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Vibration Analysis of Hyperbolic Cooling Towers due to Earthquake Excitations*

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The hyperboloidal shells of revolution supported on columns have been used effectively as natural draft cooling towers of reinforced concrete. The purpose of this paper is to analyze the dynamic response of hyperbolic cooling towers to earthquake excitations. Taking the propagating velocity of earthquake excitations into account, the author shows that hyperbolic cooling towers oscillate not only in beam vibration modes but also in such higher vibration modes as ovalization of their cross sections. When the value $\alpha R/V$ is considerably large, where α and V are circular frequency and propagating velocity of the earthquake excitation, respectively, and R is the radius of hyperbolic cooling towers on the foundation level, numerical example shows that owing to horizontal earthquake excitations an hyperbolic cooling tower oscillates mainly in an ovalization mode and that owing to vertical earthquake excitations it oscillates mainly in a rocking mode.

1. Introduction

Hyperboloidal shells of revolution supported on columns, referred to as hyperbolic cooling towers in the following, have been constructed by thousands in Europe and America for use as reinforced concrete cooling towers of power plants, because of their geometry efficient against wind load. Owing to the large diameter of a hyperbolic cooling tower on the foundation level, for example, 100 m diameter versus 130 m height, it is necessary to consider the phase difference of earthquakes in its structural analysis.

Gould et al. obtained stress distributions under seismic design load⁽¹⁾, using Novozhilov's method⁽²⁾. Davenport et al.⁽³⁾ developed a stochastic aseismic design method using design response spectra with respect to velocity.

Recently, in Japan, architectural engineers have begun structural analysis of hyperbolic cooling towers⁽⁴⁾, neglecting the phase difference of earthquakes. As for studies on the effect of the phase difference of earthquakes on structures other than hyperbolic cooling tower, Tajimi⁽⁵⁾, Kobori et al.⁽⁶⁾ and Iguchi⁽⁷⁾ analyzed dynamic response and frequency response of circular and rectangular footings subjected to propagating seismic waves.

The author expresses earthquake excitation as propagating seismic waves in order to analyze dynamic response of hyperbolic cooling towers, the deformations of which start at the position

where the wave front meets, travelling in sequence as the wave front proceeds. On the supposition that it is possible to neglect the interaction between the foundation and the hyperbolic cooling tower, vibrations of the latter can be analyzed by dividing deformations into two parts. The first part consists of such a deformation that satisfies static equilibrium equations and also satisfies boundary conditions of the hyperbolic cooling tower which is ring-supported at upper end and is constrained to move in coincidence with earthquake excitations at lower end. Since this deformation changes continuously with earthquakes, inertia forces arise as unbalance forces. The second part of deformation is caused by the inertia forces of the first part of deformation. The second part is solved by the modal analysis method. According to numerical examples, owing to horizontal earthquakes a hyperbolic cooling tower oscillates in addition to beam-bending mode prominently in such a higher mode as ovalization of its cross section, and owing to vertical earthquake excitations it oscillates in addition to heaving motion prominently in a rocking mode and in an ovalization mode.

X-figured support columns are shown to be subjected to considerably large deformations due to the phase difference of earthquakes, which necessitate adequate reinforcement at the lower end of the hyperbolic cooling tower with a concrete foundation bed or the like.

2. Fundamental equations and method of solution

On the supposition that the interaction between the foundation and the hyperbolic cooling tower can be neglected, the lower end of the latter is constrained to move in coincidence with propagating seismic

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wave. If the interaction has to be considered, reaction forces at the lower end of the hyperbolic cooling tower obtained in this paper should act upon the unity of the foundation and the hyperbolic cooling tower. The upper end of the hyperbolic cooling tower is provided with a ring where no reaction occurs. Since in this paper an energy method with respect to the displacement is used, the dynamical boundary conditions are satisfied spontaneously. If the displacement that satisfies the inhomogeneous boundary conditions at the lower end of the hyperbolic cooling tower is obtained, vibration analysis of the hyperbolic cooling tower under earthquake excitations leads to integration of inhomogeneous equations of motion with homogeneous boundary conditions. Seismic waves propagating on the ground surface which are decomposed into three components of displacement with respect to the coordinate system of the hyperbolic cooling tower, are expressed in reference to Fig. 1 as follows;

Fig. 1 A hyperbolic cooling tower supported on columns

$$\begin{aligned} \bar{u}^{(1)}|_{\theta=\theta_1} = & \cos \theta_1 \cos \varphi F[t + (R \cos \varphi - R)/V_h] \\ & + \sin \theta_1 G[t + (R \cos \varphi - R)/V_v] \end{aligned} \quad (1)$$

$$\bar{u}^{(2)}|_{\theta=\theta_1} = -\sin \varphi F[t + (R \cos \varphi - R)/V_h] \quad (2)$$

$$\begin{aligned} \bar{u}^{(3)}|_{\theta=\theta_1} = & \sin \theta_1 \cos \varphi F[t + (R \cos \varphi - R)/V_h] \\ & - \cos \theta_1 G[t + (R \cos \varphi - R)/V_v] \end{aligned} \quad (3)$$

where θ_1 denotes the meridional angle at its lower end, φ the latitudinal angle, F and G functions determined by time history of horizontal and vertical earthquake motions, respectively, t time, R the radius of the hyperbolic cooling tower on the foundation level, V_h and V_v denote propagating velocities of horizontal and vertical earthquake motions respectively. The first term of the left hand side of Eqs. (1) through (3) expresses horizontal earthquake motions travelling in the direction

from $\varphi=0$ to $\varphi=\pi$, and the second term expresses similar vertical earthquake motions, such that

$$\begin{aligned} \text{if } t + (R \cos \varphi - R)/V_h \leq 0 \\ F[t + (R \cos \varphi - R)/V_h] = 0 \\ \text{if } t + (R \cos \varphi - R)/V_v \leq 0 \\ G[t + (R \cos \varphi - R)/V_v] = 0 \end{aligned}$$

Firstly, the case is treated where only horizontal earthquake excitations exist. Equations (1) through (3) are expanded in Fourier series with respect to φ .

$$\left. \begin{aligned} \bar{u}^{(1)}|_{\theta=\theta_1} &= \cos \theta_1 \sum_{n \neq 0} \cos n\varphi F_n^c(t) \\ \bar{u}^{(2)}|_{\theta=\theta_1} &= -\sum_{n \neq 1} \sin n\varphi F_n^s(t) \\ \bar{u}^{(3)}|_{\theta=\theta_1} &= \sin \theta_1 \sum_{n \neq 0} \cos n\varphi F_n^c(t) \end{aligned} \right\} \dots\dots\dots (4)$$

where

$$\begin{aligned} F_n^c(t) &= \frac{\varepsilon_n}{\pi} \int_{-\varphi_k}^{\varphi_k} \cos \varphi F\left(t + \frac{R \cos \varphi - R}{V_h}\right) \cos n\varphi d\varphi \\ F_n^s(t) &= \frac{1}{\pi} \int_{-\varphi_k}^{\varphi_k} \sin \varphi F\left(t + \frac{R \cos \varphi - R}{V_h}\right) \sin n\varphi d\varphi \end{aligned} \quad (5)$$

$$\begin{aligned} \varphi_k &= \min\{\pi, \cos^{-1}(1 - tV_h/R)\} \\ \varepsilon_n &= 0.5(n=0), \quad \varepsilon_n = 1(n \neq 0) \end{aligned} \quad (6)$$

If the value of tV_h/R is less than 2, φ_k points to the wave front of horizontal earthquakes, and after the wave front has passed over the position $\varphi=\pi$, φ_k is set at π . Since the functions $F_n^c(t)$ and $F_n^s(t)$ are not generally equal to each other, boundary conditions at $\theta=\theta_1$ should be divided into ones which are proportional either to $F_n^c(t)$ or to $F_n^s(t)$. Let quantities which are proportional to $F_n^c(t)$ be given superscript c . Displacement u^c which satisfies equilibrium condition together with boundary conditions at $\theta=\theta_1$, may be put, with circumferential wave number n fixed arbitrary, as follows,

$$\begin{aligned} u^c &= F_n^c(t) [\bar{u}_n^{c(1)}(\theta) \cos n\varphi, \\ & \bar{u}_n^{c(2)}(\theta) \sin n\varphi, \bar{u}_n^{c(3)}(\theta) \cos n\varphi] \end{aligned} \quad (7)$$

and should be subjected to the following equations

$$L_i(u^c) = 0 \quad (i=1, 2, 3) \quad (8)$$

$$\left. \begin{aligned} \bar{u}_n^{c(1)}(\theta_1) &= \cos \theta_1, \quad \bar{u}_n^{c(2)}(\theta_1) = 0, \\ \bar{u}_n^{c(3)}(\theta_1) &= \sin \theta_1, \quad \partial \bar{u}_n^{c(3)}(\theta_1) / \partial s = 0 \end{aligned} \right\} \quad (9)$$

Here L_i are differential operators expressing equilibrium conditions which have individual forms for each structural component, that is, for a thin shell, a ring, or a straight beam of support columns. Equation (8) is for equilibrium conditions and Eq. (9) for boundary conditions. Elastic springs resisting displacements or rotation are assumed to join the foundation with the lower end of the hyperbolic cooling tower, and therefore Eq. (9) gives displacements or rotation on the foundation side of the elastic spring. The last equation of Eq. (9) is for rotation, with meridional

line element ds positive downward. Equations (8) and (9) can be easily solved by energy methods such as finite element method. In this paper, free vibration is analyzed by the finite element method with ring shell elements and lattice elements for hyperboloidal shell and support columns, respectively (8), the stiffness matrix of which can be used to solve Eqs. (8) and (9).

Similarly, let quantities which are proportional to $F_n^*(t)$ be given superscript s . Displacements u^s can be obtained by solving the following equations

$$u^s = F_n^s(t) \{ \bar{u}_n^{s(1)}(\theta) \cos n\varphi, \bar{u}_n^{s(2)}(\theta) \sin n\varphi, \bar{u}_n^{s(3)}(\theta) \cos n\varphi \} \quad (10)$$

$$L_i(u^s) = 0 \quad (i=1, 2, 3) \quad (11)$$

$$\left. \begin{aligned} \bar{u}_n^{s(1)}(\theta_1) &= 0, \quad \bar{u}_n^{s(2)}(\theta_1) = -1 \\ \bar{u}_n^{s(3)}(\theta_1) &= 0, \quad \partial \bar{u}_n^{s(3)}(\theta_1) / \partial s = 0 \end{aligned} \right\} \quad (12)$$

Let $\bar{u}$ be the displacement which satisfies equilibrium conditions together with boundary conditions at $\theta = \theta_1$. Then $\bar{u}$ is the sum of Eqs. (7) and (10).

$$\begin{aligned} \bar{u} &= (\bar{u}^{(1)}, \bar{u}^{(2)}, \bar{u}^{(3)}) = \sum_n (u^s + u^*) \\ &= \sum_{n \neq 0} [F_n^s(t) \{ \bar{u}_n^{s(1)}(\theta) \cos n\varphi, \bar{u}_n^{s(2)}(\theta) \sin n\varphi, \bar{u}_n^{s(3)}(\theta) \cos n\varphi \} + F_n^*(t) \{ \bar{u}_n^{*(1)}(\theta) \cos n\varphi, \bar{u}_n^{*(2)}(\theta) \sin n\varphi, \bar{u}_n^{*(3)}(\theta) \cos n\varphi \}] \quad (13) \end{aligned}$$

Since $\bar{u}$ changes with t proportionally to $F_n^s(t)$ or $F_n^*(t)$, inertia forces arise as unbalance forces, which cause vibration u . Let u^* be total displacements. The equations of motion in reference to the absolute quiescent coordinate system are as follows.

$$L_i(u^*) = \lambda (\partial \bar{u}^{(i)} / \partial t) + \rho h (\partial^2 \bar{u}^{(i)} / \partial t^2) \quad (i=1, 2, 3) \quad (14)$$

where

$$\begin{aligned} u^* &= \bar{u} + u, \quad u^* = (\bar{u}^{(1)}, \bar{u}^{(2)}, \bar{u}^{(3)}), \\ u &= (u^{(1)}, u^{(2)}, u^{(3)}) \quad (15) \end{aligned}$$

Dividing u^* of Eq. (15) into $\bar{u}$ and u yields

$$\begin{aligned} L_i(u) &= \lambda (\partial u^{(i)} / \partial t) + \rho h (\partial^2 u^{(i)} / \partial t^2) \\ &+ \lambda (\partial \bar{u}^{(i)} / \partial t) + \rho h (\partial^2 \bar{u}^{(i)} / \partial t^2) - L_i(\bar{u}) \quad (i=1, 2, 3) \quad (16) \end{aligned}$$

Here, λ denotes damping coefficient and ρ and h denote density and thickness of hyperboloidal shell, respectively, whereas ρh is evaluated discretely for ring beam and support columns. The last term of the right hand side of Eq. (16) is interpreted as follows: when displacement $\bar{u}$ is forced to occur on the hyperbolic cooling tower, which changes with time, there appear inertia forces $\rho h (\partial^2 \bar{u}^{(i)} / \partial t^2)$ and damping forces $\lambda (\partial \bar{u}^{(i)} / \partial t)$ which cause a damping vibration u . The boundary conditions i.e., free at upper end and spring-supported at the lower end, are

obtained by free vibration analysis in the following form.

$$\left. \begin{aligned} u^{(1)} &= \sum_n \{ \sum_m q_{mn}(t) u_{mn}^{(1)}(\theta) \} \cos n\varphi, \\ u^{(2)} &= \sum_n \{ \sum_m q_{mn}(t) u_{mn}^{(2)}(\theta) \} \sin n\varphi \\ u^{(3)} &= \sum_n \{ \sum_m q_{mn}(t) u_{mn}^{(3)}(\theta) \} \cos n\varphi \end{aligned} \right\} \quad (17)$$

Here, $q_{mn}(t)$ is an unknown function of t and $u_{mn}^{(i)}(\theta)$ is the lower m -th natural vibration mode with circumferential wave number fixed to n . $u_{mn}^{(i)}(\theta)$ satisfies equations of free vibration as follows

$$\begin{aligned} L_i(u_{mn}^{(1)} \cos n\varphi, u_{mn}^{(2)} \sin n\varphi, u_{mn}^{(3)} \cos n\varphi) \\ + \rho h \lambda_{mn}^2 u_{mn}^{(i)} \begin{pmatrix} \cos \\ \sin \end{pmatrix} n\varphi = 0 \quad (i=1, 2, 3) \quad (18) \end{aligned}$$

Terms in the above parenthesis are assigned to $\cos$ for $i=1, 3$ and to $\sin$ for $i=2$. λ_{mn} denotes circular eigen-frequency. Among natural vibration modes the orthogonal condition holds good, with the Kronecker symbol δ_i^k

$$\begin{aligned} \int \int (u_{ij}^{(1)} \cos j\varphi u_{mn}^{(1)} \cos n\varphi + u_{ij}^{(2)} \sin j\varphi u_{mn}^{(2)} \sin n\varphi \\ + u_{ij}^{(3)} \cos j\varphi u_{mn}^{(3)} \cos n\varphi) \rho h dA = \delta_m^i \delta_n^j \quad (19) \end{aligned}$$

where

$$dA = R_1 R_2 \sin \theta d\theta d\varphi = R_1 d\theta r d\varphi, \quad r = R_2 \sin \theta \quad (20)$$

To single out $q_{mn}(t)$, Eq. (16) is multiplied on both sides by variation of displacement $\delta u^{(i)}$, summed over all i , and integrated on the surface of the hyperbolic cooling tower. Considering $L_i(\bar{u}) = 0$, we obtain the following equation

$$\begin{aligned} \sum_{i=1}^3 \int \int \delta u^{(i)} L_i(u) dA = \sum_{i=1}^3 \int \int \delta u^{(i)} \left(\frac{\lambda}{\rho h} \frac{\partial u^{(i)}}{\partial t} + \frac{\partial^2 u^{(i)}}{\partial t^2} \right. \\ \left. + \frac{\lambda}{\rho h} \frac{\partial \bar{u}^{(i)}}{\partial t} + \frac{\partial^2 \bar{u}^{(i)}}{\partial t^2} \right) \rho h dA \quad (21) \end{aligned}$$

Since damping is mainly due to dispersion of energy into the foundation, it may be well assumed that λ is proportional to ρh . Eliminating the operator L_i by means of Eq. (18), and using the orthogonality condition of Eq. (19), we can single out $q_{mn}(t)$ from Eq. (21). For this purpose $\delta u^{(i)}$ is taken as follows

$$\begin{aligned} \delta u^{(1)} &= u_{mn}^{(1)} \cos n\varphi, \quad \delta u^{(2)} = u_{mn}^{(2)} \sin n\varphi, \\ \delta u^{(3)} &= u_{mn}^{(3)} \cos n\varphi \quad (22) \end{aligned}$$

Finally we obtain the equation for $q_{mn}(t)$ with the notation (dot) for time-derivative, as follows;

$$\lambda_{mn}^2 q_{mn} + (\lambda / \rho h) \dot{q}_{mn} + \ddot{q}_{mn} = f_{mn}(t) \quad (23)$$

$$\begin{aligned} f_{mn}(t) &= \frac{-\pi}{\varepsilon_n} \left(\frac{\lambda}{\rho h} \dot{F}_n^s + \ddot{F}_n^s \right) \int_{\theta_1}^{\theta_2} (u_n^{s(1)} u_{mn}^{(1)} \\ &+ \bar{u}_n^{s(2)} u_{mn}^{(2)} + \bar{u}_n^{s(3)} u_{mn}^{(3)}) \rho h R_1 r d\theta \\ &- \pi \left(\frac{\lambda}{\rho h} \dot{F}_n^* + \ddot{F}_n^* \right) \int_{\theta_1}^{\theta_2} (\bar{u}_n^{*(1)} u_{mn}^{(1)} + \bar{u}_n^{*(2)} u_{mn}^{(2)} \\ &+ \bar{u}_n^{*(3)} u_{mn}^{(3)}) \rho h R_1 r d\theta \quad (24) \end{aligned}$$

where θ_2 denotes the meridional angle of the upper end. Since Eq. (23) is a linear ordinary differential equation of second order with constant coefficients, the unknown $q_{nn}(t)$ is easily integrated, with initial values of $q_{nn}(0)$ and $\dot{q}_{nn}(0)$ denoted by $q_{nn}(0)$ and $\dot{q}_{nn}(0)$ respectively, viz.

$$q_{nn}(t) = \left\{ q_{nn}(0) \cos \gamma_{nn} t + \frac{1}{\gamma_{nn}} \left(\frac{\lambda}{2\rho h} q_{nn}(0) + \dot{q}_{nn}(0) \right) \sin \gamma_{nn} t \right\} \exp\left(-\frac{\lambda t}{2\rho h}\right) + \frac{1}{\gamma_{nn}} \int_0^t \exp\left\{-\frac{\lambda(t-\tau)}{2\rho h}\right\} \sin \gamma_{nn}(t-\tau) f_{nn}(\tau) d\tau \quad (25)$$

where $\gamma_{nn} = \{\lambda_{nn}^2 - (\lambda/2\rho h)^2\}^{1/2}$

In the foregoing we exclude vertical earthquake excitations by putting $G(\cdot) = 0$. We can likewise treat the case where only vertical earthquake excitations exist.

$$G_n(t) = \frac{\varepsilon_n}{\pi} \int_{-\varphi_n}^{\varphi_n} G\left(t + \frac{R \cos \varphi - R}{V_n}\right) \cos n\varphi d\varphi \quad (26)$$

$$\varphi_n = \min(\pi, \cos^{-1}(1 - V_n/R)) \quad (27)$$

$$\bar{u}^* = G_n(t) \{ \bar{u}_n^{*(1)}(\theta) \cos n\varphi, \bar{u}_n^{*(2)}(\theta) \sin n\varphi, \bar{u}_n^{*(3)}(\theta) \cos n\varphi \}, \bar{u} = \sum_n \bar{u}^* \quad (28)$$

$$L_i(\bar{u}^*) = 0 \quad (i=1, 2, 3) \quad (29)$$

$$\bar{u}_n^{*(1)}(\theta_1) = \sin \theta_1, \quad \bar{u}_n^{*(2)}(\theta_1) = 0, \quad \bar{u}_n^{*(3)}(\theta_1) = -\cos \theta_1, \quad \partial \bar{u}_n^{*(3)}(\theta_1) / \partial s = 0 \quad (30)$$

$$\lambda_{nn}^2 q_{nn} + (\lambda/\rho h) \dot{q}_{nn} + \ddot{q}_{nn} = g_{nn}(t) \quad (31)$$

$$g_{nn}(t) = \frac{-\pi}{\varepsilon_n} \left(\frac{\lambda}{\rho h} \dot{G}_n + \ddot{G}_n \right) \int_{\theta_1}^{\theta_2} (\bar{u}_n^{*(1)} \bar{u}_{nn}^{(1)} + \bar{u}_n^{*(2)} \bar{u}_{nn}^{(2)} + \bar{u}_n^{*(3)} \bar{u}_{nn}^{(3)}) \rho h R_1 r d\theta \quad (32)$$

$$q_{nn}(t) = \left\{ q_{nn}(0) \cos \gamma_{nn} t + \frac{1}{\gamma_{nn}} \left(\frac{\lambda}{2\rho h} q_{nn}(0) + \dot{q}_{nn}(0) \right) \sin \gamma_{nn} t \right\} \exp\left(-\frac{\lambda t}{2\rho h}\right) + \frac{1}{\gamma_{nn}} \int_0^t \exp\left\{-\frac{\lambda(t-\tau)}{2\rho h}\right\} \sin \gamma_{nn}(t-\tau) g_{nn}(\tau) d\tau \quad (33)$$

Let us determine $q_{nn}(0)$ and $\dot{q}_{nn}(0)$ of Eqs. (25) and (33) from the initial conditions. We assume total displacement u^* and $\bar{u}$ to be quiescent initially, taking $t=0$ to be the instant when the wave front has reached the position $\varphi=0$.

When $t=0$

$$\bar{u}^{(1)} = 0, \quad \partial \bar{u}^{(1)} / \partial t = 0 \quad (i=1, 2, 3) \quad (34)$$

$$\bar{u}^{(1)} = 0, \quad \partial \bar{u}^{(1)} / \partial t = 0 \quad (i=1, 2, 3) \quad (35)$$

Putting $i=1, 2, 3$ in the first of Eq. (34), multiplying each of them by $u_{nn}^{(1)} \cos n\varphi$, $u_{nn}^{(2)} \sin n\varphi$, $u_{nn}^{(3)} \cos n\varphi$ and $u_{nn}^{(3)} \cos n\varphi$ in sequence, adding all of them and integrating with respect to mass, we obtain the following equation, by substitution of Eq. (15), i.e., $u^* = \bar{u} + u$, and of Eq. (19) of orthogonality conditions.

$$\int \int (\bar{u}^{(1)} u_{nn}^{(1)} \cos n\varphi + \bar{u}^{(2)} u_{nn}^{(2)} \sin n\varphi + \bar{u}^{(3)} u_{nn}^{(3)} \cos n\varphi) |_{t=0} \rho h dA + q_{nn}(0) = 0 \quad (36)$$

Likewise, we obtain the following equation from the second of Eq. (34).

$$\int \int \left(\frac{\partial \bar{u}^{(1)}}{\partial t} u_{nn}^{(1)} \cos n\varphi + \frac{\partial \bar{u}^{(2)}}{\partial t} u_{nn}^{(2)} \sin n\varphi \right)$$

$$+ \frac{\partial \bar{u}^{(3)}}{\partial t} u_{nn}^{(3)} \cos n\varphi \Big|_{t=0} \rho h dA + \dot{q}_{nn}(0) = 0 \quad (37)$$

If we substitute Eq. (35) into Eqs. (36) and (37), we immediately obtain;

$$q_{nn}(0) = 0, \quad \dot{q}_{nn}(0) = 0 \quad (38)$$

Nevertheless, $\bar{u}^{(1)}$ and so on are retained in Eqs. (36) and (37) for subsequent reference. We apply Eq. (38) to both Eqs. (25) and (33).

When there exist both horizontal and vertical earthquake excitations u is obtained by substituting the sum of Eqs. (25) and (33) into Eq. (17), and finally u^* is obtained from Eq. (15). From thus obtained u^* , we can evaluate stress resultants or couples with help of kinematic relations and constitutive relations.

Let us consider special cases where V_h and V_v are both infinite. These are realized when the wave lengths $V_h/(\alpha/2\pi)$ and $V_v/(\beta/2\pi)$ are large enough compared with the diameter of the lower end of the hyperbolic tower $2R$ to make it possible to neglect the phase difference of earthquakes. Since in these cases $F(\cdot)$ of Eq. (5) and $G(\cdot)$ of Eq. (26) become functions of t only, and φ_h of Eq. (6) and φ_v of Eq. (27) are taken both to be π , the following expressions are derived.

$$F_n^*(t) = F_n^*(t) = \delta_n^1 F(t), \quad G_n(t) = \delta_n^0 G(t) \quad (39)$$

Expression (39) shows that due to horizontal earthquake excitations remains only the term of wave number $n=1$ which expresses such deformations that cross sections of the hyperbolic cooling tower remain circular, and that due to vertical earthquake excitations remains only the term of wave number $n=0$ which stands for such deformations that its cross sections expand uniformly. These coincide with usual assumptions. In these cases, we can employ for $\bar{u}$ those expressions which are derived from Eqs. (1) through (3) by making V_h and V_v tend to infinity and by replacing θ ($\theta_1 < \theta < \theta_2$) for θ_1 .

$$\bar{u}^{(1)} = \cos \theta \cos \varphi F(t) + \sin \theta G(t), \quad \bar{u}^{(2)} = -\sin \varphi F(t), \quad \bar{u}^{(3)} = \sin \theta \cos \varphi F(t) - \cos \theta G(t) \quad (40)$$

Expression (40), which is known to express the rigid body motion, satisfies the boundary conditions at the lower end of the hyperbolic cooling tower and static equilibrium conditions. The unknown $q_{nn}(t)$ may be integrated similarly as the one of Eqs. (25) and (33).

$$q_{nn}(t) = \left\{ q_{nn}(0) \cos \gamma_{nn} t + \frac{1}{\gamma_{nn}} \left(\frac{\lambda}{2\rho h} q_{nn}(0) + \dot{q}_{nn}(0) \right) \sin \gamma_{nn} t \right\} \exp\left(-\frac{\lambda t}{2\rho h}\right) + \frac{1}{\gamma_{nn}} \int_0^t \exp\left\{-\frac{\lambda(t-\tau)}{2\rho h}\right\} \sin \gamma_{nn}(t-\tau) \{ f_{nn}(\tau) + g_{nn}(\tau) \} d\tau \quad (41)$$

where

$$\left. \begin{aligned} f_{mn}(t) &= -\pi \delta_n \left(\frac{\lambda}{\rho h} \dot{F} + \ddot{F} \right) \int_{\theta_1}^{\theta_2} (\cos \theta u_{mn}^{(1)} - u_{mn}^{(2)} \\ &\quad + \sin \theta u_{mn}^{(3)}) \rho h R_1 r d\theta \\ g_{mn}(t) &= -2\pi \delta_n \left(\frac{\lambda}{\rho h} \dot{G} + \ddot{G} \right) \int_{\theta_1}^{\theta_2} (\sin \theta u_{mn}^{(1)} \\ &\quad - \cos \theta u_{mn}^{(3)}) \rho h R_1 r d\theta \end{aligned} \right\} \dots\dots\dots (42)$$

Initial values $q_{mn}(0)$ and $\dot{q}_{mn}(0)$ are determined from Eqs. (36) and (37), respectively. Taking $F(0) = G(0) = 0$ in these cases, we set the initial conditions as follows.

$$\text{When } t=0 \quad \ddot{u}^{(1)} = \ddot{u}^{(2)} = \ddot{u}^{(3)} = 0 \quad \dots\dots\dots (43)$$

$$\left. \begin{aligned} \partial \ddot{u}^{(3)} / \partial t &= \sin \theta \cos \varphi \dot{F}(0) - \cos \theta \dot{G}(0) \\ \partial \ddot{u}^{(2)} / \partial t &= -\sin \varphi \dot{F}(0) \\ \partial \ddot{u}^{(1)} / \partial t &= \cos \theta \cos \varphi \dot{F}(0) + \sin \theta \dot{G}(0), \end{aligned} \right\} \dots\dots\dots (44)$$

Then, $q_{mn}(0)$ and $\dot{q}_{mn}(0)$ are evaluated as follows.

$$\left. \begin{aligned} q_{mn}(0) &= 0, \quad \dot{q}_{mn}(0) = -\pi \delta_n \dot{F}(0) \int_{\theta_1}^{\theta_2} (\cos \theta u_{mn}^{(1)} - u_{mn}^{(2)} \\ &\quad + \sin \theta u_{mn}^{(3)}) \rho h R_1 r d\theta - 2\pi \delta_n \dot{G}(0) \int_{\theta_1}^{\theta_2} (\sin \theta u_{mn}^{(1)} \\ &\quad - \cos \theta u_{mn}^{(3)}) \rho h R_1 r d\theta \end{aligned} \right\} \dots\dots\dots (45)$$

The second of Eq. (45) causes shock response due to abrupt velocity change.

For numerical examples we use sine curves of earthquake motions, of which we evaluate $F_n^e(t)$, $F_n^s(t)$ and $G_n(t)$ for finite values of V_k and V_v . Referring to the following formulas for Bessel functions

$$\left. \begin{aligned} \cos(z \cos \varphi) &= 2 \sum_{k=0}^{\infty} (-1)^k \varepsilon_k J_{2k}(z) \cos 2k\varphi, \\ \sin(z \cos \varphi) &= 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(z) \cos(2k+1)\varphi \end{aligned} \right\} \dots\dots\dots (46)$$

and putting $F(\)$ and $G(\)$ into following form

$$\left. \begin{aligned} F[t + (R \cos \varphi - R)/V_k] &= \bar{a} \sin[\alpha \{t + (R \cos \varphi - R)/V_k\}] \\ G[t + (R \cos \varphi - R)/V_v] &= \bar{b} \sin[\beta \{t + (R \cos \varphi - R)/V_v\}] \end{aligned} \right\} \dots\dots\dots (47)$$

we obtain formulas for $F_n^e(t)$, $F_n^s(t)$ and $G_n(t)$

$$\left. \begin{aligned} F_n^e(t) &= \frac{\varepsilon_n}{\pi} \bar{a} \sin\left(\alpha t - \frac{\alpha R}{V_k}\right) \int_{-\varphi_k}^{\varphi_k} 2 \sum_{k=0}^{\infty} (-1)^k \varepsilon_k J_{2k} \\ &\quad \left(\frac{\alpha R}{V_k}\right) \cos 2k\varphi \left\{ \frac{\cos \varphi \cos n\varphi}{\sin \varphi \sin n\varphi} \right\} d\varphi \\ &\quad + \frac{\varepsilon_n}{\pi} \bar{a} \cos\left(\alpha t - \frac{\alpha R}{V_k}\right) \int_{-\varphi_k}^{\varphi_k} 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1} \\ &\quad \left(\frac{\alpha R}{V_k}\right) \cos(2k+1)\varphi \left\{ \frac{\cos \varphi \cos n\varphi}{\sin \varphi \sin n\varphi} \right\} d\varphi \quad \dots\dots\dots (48) \\ G_n(t) &= \frac{\varepsilon_n}{\pi} \bar{b} \sin\left(\beta t - \frac{\beta R}{V_v}\right) \int_{-\varphi_v}^{\varphi_v} 2 \sum_{k=0}^{\infty} (-1)^k \varepsilon_k J_{2k} \\ &\quad \left(\frac{\beta R}{V_v}\right) \cos 2k\varphi \cos n\varphi d\varphi \\ &\quad + \frac{\varepsilon_n}{\pi} \bar{b} \cos\left(\beta t - \frac{\beta R}{V_v}\right) \int_{-\varphi_v}^{\varphi_v} 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1} \\ &\quad \left(\frac{\beta R}{V_v}\right) \cos(2k+1)\varphi \cos n\varphi d\varphi \quad \dots\dots\dots (49) \end{aligned} \right\}$$

where $\bar{a}$ and α denote the amplitude and circular frequency of horizontal earthquakes, respectively, and $\bar{b}$ and β denote

those of vertical earthquakes. Equations (48) and (49) are simplified when $\varphi_k = \pi$ and $\varphi_v = \pi$.

for $n = 1, 3, 5, \dots$

$$F_n^e(t) = \bar{a} \sin\left(\alpha t - \frac{\alpha R}{V_k}\right) (-1)^{(n-1)/2} \left\{ J_{n-1}\left(\frac{\alpha R}{V_k}\right) \pm J_{n+1}\left(\frac{\alpha R}{V_k}\right) \right\} \dots\dots\dots (50)$$

$$G_n(t) = \bar{b} \cos\left(\beta t - \beta R/V_v\right) (-1)^{(n-1)/2} J_n(\beta R/V_v) \dots\dots\dots (51)$$

for $n = 0, 2, 4, \dots$

$$F_n^s(t) = \bar{a} \cos\left(\alpha t - \frac{\alpha R}{V_k}\right) (-1)^{n/2} \varepsilon_n \left\{ J_{n-1}\left(\frac{\alpha R}{V_k}\right) \pm J_{n+1}\left(\frac{\alpha R}{V_k}\right) \right\} \dots\dots\dots (52)$$

$$G_n(t) = \bar{b} \sin\left(\beta t - \beta R/V_v\right) (-1)^{n/2} \varepsilon_n J_n(\beta R/V_v) \dots\dots\dots (53)$$

Due to the properties of Bessel functions, $F_n^e(t)$ and $F_n^s(t)$ are less than $\bar{a}$ and $F_n^s(t)$ is in most cases less than $\bar{b}$.

Therefore, if it happens that deformations in the case where $V_k, V_v \rightarrow \infty$ are greater than those in the case where $V_k, V_v = \infty$ it is due to difference in displacement $\bar{u}$ between both cases.

Let us consider $\bar{u}$ of a hyperboloidal shell in particular.

Since the upper end of the hyperboloidal shell is provided with a circular ring where no external forces are exerted, there exist membrane forces which are nearly zero. Since the in-plane rigidity of the shell is larger than its bending rigidity and since bending deformations damp rapidly away from edges, in most parts of the shell there prevail such deformations that inplane deformations are minimum, namely, inextensional deformations. Rigid body motions are also included within them. Equations of inextensional deformations for revolutionary shells are given as follows(2).

$$\left. \begin{aligned} \partial \ddot{u}^{(1)} / R_1 \partial \theta + \ddot{u}^{(3)} / R_1 &= 0 \\ \partial \ddot{u}^{(2)} / r \partial \varphi + \ddot{u}^{(1)} \cot \theta / R_1 + \ddot{u}^{(3)} / R_2 &= 0 \\ \partial \ddot{u}^{(1)} / r \partial \varphi + \partial \ddot{u}^{(2)} / R_1 \partial \theta - \ddot{u}^{(2)} \cot \theta / R_2 &= 0 \end{aligned} \right\} \dots\dots\dots (54)$$

Equation (54) shows, in sequence, zero of meridional or latitudinal normal strain or in-plane shear strain. Out of the geometry of the hyperboloid, the following relations hold good.

$$\left. \begin{aligned} R_2 &= a(1 - \beta^2 \cos^2 \theta)^{-1/2}, \quad R_1 = -\bar{a}^2 R_2^3 / a^2, \\ \text{where} \quad \bar{a} &= b/a, \quad \beta^2 = \bar{a}^2 + 1, \quad b = a z_b (R_b^2 - a^2)^{-1/2} \end{aligned} \right\} \dots\dots\dots (55)$$

Here, a denotes the radius of hyperboloid at the throat, R_1 the lower end of the hyperboloid, and z_b denotes the height between its throat and its lower end. Considering geometric relations of Eq. (55), we introduce a new variable η in order to integrate Eq. (54)(2)

$$\left. \begin{aligned} d\eta/d\theta &= -\bar{a} R_2 / a \sin \theta \\ \therefore \quad \eta &= \sin^{-1}(\bar{a} \cot \theta) \end{aligned} \right\} \dots\dots\dots (56)$$

$$\ddot{u}^{(1)} = (a/\bar{a}) \sin \theta (c_{n1} \cos n\eta - c_{n2} \sin n\eta) \cos n\varphi \quad \dots\dots\dots (57)$$

$$\ddot{u}^{(2)} = R_2 \sin \theta (c_{n1} \sin n\eta + c_{n2} \cos n\eta) \sin n\varphi \quad \dots\dots\dots (58)$$

$$\left. \begin{aligned} \ddot{u}^{(3)} &= -(R_2 n (c_{n1} \sin n\eta + c_{n2} \cos n\eta) \\ &\quad + (a/\bar{a}) \cos \theta (c_{n1} \cos n\eta - c_{n2} \sin n\eta)) \cos n\varphi \end{aligned} \right\} \dots\dots\dots (59)$$

Here, n denotes circumferential wave number as before, and c_{n1} and c_{n2} denote integral constants which are determined by the boundary conditions at the lower end of the shell. In this case, the boundary conditions are for $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$ only, which are given by Eqs. (9), (12) and (30), separately.

$$\left. \begin{aligned} c_{n1} &= \left(\frac{\bar{u}_n^{(2)}}{R_2 \sin \theta} \sin n\gamma + \frac{\bar{\alpha} \bar{u}_n^{(1)}}{a \sin \theta} \cos n\gamma \right) \Big|_{\theta=\theta_1} \\ c_{n2} &= \left(\frac{\bar{u}_n^{(2)}}{R_2 \sin \theta} \cos n\gamma - \frac{\bar{\alpha} \bar{u}_n^{(1)}}{a \sin \theta} \sin n\gamma \right) \Big|_{\theta=\theta_1} \end{aligned} \right\} \dots\dots\dots (60)$$

In Eq. (60), $\bar{u}_n^{(1)}$, $\bar{u}_n^{(2)}$ are taken as $(\cos \theta_1, 0)$ in the case where Eq. (9) is used, and similarly in the case of Eq. (12) or (30). Substituting Eq. (60) into Eqs. (57) through (59), we obtain $\bar{u}^{(i)}$ ($i = 1, 2, 3$). For the case where $n = 1$, the following identity holds good.

$$(\bar{u}^{(1)} \cos \theta + \bar{u}^{(3)} \sin \theta) / \cos \varphi + \bar{u}^{(2)} / \sin \varphi = 0 \dots\dots\dots (61)$$

Equation (61) expresses rigid body translation and rotation. For the case where $n = 0$, integration of Eq. (54) gives the following

$$\bar{u}^{(1)} = c_{n0} \sin \theta, \quad \bar{u}^{(2)} = 0, \quad \bar{u}^{(3)} = -c_{n0} \cos \theta \dots\dots\dots (62)$$

Here, c_{n0} relates to vertical rigid body translation.

Quantities in Eqs. (57) through (59) and (62) are approximately equal to those of Eqs. (7), (10) and (28), examples of which are shown in Fig. 2. The displacement $\bar{u}$ according to Eqs. (7), (10) and (28) in the part of the support columns is as follows. Due to the horizontal earthquake excitations, $\bar{u}^e + \bar{u}^r$ for wave number $n = 1$ expresses rigid body motions, and $\bar{u}^e + \bar{u}^r$ for $n \neq 1$ yields high stresses proportional to the amplitude of earthquakes. Due to vertical earthquake excitations, $\bar{u}^v$ for $n = 0$ expresses rigid body motions, and $\bar{u}^v$ for $n \neq 0$ yields high stress proportional to the amplitude of earthquakes.

3. Numerical example

For numerical calculation, dimensions of a hyperbolic cooling tower of reinforced concrete are set as follows. Total height is 130 m, diameter of its lower end is 100 m, height and diameter at the throat are 100 m, 60 m, respectively, and thickness of the shell is 0.2 m at the upper end, being tapered to 0.6 m at its lower end. X-shaped support columns consist of 64 x 2 pieces of straight column with height of 18 m and cross sectional area of 1 m². Upper and lower ends of the hyperboloidal shell are stiffened with circular rings of cross sectional areas of 0.5 m² and 2 m² respectively. The lower end of the support columns are connected to the foundation by circular rings of cross sectional area of 12 m² with elastic spring's constants of 15 kg/cm²/cm for displacements and 300 kg/cm²/rad for

rotation. Fig. 2 shows the first part of displacements in the normal direction which occur due to horizontal or vertical earthquake excitations of unit amplitude.

Fig. 2 Normal displacement $u^{(3)}$ consistent with foundation motion

When wave number n is fixed to 2, $\bar{u}_2^{e(3)}$, $\bar{u}_2^{r(3)}$ and $\bar{u}_2^{v(3)}$ of Eqs. (7), (10) and (28), respectively, are shown in sequence from left in the figure with symbol B. Symbol M denotes those of inextensional deformation given by Eq. (59). Curves B well agree with curves M, hence curves B are reasonable. $\bar{u}_2^{v(3)}$ is considerably larger than unity, which causes deformations for $V_v \approx \infty$ greater than those for $V_v = \infty$, due to vertical earthquakes. Sine curves of earthquake motions are chosen for numerical analysis, because only two or three waves are prominent in the record of seismographs, recurrence of which will be probably poor, and because earthquake motions may be represented by appropriate superposition of sine waves. Horizontal earthquake motions have circular frequency α of 2π c/s and amplitude $\bar{a}$ of -0.07447 m corresponding to maximum acceleration of 0.3 g. Vertical earthquake motions have circular frequency β of 2π c/s and amplitude $\bar{b}$ of -0.07447 m.

Figure 3 shows $\bar{u}^{(3)}$, $\dot{\bar{u}}^{(3)}$ and T_1 versus height at the instants of 0.2s, 0.4s and 0.6s, where $\bar{u}^{(3)}$, $\dot{\bar{u}}^{(3)}$ and T_1 are normal displacement consistent with foundation motion, total

Fig. 3 Normal displacements $\bar{u}^{(3)}$ and $\dot{\bar{u}}^{(3)}$ and membrane force T_1 versus height (horizontal earthquake motion, $V_v = 400$ m/s, $\varphi = 0$)

Fig. 4 Time history of membrane force T_1 (horizontal earthquake motion, height = 18 m, $\varphi = 0$)

Fig. 5 Distribution of membrane forces T_1 on a cross section (horizontal earthquake motion, height = 18 m)

Fig. 6 Distribution of normal displacements $\bar{u}^{(3)}$ on a cross section (horizontal earthquake motion, height = 130 m)

Fig. 7 Normal displacements $\bar{u}^{(3)}$ and $\bar{u}^{(3)}$ membrane force T_1 versus height (vertical earthquake motion, $V_v = 400$ m/s, $\varphi = 0$)

Fig. 8 Time history of membrane force T_1 (vertical earthquake motion, height = 18 m, $\varphi = 0$)

Fig. 9 Distribution of membrane forces T_1 on a cross section (vertical earthquake motion, height = 18 m)

Fig. 10 Distribution of normal displacements $\bar{u}^{(3)}$ on a cross section (vertical earthquake motion, height = 130 m)

normal displacement and meridional membrane force, respectively on the meridian of $\varphi = 0$, due to horizontal earthquake excitations. Figure 4 shows time history of meridional membrane force T_1 at height of 18 m on the meridian of $\varphi = 0$. The curve for propagating velocity $V_A = \infty$ is known to display shock response due to abrupt velocity-change. Figure 5 shows a distribution of membrane forces T_1 due to horizontal earthquake excitations of propagating velocity $V_A = 400$ m/s on the cross section of height of 18 m at the instants of 0.2s, 0.4s and 0.6s. Similarly, Fig. 6 shows total normal displacement $\bar{u}^{(3)}$ on the cross section of height of 130 m. Figure 5 and 6 are known to comprise such higher modes as ovalization of cross section besides beam bending mode.

Figure 7 through 10 are for vertical earthquake excitations.

Figure 7 shows $\bar{u}^{(3)}$, $\bar{u}^{(3)}$, and T_1 , which are of the same meaning as in Fig. 3, versus height at the instants of 0.2s, 0.4s and 0.6s on the meridian of $\varphi = 0$. Figure 8 shows time history of meridional membrane force T_1 at height of 18 m on the meridian of $\varphi = 0$. The curve for propagating velocity $V_A = \infty$ is known to display shock response, and has less magnitude than the one for $V_A = 400$ m/s. Figure 9 shows distributions of T_1 due to vertical earthquake excitation of propagating velocity $V_A = 400$ m/s on the cross section of height of 18 m at the instants of 0.2s, 0.4s and 0.6s. Similarly, Fig. 10 shows total displacement $\bar{u}^{(3)}$ on the cross section of height of 130 m. Figures 5 and 6 are known to show that beam bending mode and ovalization mode are prominent.

4. Conclusions

Time history of dynamic response of a hyperboloidal shell of revolution supported on columns is analyzed taking into account the phase difference of earthquakes. When a hyperbolic cooling tower of large radius at its lower end is subjected to horizontal earthquake excitations of slow propagating velocity and of short period, such higher modes of vibration as ovalization of its cross section may occur prominently. Due to vertical earthquake excitations of similar nature, there may appear vibrations of beam bending mode and of ovalization mode in addition to heaving motion.

Sufficient attention should be paid to large deformations due to vertical earthquake excitations of finite propagating velocity than those due to vertical earthquake excitations of infinite propagating velocity. It is advisable that adequate reinforcement, such as concrete foundation bed, be provided at the lower end of hyperbolic cooling towers.

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