin \beta x dx \dots\dots\dots (9)$$

2.3 Kernel expressed as finite integral

Substituting Eq. (4) into Eq. (7), we have

$$K(x, y, \alpha) = \frac{\alpha^2}{\pi} \int_0^\infty \frac{\cos(x+y)\beta - \cos(x-y)\beta}{\sqrt{\beta^2 + \alpha^2} (\sqrt{\beta^2 + \alpha^2} + \beta)} d\beta$$

Making use of the identity (5)

$$\begin{aligned} &\int_0^\infty \frac{\cos \beta y}{\sqrt{\beta^2 + \alpha^2} (\sqrt{\beta^2 + \alpha^2} + \beta)} d\beta \\ &= \frac{1}{\sin \nu \pi} \frac{\pi}{\alpha^\nu} \left[\frac{1}{2} J_\nu(i\alpha y) + \frac{1}{2} J_\nu(-i\alpha y) \right. \\ &\quad \left. - I_\nu(\alpha y) \cos \frac{\pi}{2} \nu \right], \quad (R, \alpha > 0, R, \nu > -1) \end{aligned}$$

where

$$\begin{aligned} J_\nu(z) &= \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin\theta) d\theta \\ I_\nu(z) &= \frac{1}{\pi} \int_0^\pi e^{\nu \cos\theta} \cos\nu\theta d\theta \\ &= \frac{\sin\nu\pi}{\pi} \int_0^\infty e^{-z \cosh t - \nu t} dt \\ &\quad \left(R_\nu \nu > -\frac{1}{2}, R_\nu z > 0 \right) \end{aligned}$$

we obtain for $\nu=1$

$$\begin{aligned} &\int_0^\infty \frac{\cos\beta y}{\sqrt{\beta^2 + a^2}(\sqrt{\beta^2 + a^2} + \beta)} d\beta \\ &= \frac{1}{a} \int_0^{\pi/2} \cos\varphi e^{-ay \cos\varphi} d\varphi, \quad (R_\nu a > 0, y \geq 0) \end{aligned}$$

Using the above results, the kernel is represented by the finite integral

$$\begin{aligned} K(x, y, \alpha) &= \frac{\alpha}{\pi} \int_0^{\pi/2} \cos\varphi [e^{-|\alpha \cos\varphi|} \alpha \cos\varphi \\ &\quad - e^{-|\alpha \cos\varphi|} d\varphi] \dots \dots \dots (10) \end{aligned}$$

3. Stress and displacement for a special case

3.1 Final values of stress and displacement

According to the final value theorem of Laplace transformation, the behavior of function in the neighbourhood of $t = \infty$ corresponds to $\lambda = 0$.

From Eq. (7) or Eq. (10),

$$\lim_{\alpha \rightarrow 0} K(x, y, \alpha) = 0$$

so, from Eq. (8)

$$\lim_{\alpha \rightarrow 0} \psi(x, \alpha) = x \quad (0 \leq x \leq 1)$$

If, $f(\eta)$ which represents time history of rotational angle of the rigid cylinder, tends to unity as η tends to infinity, we have

$$\lim_{\alpha \rightarrow 0} \alpha f(\alpha) = 1$$

So,

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \alpha A(\beta, \alpha) &= \frac{4\theta_0}{\pi} \int_0^1 x \sin\beta x dx \\ &= \frac{4\theta_0}{\pi} \left(\frac{\sin\beta}{\beta^2} - \frac{\cos\beta}{\beta} \right) \end{aligned}$$

Then, from Eq. (1), final values of stresses and displacement coincide with the static ones (6) as shown below.

$$\begin{aligned} (v_2)_{st} &= \frac{4\theta_0}{\pi} \int_0^\infty \left(\frac{\sin\beta}{\beta^2} - \frac{\cos\beta}{\beta} \right) J_1(\beta\rho) e^{-\beta t} d\beta \\ (\tau_{\theta t})_{st} &= -\frac{4\theta_0}{\pi} \int_0^\infty \left(\frac{\sin\beta}{\beta} - \cos\beta \right) J_1(\beta\rho) e^{-\beta t} d\beta \\ (\tau_{\rho t})_{st} &= -\frac{4\theta_0}{\pi} \int_0^\infty \left(\frac{\sin\beta}{\beta} - \cos\beta \right) J_2(\beta\rho) e^{-\beta t} d\beta \end{aligned} \quad \dots \dots \dots (11)$$

The infinite integrals in the right hand side of Eqs. (11) can be represented in closed forms.

On the surface $\xi=0$ of the semi-infinite body, static results of the stresses and displacement are as follows.

$$\begin{aligned} \frac{\pi}{4\theta_0} (v_2)_{st} &= \begin{cases} \frac{1}{2} \left\{ \rho \sin^{-1} \left(\frac{1}{\rho} \right) - \frac{\sqrt{\rho^2 - 1}}{\rho} \right\} & (1 < \rho < \infty) \\ \frac{\pi}{4} \rho & (0 \leq \rho \leq 1) \end{cases} \\ -\frac{\pi}{4\theta_0} (\tau_{\theta t})_{st} &= \begin{cases} 0 & (1 < \rho < \infty) \\ \frac{\rho}{\sqrt{1 - \rho^2}} & (0 \leq \rho < 1) \end{cases} \\ -\frac{\pi}{4\theta_0} (\tau_{\rho t})_{st} &= \begin{cases} \frac{1}{\rho^2 \sqrt{\rho^2 - 1}} & (1 < \rho < \infty) \\ 0 & (0 < \rho < 1) \end{cases} \end{aligned}$$

From the above results, stress intensity factors for $(\tau_{\theta t})_{st}$ and $(\tau_{\rho t})_{st}$ are equal to $(k_s)_{st} = -4\theta_0/\pi$.

If the rigid cylinder is statically rotated by an angle θ_0 , nondimensional torque $(T)_{st}$ is $-16\theta_0/3$. In this case, unit torque is taken to be $G\alpha^3$.

3.2 Displacement at the surface

Substituting Eqs. (4) and (9) into the first equation of Eq. (1) and putting $\xi=0$, we have

$$\begin{aligned} \frac{\pi}{4\theta_0} \frac{(\bar{v}_2)_{\xi=0}}{f(\alpha)} &= \int_0^1 \psi(x, \alpha) \left\{ \int_0^\infty \sin\beta x J_1(\beta\rho) d\beta \right\} dx \\ &\quad - \int_0^1 \psi(x, \alpha) \left\{ \int_0^\infty g(\beta, \alpha) \sin\beta x J_1(\beta\rho) d\beta \right\} dx \end{aligned}$$

Substituting Eq. (5) into the first term and Eq. (6) into the second term of the right hand side of the above equation, $(\bar{v}_2)_{\xi=0}$ is represented by using finite integral as follows

$$\begin{aligned} \frac{\pi}{4\theta_0} \frac{(\bar{v}_2)_{\xi=0}}{f(\alpha)} &= \int_0^1 \psi(x, \alpha) \left[\frac{x}{\rho \sqrt{\rho^2 - x^2}} \right. \\ &\quad \left. + \frac{\pi}{2} \int_0^1 K\left(x, \rho \sin \frac{\pi y}{2}, \alpha\right) \sin \frac{\pi y}{2} dy \right] dx \\ &\quad (1 < \rho < \infty) \quad \dots \dots \dots (12) \end{aligned}$$

3.3 Stress, stress intensity factor and torque at the surface

Substituting Eq. (9) into the second formula of Eq. (1) and putting $\xi=0$, stress $(\tau_{\theta t})_{\xi=0}$ at the surface is expressed as follows.

$$\begin{aligned} \frac{\pi}{4\theta_0} \frac{(\bar{\tau}_{\theta t})_{\xi=0}}{f(\alpha)} &= - \int_0^\infty \beta \left[\int_0^1 \psi(x, \alpha) \sin\beta x dx \right] J_1(\beta\rho) d\beta \\ &= \frac{\partial}{\partial \rho} \int_0^1 \psi(x, \alpha) \int_0^\infty \sin\beta x J_0(\beta\rho) d\beta dx \\ &\quad (0 \leq x \leq 1) \end{aligned}$$

By using the identity (5)

$$\int_0^\infty \sin \beta x f_0(\beta \rho) d\beta = \begin{cases} 0 & (x < \rho) \\ \frac{1}{\sqrt{x^2 - \rho^2}} & (\rho < x) \end{cases}$$

we have

$$\begin{aligned} \frac{\pi}{4\theta_0} \frac{(\bar{\tau}_{\theta t})_{t=0}}{f(\alpha)} &= \frac{\partial}{\partial \rho} \int_0^1 \psi(x, \alpha) \frac{1}{\sqrt{x^2 - \rho^2}} dx \\ &= -\frac{\rho}{\sqrt{1 - \rho^2}} \psi(1, \alpha) - \frac{1}{\rho} \left[\sqrt{1 - \rho^2} \psi(1, \alpha) \right. \\ &\quad \left. - \sqrt{1 - \rho^2} \psi'(1, \alpha) + \int_0^1 \psi''(x, \alpha) \sqrt{x^2 - \rho^2} dx \right] \\ (0 < \rho < 1, 0 \leq x \leq 1) \end{aligned} \quad (13)$$

where, prime means partial derivative with respect to x and

$$\psi'(1, \alpha) = [\partial \psi(x, \alpha) / \partial x]_{x=1}$$

The first term of Eq.(13) is a singular and dynamic stress intensity factor $(k_3)_{dy}$ for $\tau_{\theta t}$ is represented as follows

$$(k_3)_{dy} = -\frac{4\theta_0}{\pi} f(\alpha) \psi(1, \alpha) \quad (14)$$

Nondimensional torque is

$$\begin{aligned} \bar{T} &= \int_0^1 2\pi \rho^2 (\bar{\tau}_{\theta t})_{t=0} d\rho \\ &= -2\pi \frac{4\theta_0}{\pi} f(\alpha) \int_0^1 \psi(x, \alpha) \left\{ \int_0^\infty \sin \beta x f_0(\beta \rho) d\beta \right\} dx \end{aligned}$$

By using identity (5)

$$\int_0^\infty \sin \beta x f_0(\beta \rho) d\beta = \begin{cases} \frac{2x}{\rho^2} & (x < \rho) \\ \frac{2x}{\rho^2} + \frac{\rho^2 - 2x^2}{\rho^2(x^2 - \rho^2)^{3/2}} & (\rho < x) \end{cases}$$

$\bar{T}$ is expressed as

$$\bar{T} = -16\theta_0 f(\alpha) \int_0^1 x \psi(x, \alpha) dx \quad (15)$$

4. Numerical inversion of Laplace transform

It is difficult to get the solution $\psi(x, \alpha)$ of integral equation (8) in closed form.

In this study, numerical method is employed for inversion of Laplace transforms. The formula (7) used is as follows:

$$\sum_{i=1}^N w_i x_i^{s-1} F(-S \log x_i) = \frac{1}{S} \bar{F}\left(\frac{\alpha}{S}\right)$$

$$w_i = \int_0^1 \frac{P_N(1-2x)}{(x-x_i)[P_N'(1-2x)]_{x=x_i}} dx$$

where $P_N(x)$ is Legendre polynomial of order N , $P_N'(1-2x)$ is used for $dP_N(1-2x)/d(1-2x)$, $x_i (i=1, N)$ is the root of $P_N(1-2x)=0$, and S is a parameter to determine the time scale.

In the case of $N=7$, values of the function are obtained for $\eta_i/S = -\log_e x_i = 0.025775, 0.138382, 0.352509, 0.693147, 1.213762, 2.046127, 3.671195$ respectively.

Results shown in Fig.3 are obtained for seven integral values of α from 1 to 7.

5. Impulsive response

5.1 One dimensional theory of torsional impact

The results of one dimensional theory of torsional impact of a cylindrical bar show that the intensity of the shear stress in the bar produced by the torsional wave is proportional to the angular velocity at that point as shown in Fig.2.

Fig.2 Stress and displacement produced on the end of bar due to one dimensional torsional impact

5.2 Time history of displacement

Fig.3 shows time history of displacement at the surface of the semi-infinite body. Numerical calculations are carried out for $\rho=1.05$ and for the rotational angle-time history shown in Fig.2(c). The value of $\eta_0=2$ means time for shear wave to travel the distance $2a$ which is equal to the diameter of a rigid cylinder. The calculated points are shown as symbols in the figure, and the solid line shows a smooth curve drawn to pass through these points. Those results are obtained for the parameter $N=7$ of numerical inversion of Laplace transformation. The results for $N=5,6$ closely lie on the solid curve. The error of the numerical results is within 2%.

Fig.3 Example of numerical results

In Fig.4, solid line shows the same curve as in Fig.3. Dotted line in Fig.4 shows the results for the rotational angle-time history shown in Fig.2(b). The peak value of the solid line is a little larger than that of the dashed one. The reason is that the maximum value of rotational angular velocity for $f_1(\eta)$ is a little larger than that for $f_2(\eta)$.

Fig.4 Effect of angular velocity on time history of displacement

Figure 5 shows the displacements $(u_2)_{\xi=0}$ for $\rho=1.05$ and for the rise time $\eta_0=1.0, 2.0, 3.0$ of $f_1(\eta)$, respectively.

Figure 6 shows $(u_2)_{\xi=0}$ for $\rho=1.05, 1.20, 1.40, 1.60$ and 1.80 .

Figure 7 shows the displacement in elastic body for $\rho=1$ and for $\xi=0.8, 1.0, 1.2$, respectively.

Fig.5 Effect of rise time on time history of displacement

Fig.6 Time history of displacement at the surface

Fig.7 Time history of displacement in the elastic body

5.3 Time history of stress

Shear stresses are calculated for the time function $f_1(\eta)$ shown in Fig.2(c).

In Figs.8,9,10, value of η_0 is constant and equal to 2.0.

Figure 8 shows the time history of shear stress on the surface for $\rho=0.7, 0.8, 0.9$ calculated by Eq.(13). Initial response of these stresses coincides with that obtained from one dimensional theory. For instance, the curve for $\rho=0.8$ in Fig.8 coincides with the sine curve in Fig.2(f) from the time $\eta=0$ to $\eta=1-\rho=0.2$.

Fig.8 Time history of stress at the surface

Fig.9 Time history of stress in the elastic body

$\eta=1-\rho$ shows the time for shear wave produced on the periphery of the contact region to travel the distance $1-\rho$. The above fact shows that numerical results are sufficiently accurate.

In Fig.9, stress $\tau_{\theta\xi}$ in the elastic body is shown for $\xi=0.8, 1.0, 1.2$ and $\rho=1$.

Figure 10 shows the time history of $\tau_{\rho\theta}$ at the same position as shown in Fig. 9. From Figs.9 and 10, we can see that the value of $\tau_{\theta\xi}$ is larger than that of $\tau_{\rho\theta}$.

Fig.10 Time history of stress in the elastic body

5.4 Stress intensity factor and torque

Shear stresses are singular on the periphery of the contact surface.

Figure 11 (a),(b) show the time histories of stress intensity factor calculated by Eq.(14). Dynamical stress intensity factor monotonically increases from zero to static results with time.

(a)

(b)

Fig.11 Time history of stress intensity factor

Fig.12 (a),(b) show time histories of torque calculated by Eq.(15).

As shown in Figs.11,12, dynamic stress intensity factor is less than static one for this problem. It may be the reason why to control the angular rotation of the rigid cylinder corresponds to restricting the motion of each particle in the elastic solid.

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(a)

(b)

Fig.12 Time history of torque