621-253:532.5:518

Paper No. 156 - 7

Finite Element Analysis of Cascade Flow with Varying Flow Rate*

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The finite element method for the unsteady two-dimensional inviscid rotational flow through a cascade proposed in the previous paper(1) was extended in order to apply to the varying flow rate case. The most important improved point is the treatments of the trailing vortex wake. Two basic oscillatory flows through flat plate cascades which have either parallel or normal velocity disturbance to the cascade were analysed. The calculated results agreed substantially with the existing one. Then the lift forces of a double-circular arc cascade were compared with that of the corresponding flat plate cascade. The present method is applicable to the flows through arbitrarily formed cascades with relatively large periodic and non-periodic variations of flow rate.

1. Introduction

In a turbo-pump installed in the piping systems which oscillate owing to water hammer and surging, and in an axial flow compressor the front stages of which have fallen into the rotating stall condition, the local flow rate through the rotor cascade varies remarkably. The purpose of this paper is to improve the finite element solution developed in the previous paper(1), so as to elucidate the unsteady characteristics of such largely disturbed flows.

In the existing theories for the unsteady characteristics of cascade in distorted and oscillatory periodic flows, the disturbances of flow have been generally treated separately dividing them into two sorts of components: one is a sinusoidal wave gust travelling with the main stream velocity, and the other is a uniformly oscillating wave gust in the upstream region. The former has been studied by Kemp and Sears(2)(3) and many other workers(4)-(8). But in the latter problem which has to be solved in the case of varying flow rate only a few theories become known, that is, the theory of flat plate cascade without stagger(4) and the linearized theory of staggered cascade(7). There have been several other approximate analyses(9)(10) which dealt with the moving cascade subject to a non-uniform inlet flow. The present numerical method was developed to analyze in more detail the flow through an arbitrary cascade with a largely varying flow rate.

The finite element method previously presented(1) can be sufficiently applied to the largely oscillating distorted flow which

is difficult to use the linearized theory. But for the varying flow rate it needs to be improved as to the calculations of the trailing vortex wakes and the fluid-dynamic forces acting on the blades. In the following we shall describe the details of the method laying stress upon improved points, show the comparison between the present and existing results of flat plate cascades, and discuss the effects of the frequency and the amplitude of upstream boundary flow and its position on the characteristics of a double-circular arc cascade.

Nomenclature

Let all the following quantities be made dimensionless by referring them to the proper characteristic values.

- A : amplitude of velocity
- c : circumferential velocity
- D : drag force per unit span, Eq. (27)
- $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$
- $K = \ell\omega/2\pi U$: nondimensional frequency
- L : lift force per unit span, Eq. (27)
- ℓ : chord length
- $P = p + u^2/2$
- p : static pressure
- q : flow rate, Eq. (5)
- s : spacing
- t : time
- U : main stream velocity
- $\mathbf{u} = (u, v)$: relative velocity
- $\bar{u}$: average value of u with respect to y outside of cascade, Eq. (5)
- $\mathbf{X} = (X, Y)$: fluid-dynamic forces acting on blade per unit span, Eq. (26)
- $\mathbf{x} = (x, y)$: Cartesian coordinates
- α, β : absolute and relative flow angles
- γ : strength of shed vortices (trailing vortex wake) per unit length of x
- $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$: Laplacian
- δt : time interval
- $\delta x, \delta y$: distances in x- and y-directions between neighbouring nodal points
- θ : stagger
- $\zeta = v_x - u_y$: vorticity

* Received 16th May, 1977.

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ψ : stream function ($\psi_x = -v$, $\psi_y = u$)
 ω : frequency
 Superscripts
 n : n -th time step
 $\sim$: trailing vortex wake
 Subscripts
 A : blade surface
 i, j : i, j -th nodal point
 x, y, t : partial derivatives with respect to x, y and t , respectively
 U, D : upstream and downstream boundaries
 $1, 0$: upper and lower boundaries outside of cascade

2. Rotating cascade flow with varying flow rate

We consider here an incompressible inviscid time-dependent two-dimensional flow through a rotating cascade on a cylindrical surface. Assume that the cascade is rotating with a constant circumferential speed c , and the flow discharged from the inlet guide vanes has a constant flow angle α_U at the upstream boundary irrespective of the flow rate, Fig.1. The governing equations of the flow are

$$\frac{du}{dt} \equiv u_t + \frac{1}{2} \nabla u^2 - u \times \zeta = -\nabla p \quad (1)$$

$$\nabla \cdot u = 0 \quad (2)$$

For the numerical calculation Poisson's equation of stream function and the vorticity transfer equation

$$\Delta \psi = -\zeta \quad (3)$$

$$d\zeta/dt = 0 \quad (4)$$

are used, which are derived from the above equations. The flow rate through a passage between the blades is

$$q(t) = \bar{u}(t)s = \int_{y_0}^{y_1} u dy \quad (5)$$

In the case of varying flow rate the flow field is in general rotational and from Eq. (4) the vorticities are preserved along a path line without diffusion.

Upstream boundary: Assuming that the inlet flow is uniform with respect to y , we obtain the boundary condition

$$\psi_x(x_U, y, t) = c - q(t) \tan \alpha_U / s \quad (6)$$

Since the circulation $\int_{y_0}^{y_1} v dy$ is preserved, $\zeta = v_x$ is a function only depending on $[t - (x - x_U) / \bar{u}]$. Therefore the starting values

of vorticity are given as

$$\zeta(x_U, y, t) = -(\tan \alpha_U) q'(t) / q(t) \quad (7)$$

Downstream boundary: Assuming that the outlet flow angle β_D does not depend on y , we obtain the boundary condition

$$\psi_x(x_D, y, t) = -u(x_D, y, t) \tan \beta_D(t) \quad (8a)$$

where

$$\left. \begin{aligned} u(x_D, y, t) &= \bar{u}(t) - \cos^2 \beta_D(t) \\ &\times \left[\int_{y_0}^y \zeta(x_D, \eta, t) d\eta - C(t) \right] \end{aligned} \right\} \quad (8b)$$

$$C(t) = \frac{1}{s} \int_{y_0}^y dy \int_{y_0}^{y_1} \zeta(x_D, \eta, t) d\eta$$

and may simplify as

$$\psi_x(x_D, y, t) \approx -\bar{u}(t) \tan \beta_D(t) \quad (8c)$$

Upper and lower boundaries outside of cascade: From the periodicity of the flow

$$\left. \begin{aligned} y_1 &= y_0 + s, \\ \psi(x, y_1, t) &= \psi(x, y_0, t) + q(t) \\ \psi_y(x, y_1, t) &= \psi_y(x, y_0, t) \\ \zeta(x, y_1, t) &= \zeta(x, y_0, t) \end{aligned} \right\} \quad (9)$$

Surface of blades:

$$\psi(x_A, y_A, t) = 0 \quad \text{and} \quad q(t) \quad (10)$$

3. Numerical method

3.1 Outline of the method

The present method is based on the finite element solution developed in the previous paper⁽¹⁾, and an outline of the procedure is illustrated in the flow chart of Fig.2. In the method the mixed boundary value problem of Eq. (3) under the conditions of Eqs. (6), (8)-(10) is replaced by an equivalent variational problem, and by solving the latter problem using the finite element method, the values of stream function $\psi(x, y)$ are determined. Meanwhile by solving the initial value problem of Eq. (4) with the starting values of Eq. (7) by means of the convective-difference schemes, the values of vorticity $\zeta(x, y, t)$ are determined.

When the conditions of flow on the upstream boundary are prescribed, the outlet flow angle β_D and the strength of shed vortices from the trailing edge y_1 become unknown variables. The values of these variables can be determined in the processes of the iterative calculation. That is, first the predictors of β_D and y_1 are adequately estimated and then the correctors are calculated so that both the Kutta condition and the one-valued pressure condition are satisfied.

In the previous paper, the distorted periodic flow owing to the wakes of the inlet guide vanes was treated. The disturbance on the upstream boundary has the wave-length s and propagates at the speed $-c$ in the y -direction. The averaged-values of the vorticity with respect to y in the downstream region are nearly equal to zero. For this reason the trailing vortex wakes lie approximately along the stagnation streamlines of the corresponding steady potential flow without disturbances. However, in the periodic

Fig.1 Computational domain and the inlet velocity diagram

flow with the varying flow rate treated in the present paper, the vorticities take in general the same sign in the y -direction, and the trailing vortex wakes are considerably curved. Therefore, it is necessary to consider the shift of the wake. In the following section, we shall describe in detail only some improved points of the method.

3.2 Trailing vortex wakes

If the vortices shed by the blade are carried away along path lines without diffusion, then they form the vortex sheet as a discontinuity on the streak line through the trailing edge. Let us consider an element of the sheet which is composed of the same fluid particles. Denoting the x -length of the element at time t by Δx and the strength per unit x -length by γ , the length at time $(t+\Delta t)$ is expressed as $\{1+(d\tilde{u}/dx)\Delta t\}\Delta x$ and the strength as $\gamma+(d\gamma/dt)\Delta t$. Therefore, from the conservation of vortex, it follows that

$$\gamma \Delta x = (\gamma + \frac{d\gamma}{dt} \Delta t) (1 + \frac{d\tilde{u}}{dx} \Delta t) \Delta x$$

Supposing that the value of Δt is sufficiently small, the relation is rewritten as

$$\frac{d\gamma}{dt} = -\gamma \frac{d\tilde{u}}{dx} \quad (11)$$

where $d/dx \approx \partial/\partial x + (\tilde{v}/\tilde{u})\partial/\partial y$. Introducing the approximate relation $\tilde{u}_t/\tilde{u} \approx \tilde{u}_t/\bar{u}$ into Eq. (11), we get

$$\frac{d}{dt} \left(\frac{\tilde{u}\gamma}{\bar{u}} \right) \approx 0 \quad (12)$$

i.e. the value of $\tilde{u}\gamma/\bar{u}$ is kept constant along a path line.

The position of vortex sheet $\tilde{y}(x)$ together with the velocity $\tilde{u}(x)$ and the di-

rection $\tilde{\beta}(x)$ can be successively obtained from the trailing edge to the downstream boundary by using the following predictor-corrector-method. Let the flow field in the downstream region be discretized into the linear triangular elements of a size, Fig.3. Since the vortex sheet is a streak line, we have

$$\left. \begin{aligned} \tilde{y}_i^{n+1}(p) &= \tilde{y}_{i-1}^{n+1} + \delta x \tan \tilde{\beta}_{i-1}^{n+1} \\ \tilde{y}_i^{n+1}(c) &= \tilde{y}_p + \delta x \tan \tilde{\beta}_m \end{aligned} \right\} \dots (13)$$

where

$$\tilde{f}_m = \frac{1}{2} (\tilde{f}_p + \tilde{f}_i^{n+1}) \dots (14)$$

And $\tilde{f}_p$ is obtained by using Lagrangian interpolation polynomial

$$\tilde{f}_p = \frac{1}{2} \tau (1+\tau) \tilde{f}_{i-1}^{n-1} + (1-\tau^2) \tilde{f}_{i-1}^n - \frac{1}{2} \tau (1-\tau) \tilde{f}_{i-1}^{n+1} \dots (15)$$

where

$$\tau = \delta x / \tilde{u}_m \delta t - 1 \dots (16)$$

The velocity u has a jump $-\gamma \cos^2 \tilde{\beta}$ across the vortex sheet in the direction of increasing y , and the stream function ψ also becomes discontinuous along it. Hence, in order to calculate the accurate interpolated values and differences of ψ , it is adequate to employ the stream function without these discontinuities

$$\Psi(y) \equiv \psi(y) + 1(y-\tilde{y})(y-\tilde{y})\gamma \cos^2 \tilde{\beta} \dots (17)$$

along an $x = \text{const.}$ line. Where $1(y)$ denotes the Heaviside function. Then the velocity is

$$u(y) = \Psi'(y) - 1(y-\tilde{y})\gamma \cos^2 \tilde{\beta} \quad (18)$$

especially

$$\tilde{u}_i = \Psi'(\tilde{y}_i) - \frac{1}{2} \gamma_i \cos^2 \tilde{\beta}_i$$

and the flow angle is

$$\tan \tilde{\beta}_i = \frac{1}{2} (\tilde{y}_{i+1} - \tilde{y}_{i-1}) / \delta x \dots (19)$$

where the position $\tilde{y}_{i+1}$ on the stream line $\psi = \psi(\tilde{y}_i)$ is calculated by using Newton's method, that is

$$\tilde{y}_{i+1} = \tilde{y}_i + [\psi(\tilde{y}_i) - \psi(\tilde{y}_{i+1})] / u(\tilde{y}_{i+1}) \dots (20)$$

The strength of trailing vortices γ_i^{n+1} is determined by the relation

$$(\tilde{u}\gamma/\bar{u})_i^{n+1} = (\tilde{u}\gamma/\bar{u})_p \quad (21)$$

Fig.2 Simplified flow chart

Fig.3 Trailing vortex wake

deduced from Eq.(12). In the calculation of ψ , the value of γ_i^{n+1} is divided into values at the nodal points neighbouring the vortex sheet.

$$\begin{aligned}\Delta\psi_{i,j}\delta x\delta y &= -\zeta_{i,j}\delta x\delta y - (1-\eta)\gamma_i\delta x \\ \Delta\psi_{i,j+1}\delta x\delta y &= -\zeta_{i,j+1}\delta x\delta y - \eta\gamma_i\delta x\end{aligned}\quad (22)$$

3.3 Fluid-dynamic forces acting on a blade

From the equation of motion (1) we have

$$dP = -\mathbf{u}_t \cdot d\mathbf{s} - \zeta d\psi \quad (23)$$

Integrating the equation along an integral path yields

$$P_2 - P_1 = -\int_1^2 (\mathbf{u}_t dx + v_t dy + \zeta d\psi) \pm [\tilde{u}\gamma] \quad (24)$$

The last term is to be added across the vortex sheet and to take the minus sign toward the upper side. The distribution of pressures P can be determined from Eq.(24).

Integrating the equation of motion (1) throughout the region inside the closed curve $\Gamma_1 + \Gamma_A$ shown in Fig.4, we get

$$\int [\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p] dA = 0$$

By using the relation $\nabla \cdot (\mathbf{u}\mathbf{u}) = (\mathbf{u} \cdot \nabla) \mathbf{u}$ and applying Gauss's theorem, the fluid-dynamic force acting on a blade becomes

$$\begin{aligned}\int_{\Gamma_A} \mathbf{n} p ds &= \mathbf{X} \\ &= -\int_{\Gamma_A} \mathbf{u}_t dA - \int_{\Gamma_1} [\mathbf{u}(\mathbf{n} \cdot \mathbf{u}) + \mathbf{n} p] ds \\ &= \int_{\Gamma_1 + \Gamma_A} \psi_t d\mathbf{s} - \int_{\Gamma_1} (\mathbf{u} d\psi + \mathbf{n} p ds)\end{aligned}\quad (25)$$

Here, taking account of the periodicity of flow, the x- and y-components of the force are

$$\begin{aligned}X &= \int_0^1 [u d\psi + p dy]_{x=x_3} \\ &\quad - \int_0^1 [u d\psi + p dy]_{x=x_4} - q'(t)(x_4 - x_3)\end{aligned}\quad (26a)$$

$$\begin{aligned}Y &= \int_0^1 [v d\psi - \psi_t dy]_{x=x_3} \\ &\quad - \int_0^1 [v d\psi - \psi_t dy]_{x=x_4} - q'(t)l \sin \theta\end{aligned}\quad (26b)$$

And the normal and parallel components to the chord are

$$\begin{aligned}L &= Y \cos \theta - X \sin \theta \\ D &= X \cos \theta + Y \sin \theta\end{aligned}\quad (27)$$

Fig.4 Fluid-dynamic forces acting on blade

4. Numerical examples

4.1 Flat plate cascades

We first consider two basic oscillatory flows through flat plate cascades to compare the calculating results to the existing one. The pitch-chord ratio $s/l = 1.0$, and the stagger $\gamma = 30^\circ$ and 45° .

(a) The flow having the y-directional velocity disturbance alone

$$\begin{aligned}u_U &= 1 \\ v_U &= \tan \theta + A \sin \omega t\end{aligned}$$

(b) The flow having the x-directional velocity disturbance alone

$$\begin{aligned}u_U &= 1 + A \sin \omega t \\ v_U &= \tan \theta\end{aligned}$$

The flow (a) that corresponds to the sinusoidal wave gust travelling with the main stream velocity in the existing theories is a rotational flow with a constant flow rate. And the flow (b) that corresponds to the uniformly oscillating wave gust in the upstream region is an irrotational flow with varying flow rate. The numerical calculation was carried out for the amplitude $A = 0.1$ and the frequencies $K = 0$ (steady), 0.25 and 0.5. K of (a) is nearly equal to the wave number per length $x_2 - x_1$.

In the calculation, $35 \times 12 \times 2$ linear triangular elements are used, and 20 nodal points of the elements lie on the blade contour. $\delta x = 0.1$, $\delta y = 0.09623$ and 0.11785 , $\delta t = 0.1$. The maximum and minimum values of lift and the phase angle for the steady oscillation are shown in Table 1. Where the phase angle ϕ denotes a difference of the calculating phase angle of maximum lift from the corresponding phase angle for the quasi-steady oscillation. The lift coefficients are determined from these values considering that the parallel component of the disturbance to the blade does not depend on the lift. A comparison of these lift coefficients and phase angles with the existing results⁽⁷⁾ is shown in Fig.5. The lift coefficient is defined as

$$c_L = \frac{M}{\pi \ell U a} \quad (28)$$

where $M = (L_{\max} - L_{\min})/2$, and a is the amplitude of velocity normal to the blade. The figure shows that the present results agree substantially with the existing one.

Table 1 Lift forces for flat plate cascades

	θ	K	$L_{\max}$	$\phi_{\max}$	$L_{\min}$	$\phi_{\min}$
(a)	30°	0	.132	0°	-.126	180°
		0.25	.114	26°	-.110	207°
		0.50	.105	63°	-.101	241°
	45°	0	.198	0°	-.188	180°
		0.25	.171	25°	-.165	205°
		0.50	.146	59°	-.143	237°
(b)	30°	0	.068	0°	-.081	180°
		0.25	.102	49°	-.108	223°
		0.50	.173	67°	-.173	245°
	45°	0	.179	0°	-.208	180°
		0.25	.258	48°	-.271	222°
		0.50	.427	65°	-.427	243°

4.2 Double-circular arc cascade

We next consider a flow through a moving cascade composed of double-circular arc blades with the maximum thickness 8% and the maximum camber 6%, whose pitch-chord ratio

Fig.5 Lift forces for flat plate cascades
(a) ● Present method, — Nishiyama-Matsudaira
(b) ○ Present method, — Nishiyama-Matsudaira

Fig.6 Flow through double-circular arc cascade

— $\omega t = 0$, — $\omega t = \pi/2$,
..... $\omega t = \pi$, - - - $\omega t = 3\pi/2$

$s/\ell = 1.0$ and stagger $\gamma = 30^\circ$. Assume that the flow has a constant absolute flow angle at the upstream boundary and the flow rate oscillates sinusoidally.

$$u_U = 1 + A \sin \omega t$$

$$v_U = -c + u_U \tan \alpha_U$$

The numerical calculation was carried out for the circumferential velocity $c = -1.4$, the inlet flow angle $\alpha_U = -23^\circ$, the amplitudes $A = 0, 0.1, 0.2$, the frequencies $K = 0$ (steady), 0.25, 0.5 and the positions of upstream boundary $x_U = -0.5, -1.0, -1.5$ ($x_1 = 0, x_2 = 1.0$).

In the calculation, the elements and time interval similar to the section 4.1 are used. The velocity distributions on the blade surface, the fluid dynamic forces act-

(a) $K = 0.25$

(b) $K = 0.5$

Fig.7 Time-variations of lift and drag forces

— Double-circular arc cascade,
- - - Flat plate cascade

ing on a blade, the positions and strengths of trailing vortex wake for each time step, and the nodal values of vorticity, stream function and velocity for every one-fourth period are printed.

As an example, the velocity distributions on the blade surface u_A , the stagnation streamlines and the trailing vortex wakes and their strengths γ are shown in Fig.6. It is pointed out that these streamlines and wakes are considerably shifting. The time variations of the normal and parallel components of fluid-dynamic force acting on a blade, namely lift L and drag D are shown in Fig.7, for $A=0, 0.1, 0.2$, $K=0.25, 0.5$ and $x_U=-1.0$. The figures show that the lift and drag forces oscillate around the corresponding steady values ($A=0$), and that the amplitudes have an equivalent order to the flat plate cascade with the same pitch-chord ratio and stagger.

The amplitudes of the oscillating lift generally depend on the amplitude A and frequency K of velocity disturbance and the position of upstream boundary x_U . A comparison between the amplitudes is shown in Table 2. $M=(L_{\max}-L_{\min})/2$ denotes the amplitude of lift for the double-circular arc cascade, and M_P the amplitude obtained by the superposition of the basic oscillatory flows in the preceding section. As is evident from the table, within the limits of calculation, the amplitude of the double-circular arc cascade can be approximately predicted from the values of flat plate cascades.

5. Conclusions

A finite element method has been developed for calculating the unsteady flow characteristics of an arbitrary cascade with a varying flow rate.

(1) The finite element method for the unsteady two-dimensional inviscid rotational flow through an arbitrary cascade proposed in the previous paper has been extended to be applied to the varying flow rate. The treatments of the trailing vortex wake and the fluid-dynamic forces acting on the blade were improved.

(2) Sinusoidally oscillatory flows through flat plate cascades which have ei-

Table 2 Lift amplitudes for a double-circular arc cascade

K	A	x_U	$L_{\max}$	$L_{\min}$	M	M_P
0.25	0.1	-0.5	.765	.700	.032	.116
		-1.0	.757	.688	.034	.076
		-1.5	.805	.644	.080	.073
	0.2	-1.0	.768	.617	.075	.152
		-0.5	.850	.605	.122	.151
		-1.0	.923	.530	.196	.212
0.50	0.1	-1.5	.949	.528	.210	.254
		-1.0	1.095	.312	.391	.424
		-0.5				
	0.2	-1.0				

ther parallel or normal velocity disturbance to the cascade have been analysed. The calculated results agree substantially with the existing one.

(3) The flows through a moving cascade composed of double-circular arc blades which have a constant absolute inlet flow angle and sinusoidal oscillating flow rates have been analysed. Within the limits of calculation, the lifts oscillate around the corresponding steady values and have amplitudes equivalent to the corresponding flat plate.

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