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Longitudinal Impact of a Semi-Infinite Rectangular Bar*

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Longitudinal transient strain waves in a semi-infinite rectangular bar are analyzed based on the elasticity equations. Asymptotic solutions are obtained which are valid at large distances from the impacted end. The expression of the axial strain includes the effect of warping of plane sections and completely coincides with that of a circular bar if the radius of gyration of the cross sections is the same.

1. Introduction

The analysis of longitudinal impact of a semi-infinite circular bar has been the subject of numerous theoretical and experimental investigations^{(1)~(7)}. Although it is of great interest to look into the problem of longitudinal impact of bars of noncircular cross section, apparently very few solutions are available in the literature. The only solution known to the authors is one given by Jones and Ellis⁽⁸⁾. They studied longitudinal transient waves in a wide rectangular bar assuming the plane stress condition. For a more complicated geometry of the cross section, it becomes difficult to treat the problem in the framework of elasticity theory because the frequency equation which defines dispersion relations of harmonic waves can not be obtained in a closed form.

Bars of rectangular cross section, as well as ones of circular cross section, are considered to be of the basic waveguide shape. Recently, formal solutions are presented by the present authors for harmonic waves travelling in a rectangular bar with traction-free lateral planes⁽⁹⁾. The frequency equations are obtained in the form of an infinite determinant set equal to zero. Numerical results for the dispersion relations are computed from approximate frequency equations which naturally follow from those formal solutions.

The purpose of this paper is to analyze the longitudinal transient waves in a semi-infinite rectangular bar using the knowledge of the dispersion relations of harmonic waves. The bar is assumed to be initially at rest and to be suddenly moved with a constant axial velocity at the bar end. By applying a double integral transformation to the system of the governing equations, the problems are reduced to a two dimensional boundary value problem. The solutions for the transforms of strains are obtained in a form of superposition of two single series. For the first order approximation of the trans-

forms, only the first terms of those series solutions are considered. Using saddle point methods of integration, inversions are performed for the terms. Obtained expressions for the strain waves are valid at large distances from the impacted end and describe the head of the pulse when the lowest branch is operative.

2. Statement of the Problem

The problem is to determine longitudinal transient strain waves in a semi-infinite bar of rectangular cross section with traction-free side planes. The bar is assumed to be initially at rest and to be suddenly moved with a constant velocity in the axial direction at the end of the bar where the shear stresses are always zero.

The vector equation of motion of a homogeneous, isotropic elastic solid can be written as

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) = \rho \partial^2 \mathbf{u} / \partial t^2, \quad (1)$$

where $\mathbf{u}$ is displacement, t is time, and ∇ and ∇^2 are Nabla and Laplacian operators, respectively. All the solutions of Eq.(1) can be expressed from a combination of a vector potential function $\boldsymbol{\psi}$ and a scalar potential function φ such that

$$\mathbf{u} = \nabla \varphi + (\nabla \times \boldsymbol{\psi}), \quad (2)$$

if φ and $\boldsymbol{\psi}$ are solutions to the equations

$$C_1^2 \nabla^2 \varphi = \partial^2 \varphi / \partial t^2, \quad C_2^2 \nabla^2 \boldsymbol{\psi} = \partial^2 \boldsymbol{\psi} / \partial t^2, \quad (3)$$

and

$$\nabla \cdot \boldsymbol{\psi} = 0, \quad (4)$$

where C_1 and C_2 are the velocities of the dilatational and distortional waves in an unbounded medium. They are given by

$$C_1 = \sqrt{(\lambda + 2\mu) / \rho}, \quad C_2 = \sqrt{\mu / \rho}. \quad (5)$$

Substitution of Eq.(2) into Hooke's law gives expression of stresses in terms of these potential functions.

From Eq.(2), the initial conditions

$$\mathbf{u} = \partial^2 \mathbf{u} / \partial t^2 = 0 \quad \text{at } t = 0, \quad (6)$$

are satisfied if

$$\varphi = \partial \varphi / \partial t = 0, \quad \boldsymbol{\psi} = \partial \boldsymbol{\psi} / \partial t = 0 \quad \text{at } t = 0. \quad (7)$$

Using rectangular Cartesian coordinates x , y , and z and making the z -axis coincide with the bar axis, the boundary conditions at an end of the bar are

$$u_z = Vt, \quad \tau_{zx} = \tau_{zy} = 0 \quad \text{at } z = 0, \quad (8)$$

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where u_z is the displacement in z -direction and V is the velocity of the struck end. Let a and b denote the semiwidth and semithickness of the rectangular bar, then the traction-free conditions at the side planes are

$$\begin{aligned} \sigma_x = \tau_{xy} = \tau_{xz} = 0 \text{ at } x = \pm a, \\ \sigma_y = \tau_{yz} = \tau_{yx} = 0 \text{ at } y = \pm b. \end{aligned} \quad (9)$$

It can be easily seen that the conditions (8) are satisfied if we choose the potential functions such as

$$\partial\varphi/\partial z = Vt, \quad \varphi_x = \varphi_y = \partial\varphi/\partial z = 0 \text{ at } z = 0. \quad (10)$$

In addition to these conditions, all the dependent variables must also satisfy the radiation condition, i.e. they must vanish with $z \rightarrow \infty$.

3. Double transform method of solution

To obtain the solutions to the problem, a double transform method developed by Folk²⁾ is used. Fourier exponential, sine, and cosine transforms of a function $f(x, y, z, t)$ are denoted by $f^L(x, y, z, \omega)$, $f^S(x, y, \gamma, t)$, and $f^C(x, y, \gamma, t)$, respectively, where

$$f^L(x, y, z, \omega) = \int_0^\infty f(x, y, z, t) e^{i\omega t} dt, \quad (11.1)$$

$$f^S(x, y, \gamma, t) = \int_0^\infty f(x, y, z, t) \sin \gamma z dz, \quad (11.2)$$

$$f^C(x, y, \gamma, t) = \int_0^\infty f(x, y, z, t) \cos \gamma z dz. \quad (11.3)$$

The inversion formulae of these transformations are

$$f(x, y, z, t) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} f^L(x, y, z, \omega) e^{-i\omega t} d\omega, \quad (12.1)$$

$$f(x, y, z, t) = -\frac{i}{\pi} \int_0^\infty f^S(x, y, \gamma, t) e^{i\gamma z} d\gamma, \quad (12.2)$$

$$f(x, y, z, t) = \frac{1}{\pi} \int_0^\infty f^C(x, y, \gamma, t) e^{i\gamma z} d\gamma. \quad (12.3)$$

The crucial point of the method is the proper use of sine and cosine transformations which require only the specified boundary conditions at the bar end. Eqs. (3) are thus transformed to

$$(\nabla^2 + \alpha^2) \varphi^{LC} + V/\omega^2 = 0, \quad (13)$$

and

$$(\nabla^2 + \beta^2) (\varphi_x^{LS}, \varphi_y^{LS}, \varphi_z^{LC}) = 0, \quad (14)$$

where

$$\alpha^2 = \omega^2/c_1^2 - \gamma^2, \quad \beta^2 = \omega^2/c_2^2 - \gamma^2, \quad (15)$$

and ∇^2 is written for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Similarly the condition (4) is transformed to

$$\frac{\partial \varphi_x^{LS}}{\partial x} + \frac{\partial \varphi_y^{LS}}{\partial y} - i\gamma \varphi_z^{LC} = 0. \quad (16)$$

The particular solution, $-V/\alpha^2\omega^2$, for φ^{LC} is easily found. Then, following the procedure outlined in reference (9), a general solution of Eqs. (13) and (14) satisfying the condition (16) can be written as

$$\begin{aligned} \varphi^{LC} &= \sum_{p=0}^{\infty} [a_p f^e(\alpha_1 y) \cos \xi_1 x + b_p f^e(\alpha_2 x) \cos \xi_2 y] - V/\alpha^2\omega^2, \\ \varphi_x^{LS} &= \sum_{p=0}^{\infty} [a_p x g^o(\beta_1 y) \cos \xi_1 x], \quad \varphi_y^{LS} = \sum_{p=0}^{\infty} [b_p y g^o(\beta_2 x) \cos \xi_2 y], \end{aligned} \quad (17)$$

$$\varphi_z^{LC} = \sum_{p=0}^{\infty} [a_p x g^o(\beta_1 y) \sin \xi_1 x + b_p y g^o(\beta_2 x) \sin \xi_2 y],$$

where

$$\begin{aligned} \xi_1 a &= \xi_2 b = p\pi, \\ a_p &= L_1^2 g^o(\beta_1 b) A_p, \quad a_p x = \gamma f^{e'}(\alpha_1 b) A_p, \quad a_p z = -\xi_1 f^{e'}(\alpha_1 b) A_p, \\ b_p &= L_2^2 g^o(\beta_2 a) B_p, \quad b_p y = -\gamma f^{e'}(\alpha_2 a) B_p, \quad b_p z = \xi_2 f^{e'}(\alpha_2 a) B_p; \quad (p=0, 1, 2, \dots) \\ \alpha_i^2 &= \alpha^2 - \xi_i^2, \quad \beta_i^2 = \beta^2 - \xi_i^2, \quad L_i^2 = \omega^2/2c_i^2 - \gamma^2 - \xi_i^2; \quad (i=1, 2) \end{aligned} \quad (18)$$

The notation f^e and g^o mean $\cos$ and $\sin$ ($\alpha_i^2, \beta_i^2 > 0$), or, $\cosh$ and $\sinh$ ($\alpha_i^2, \beta_i^2 < 0$), respectively, and ' denotes differentiation with respect to x or y . The solutions (17) give zero shear stresses on the side planes. Arbitrary constants, A_p and B_p , in Eqs. (18) are determined by the condition for normal stresses to vanish on the side planes, i.e., by an infinite set of linear algebraic equations

$$\begin{aligned} & \cdot + \cdot + \cdot + \cdot + \cdot + \cdot + \cdot = \cdot \\ & \cdot + \frac{1}{2} A_1 \gamma_1 + 0 + B_0 J_1^0 + B_1 J_1^1 + \cdot = 0 \\ & \cdot + 0 + A_0 \gamma_0 + B_0 J_0^0 + B_1 J_0^1 + \cdot = -\kappa V/c^2 \alpha^2 \\ & \cdot + A_1 I_0^1 + A_0 I_0^0 + B_0 x_0 + 0 + \cdot = -\kappa V/c^2 \alpha^2 \\ & \cdot + A_1 I_1^1 + A_0 I_1^0 + 0 + \frac{1}{2} B_1 x_1 + \cdot = 0 \\ & \cdot + \cdot + \cdot + \cdot + \cdot + \cdot + \cdot = \cdot \end{aligned} \quad (19)$$

where

$$I_p^q = \frac{1}{b} (-1)^{p+q} \left[\frac{L_1^2 (\kappa \omega^2/c_1^2 + \xi_1^2)}{\alpha_1^2 - \xi_1^2} - \frac{\xi_1^2 \beta_1^2}{\beta_1^2 - \xi_1^2} \right] f^{e'}(\alpha_1 b) g^o(\beta_1 b)$$

$$\begin{aligned}
J_8^P &= \frac{1}{\alpha} (-1)^{P+1} \left[\frac{L^2 (\kappa \omega^2 C_2^2 + \xi_2^2)}{\alpha_2^2 - \xi_1^2} - \frac{\xi_2^2 \beta_2^2}{\beta_2^2 - \xi_1^2} \right] f^{e'}(\alpha_2 a) g^o(\beta_2 a), \\
X_P &= -L^2 g^o(\beta_2 a) f^e(\alpha_2 a) + (\gamma^2 + \xi_2^2) f^{e'}(\alpha_2 a) g^o(\beta_2 a), \\
Y_P &= -L^2 g^o(\beta_1 b) f^e(\alpha_1 b) + (\gamma^2 + \xi_1^2) f^{e'}(\alpha_1 b) g^o(\beta_1 b), \\
\kappa &= \lambda/2\mu.
\end{aligned} \tag{20}$$

Eqs.(19) can be solved only formally as

$$A_P = -\frac{\kappa V D A^P(\gamma, \omega)}{C_1^2 \alpha^2 D(\gamma, \omega)}, \quad B_P = -\frac{\kappa V D B^P(\gamma, \omega)}{C_1^2 \alpha^2 D(\gamma, \omega)}, \tag{21}$$

where $D(\gamma, \omega)$ is the determinant of the coefficients and $-\kappa V D A^P(\gamma, \omega)/C_1^2 \alpha^2$ and $-\kappa V D B^P(\gamma, \omega)/C_1^2 \alpha^2$ are obtained from $D(\gamma, \omega)$ by replacing the elements in the column of A_P or B_P by corresponding terms on the right hand side of Eqs.(19). From Eqs.(17), transforms of strains become

$$\begin{aligned}
\varepsilon_x^{LC} &= \sum_{p=0}^{\infty} [-A_p \xi_1^2 F_{1p}(\gamma) \cos \xi_1 x + B_p G_{2p}(x) \cos \xi_2 y], \\
\varepsilon_y^{LC} &= \sum_{p=0}^{\infty} [A_p G_{1p}(\gamma) \cos \xi_1 x - B_p \xi_2^2 F_{2p}(x) \cos \xi_2 y], \\
\varepsilon_z^{LC} &= \frac{V}{C_1^2 \alpha^2} - \gamma^2 \sum_{p=0}^{\infty} [A_p F_{1p}(\gamma) \cos \xi_1 x - B_p F_{2p}(x) \cos \xi_2 y], \\
\gamma_{xy}^{LC} &= 2 \sum_{p=0}^{\infty} [A_p \xi_1 L^2 H_{1p}(\gamma) \sin \xi_1 x - B_p \xi_2 L^2 H_{2p}(x) \sin \xi_2 y], \\
\gamma_{yz}^{LS} &= 2 \gamma \sum_{p=0}^{\infty} [A_p L^2 H_{1p}(\gamma) \cos \xi_1 x + B_p \xi_2 F_{2p}(x) \sin \xi_2 y], \\
\gamma_{zx}^{LS} &= 2 \gamma \sum_{p=0}^{\infty} [A_p \xi_1 F_{1p}(\gamma) \sin \xi_1 x - B_p L^2 H_{2p}(x) \cos \xi_2 y],
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
F_{ip}(\gamma) &= L^2 g^o(\beta_i d_i) f^e(\alpha_i \gamma) + f^{e'}(\alpha_i d_i) g^o(\beta_i \gamma), \\
G_{ip}(\gamma) &= -\alpha_i^2 L^2 g^o(\beta_i d_i) f^e(\alpha_i \gamma) + (\xi_i^2 + \gamma^2) f^{e'}(\alpha_i d_i) g^o(\beta_i \gamma), \\
H_{ip}(\gamma) &= -g^o(\beta_i d_i) f^{e'}(\alpha_i \gamma) + g^o(\beta_i \gamma) f^{e'}(\alpha_i d_i), \\
(i=1, 2; d_1=b, d_2=a; \gamma=x, y).
\end{aligned} \tag{23}$$

4. Inversion of the approximate transforms

$D(\gamma, \omega)$ being an infinite determinant, it is not practicable to invert the transforms (22) directly. As shown in the preceding chapter, however, approximate solutions naturally follow from the formal solutions. In the subsequent analysis, we confine our attention to the lowest order approximation for which $D(\gamma, \omega)$ takes a simple form of

$$D(\gamma, \omega) \cong D_1(\gamma, \omega) = X_0 Y_0 - I_0^2 J_0^2. \tag{24}$$

It can be seen from Eqs.(22) that at this order of approximation γ_{xy}^{LC} is identically equal to zero. The remaining transforms of the strains become

$$\begin{aligned}
\varepsilon_x^{LC} &= B_0 G_{20}(x), \quad \varepsilon_y^{LC} = A_0 G_{10}(y), \\
\varepsilon_z^{LC} &= -\frac{V}{C_1^2 \alpha^2} - \gamma^2 [A_0 F_{10}(\gamma) + B_0 F_{20}(x)], \\
\gamma_{yz}^{LS} &= 2 \gamma A_0 L^2 H_{10}(\gamma), \quad \gamma_{zx}^{LS} = 2 \gamma B_0 L^2 H_{20}(x),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
A_0 &= -\frac{\kappa V D A^0(\gamma, \omega)}{C_1^2 \alpha^2 D_1(\gamma, \omega)}, \quad B_0 = -\frac{\kappa V D B^0(\gamma, \omega)}{C_1^2 \alpha^2 D_1(\gamma, \omega)}, \\
D A^0 &= X_0 - J_0^2, \quad D B^0 = Y_0 - I_0^2, \quad L^2 = \omega^2/2C_2^2 - \gamma^2, \\
G_{10}(\gamma) &= -\alpha^2 L^2 g^o(\beta d_i) f^e(\alpha \gamma) - \gamma^2 f^{e'}(\alpha d_i) g^o(\beta \gamma), \\
F_{10}(\gamma) &= L^2 g^o(\beta d_i) f^e(\alpha \gamma) + f^{e'}(\alpha d_i) g^o(\beta \gamma), \\
H_{10}(\gamma) &= -g^o(\beta d_i) f^{e'}(\alpha \gamma) - f^{e'}(\alpha d_i) g^o(\beta \gamma), \\
(i=1, 2; d_1=b, d_2=a; \gamma=x, y).
\end{aligned} \tag{26}$$

At first, Fourier sine and cosine inversions of the approximate transforms are performed. Using Formulae (12) $\varepsilon_x^L(x, y, z, \omega)$, for example, is expressed as

$$\varepsilon_x^L(x, y, z, \omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varepsilon_x^{LC}(x, y, z, \omega) e^{i\gamma z} d\gamma. \tag{27}$$

These integrals may be evaluated by applying Cauchy's residue theorem in the complex γ -plane. For each integral, the path of integration is closed with a semi-circle of infinite radius about the origin in the upper half of the γ -plane. Since it can be shown that the integral along the semi-circle vanishes, the original integral is equal to $2\pi i$ times the sum of the residues of the poles enclosed.

If the trigonometric and hyperbolic functions in the integrand are replaced by their respective infinite series expressions, it is found that only even powers of α and β appear. Hence there are no branch points of the integrand in the γ -plane, even though α and β are irrational functions of γ .

For a given ω , the positions of these poles are determined by the equation

$$D_1(\gamma, \omega) = X_0 \gamma_0 - I_0 J_0 = 0. \quad (28)$$

This is the lowest order approximation to the frequency equation of harmonic waves travelling in the longitudinal mode. The frequency spectrum computed from Eq.(28) is shown in Fig.1 using dimensionless variables $\Omega (=2b\omega/\pi C_2)$ and $\Gamma (=2b\gamma/\pi)$. The figure shows the case of a square bar ($b/a=1$) with Poisson's ratio $\nu=1/3$. To estimate the accuracy of the first order approximation, the results given by the second order approximate solution are also shown in the same figure. The following notations are used. Let L_i and S'_i ($i=1, 2$) specify the families of branches of the longitudinal and the first screw mode⁹, respectively, where the subscript i means the order of approximation. Further, branches of each mode are numbered in order of frequency value at $\Gamma=0$. They are designated $L_i(j)$ and $S'_i(j)$, where j is a positive integer indicating the order of the branch. In Fig.1, L_1 , S'_1 , L_2 , and S'_2 are represented by full lines, broken lines, one point chain lines, and two points chain lines, respectively.

One of the characteristic features of the frequency spectrum of a square bar is the existence of crossings between the branches of the longitudinal mode and the first screw one. In the present problem of the longitudinal impact, the first screw mode is not excited and the integrand for the strains is analytic at those branches. Hence, it can be assumed that the roots of the frequency equation define simple poles of the integrand. In Fig.1,

Fig.1

agreement of $L_1(1)$ and $L_2(1)$ is excellent for all values of Γ considered. They can not be distinguished from one another in the figure. Meanwhile, $L_1(2)$ agrees fairly well with $L_2(2)$ only in the region $\Gamma=0 \sim 1.5$. In the neighborhood of $\Gamma=1.5$, $L_2(2)$ and $L_2(3)$ become very close to each other and $L_1(2)$ can be regarded to change over from $L_2(2)$ to $L_2(3)$ at this point. In the region $\Gamma > 1.5$, $L_1(2)$ almost coincides with $L_2(3)$. $L_1(3)$ shows a similar tendency to that of $L_2(3)$ in the region $\Gamma=0 \sim 0.5$ even though the absolute values of the frequency are somewhat higher. With an increasing Γ , $L_1(3)$ crosses $L_2(4)$ and then coincides with $L_2(5)$.

Similar statements hold for higher order branches. Consequently, it can be concluded that the first order approximate solution for a square bar gives sufficiently accurate results of the first branch in the entire region of Γ and of the second branch in the region of small Γ .

Functional dependency of γ on ω is thus clarified numerically and the branches of the longitudinal mode are denoted as $\gamma = \gamma_j(\omega)$. Then the Fourier sine and cosine inversions are performed formally and we obtain the Fourier exponential transforms of strains

$$\begin{aligned} \mathcal{E}_x^L &= 2i \frac{\nu}{C_1^2} \sum_j \left[\left\{ D_1 + \kappa \gamma^2 (D_1 F_{10}(\gamma) + D_2 F_{20}(x)) \right\} \exp(i\gamma z) / \frac{\partial}{\partial \gamma} (\alpha^2 D_1) \right] \gamma = \gamma_j, \\ \mathcal{E}_x^L &= -2i \frac{\kappa \nu}{C_1^2} \sum_j \left[D_2 G_{20}(x) \exp(i\gamma z) / \frac{\partial}{\partial \gamma} (\alpha^2 D_1) \right] \gamma = \gamma_j, \\ \mathcal{E}_y^L &= -2i \frac{\kappa \nu}{C_1^2} \sum_j \left[D_1 G_{10}(\gamma) \exp(i\gamma z) / \frac{\partial}{\partial \gamma} (\alpha^2 D_1) \right] \gamma = \gamma_j, \\ \gamma_{zx}^L &= -4 \frac{\kappa \nu}{C_1^2} \sum_j \left[\gamma L^2 D_2 H_{20}(x) \exp(i\gamma z) / \frac{\partial}{\partial \gamma} (\alpha^2 D_1) \right] \gamma = \gamma_j, \\ \gamma_{zy}^L &= -4 \frac{\kappa \nu}{C_1^2} \sum_j \left[\gamma L^2 D_1 H_{10}(\gamma) \exp(i\gamma z) / \frac{\partial}{\partial \gamma} (\alpha^2 D_1) \right] \gamma = \gamma_j. \end{aligned} \quad (29)$$

where $\sum_{\gamma}$ means a summation of all residues of poles in the upper half of the γ -plane. For real values of γ_j , only the branches for which $d\gamma_j/d\omega$ is positive are included in the summation as in the reference (2).

Substitution of Eqs.(29) into Fourier inversion formula (12.1) gives integral expressions of strains as

$$\tilde{E} = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \sum_{\gamma} E_j \exp[i(\gamma_j Z - \omega t)] d\omega, \quad (30)$$

where E is a generic expression for the strains and E_j may be easily obtained from Eqs.(29). It is impossible in general to carry out these integrations by simple means, because $\gamma_j(\omega)$ in the integrands are implicitly defined through the frequency equation. However, by using saddle point methods of integration, asymptotic expressions can be obtained for large values of Z . The application of this technique to the longitudinal impact problems of a circular bar is formulated and developed by Skalak¹⁾, Folk et al.²⁾ and Jones and Norwood³⁾.

To evaluate integrals in the form of Eq.(30) approximately for large values of Z , the path of integration is so deformed as to pass through the saddle points of the function $\gamma_j(\omega)Z - \omega t$, in the direction of steepest descents. Then the asymptotic expansion of the integral is wholly determined by the behaviour of the integrand near those saddle points. These saddle points are determined by

$$\frac{d\gamma_j(\omega)}{d\omega} = \frac{t}{Z}, \quad (31)$$

where $d\gamma_j/d\omega$ is the reciprocal of the group velocity. For a given ω , there is an infinite number of γ_j ; real, imaginary, and complex. The contribution from the saddle points where γ_j has a non-real value decays exponentially with Z . Hence it is sufficient to consider real values of γ_j if the analysis is concerned only with the large values of Z .

For a given t/Z the position of the saddle points may be determined by a graph showing the relation between ω and $d\gamma_j/d\omega$. Such a graph is given in Fig.2 for the first three branches of the longitudinal

Fig.2

mode of a square bar using dimensionless variables.

The abscissa, $d\gamma_j/d\omega$ multiplied by C_0 , is the reciprocal of the group velocity in a dimensionless form where $C_0 (= \sqrt{E/\rho})$ is the velocity of longitudinal waves given by the elementary one dimensional theory. To determine the saddle points for a fixed location Z , using the relation (31) this axis may be read as indicating time. Full lines and broken lines in the figure are the results computed from the first and second order approximations, respectively. As can be seen in the figure, for the head of the pulse ($C_0 t/Z=1$) there is a saddle point only on the first branch. Namely, for large values of Z , the head of the pulse of the strain waves are approximately given by the contribution from the first branch in the region of small ω . However the group velocity takes maximum value at $\omega=0$, i.e., $d^2\gamma_j/d\omega^2=0$ at $\omega=0$. Further the integrands for the normal strains have a pole at $\omega=0$. Hence special consideration is required when applying saddle point methods in this case⁶⁾. Expanding $\gamma_j(\omega)$ and other functions of ω included in the integrands of Eq.(30) about $\omega=0$, and estimating the integrals of the resulting functions, we obtain

$$\begin{aligned} \varepsilon_Z/V_0 &= -[A(Z,T) + h_Z C(Z,T)], \\ \varepsilon_X/V_0 &= \nu[A(Z,T) + h_X C(Z,T)], \\ \varepsilon_Y/V_0 &= \nu[A(Z,T) + h_Y C(Z,T)], \\ \gamma_{ZX}/V_0 &= -\frac{1}{3}X\left(\frac{1}{R^2} - X^2\right)G(Z,T), \\ \gamma_{ZY}/V_0 &= -\frac{1}{3}Y(1-Y^2)G(Z,T), \end{aligned} \quad (32)$$

where

$$\begin{aligned} X &= x/b, Y = y/b, Z = z/b, T = C_0 t/b, R = b/a, \\ A(Z,T) &= \frac{1}{3} + \int_0^B A_i(-u) du, \\ C(Z,T) &= \left(\frac{3}{2}\nu^2 k_p^2 Z\right)^{2/3} \frac{dA_i(-B)}{dB}, \\ G(Z,T) &= \frac{1}{3\nu(1-\nu)k_p^2 Z} \frac{d^2 A_i(-B)}{dB^2}, \\ B &= \frac{T-Z}{(3\nu^2 k_p^2 Z/2)^{1/3}}, \quad k_p^2 = \frac{1}{3}\left(1 + \frac{1}{R^2}\right), \\ h_Z &= \frac{\nu}{2} [X^2 + Y^2 - k_p^2(1+\nu)], \\ h_X &= \frac{-3+2\nu+3\nu^2-\nu^3}{6R^2(1-\nu)} + \frac{\frac{1}{6}\nu(2+\nu) + \frac{1-2\nu^2}{2(1-\nu)}X^2}{}, \\ h_Y &= \frac{-3+2\nu+3\nu^2-\nu^3}{6(1-\nu)} + \frac{\frac{1}{6}\nu(2+\nu) + \frac{1-2\nu^2}{2(1-\nu)}Y^2}{}, \end{aligned} \quad (33)$$

and $A_i(u)$ is the Airy function. Substituting these solutions into Hooke's law, expressions of stresses may easily be obtained.

5. Numerical results and consideration

The approximate solutions (32) for the head of the pulse of strain waves are valid asymptotically at large distances from the impacted end. In the expressions of normal strains, the first terms including an integral of the Airy function are identical with the results given by Love's one dimensional approximate theory which accounts for the effects of lateral inertia with the assumption that the plane sections remain plane.

The second terms correspond to the warping of the cross section and nonlinear distribution of lateral displacements. As the pulse propagates down the bar, the contribution of the second terms decreases according to factor $Z^{-2/3}$.

The axial strain ϵ_z is one of the most noticeable quantities in the longitudinal impact problems. It should be emphasized that the expression (32) for ϵ_z completely coincides with that for a circular bar given by Jones and Norwood⁶⁾ if it is expressed by the radius of gyration of the cross section. Hence, we may conclude that the lateral inertia effects on ϵ_z in a rectangular bar are approximately same as those in a circular bar having an identical radius of gyration.

In Fig.3 numerical results are shown for normal strains, where $Z=10$, $R=1$ and $\nu=1/3$. The parameter $r(=\sqrt{\lambda^2+\gamma^2})$ in the figure indicates the distance from the centroid of the cross section. As can be seen in the figure, the overshootings and subsequent vibrations due to the lateral inertia effects are most pronounced at the corners of the cross section, and they decrease with a decreasing r .

Corresponding to the warping of the cross section and the nonlinear distribution of lateral displacement, shear strains exist in the cross section. In Fig.4 a shear strain history is shown at a location $Z=10$. These shear strains decay rapidly as $1/Z$ with an increase of the propagation distance. The lines of shear stresses are shown in Fig.5 for the case of $R=1$ and $R=1/2$. It is observed that when $R=1/2$, these lines become almost parallel to the longer sides of the cross section.

Fig.3

Fig.4

Fig.5

6. Concluding remarks

Asymptotic solutions are obtained for longitudinal transient strain waves in a bar of rectangular cross section with traction-free parallel side planes. These solutions describe the head of the pulse at large distances from the impacted end. Obtained expression for the axial strain contains the effect of warping of plane sections. It was found that the expression completely coincides with that of a circular bar, if it is written using the radius of gyration of the cross section. Shear stresses are found to decrease rapidly with an increase in the propagation distances.

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