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Research on Wave Phenomena in Hydraulic Lines*

(3rd Report, Frequency Characteristics between Pressure and Flow Rate)

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The impedance properties of a wave transmission line containing viscous liquid are experimentally investigated. The pulsating flow rate is measured with an accelerometer connected to a light piston placed at the inlet of a pipe with a closed end. There is good agreement between the experimental results and the theoretical ones given by the axisymmetrical viscous wave equations. Another wave equations considering also a radial pressure gradient are introduced to explain the transition of velocity profile in front of the piston. As the result it is thought that the influence of the piston interposed in oil column decreases rapidly with an increasing distance from it, and is negligible for the present purpose. According to the equations proposed here, dynamic pressure and flow rate in a circular liquid pipe are expressed in infinite series with respect to eigenvalues. A numerical procedure to obtain the eigenvalues for oscillation is developed, too.

1. Introduction

Wave transmission in oil contained in a circular pipe is one of the most interesting problems among the unsteady phenomena in hydraulic systems. The axisymmetrical viscous wave equations taking account of the microscopic contribution of viscosity have been introduced to express the wave flow⁽¹⁾, and experimentally examined by many investigators⁽²⁾⁻⁽⁷⁾. On the whole the results of their experiments show that the equations can well express the wave phenomena in oil pipes. However between the two physical quantities which represent the wave transmission, that is, pressure and flow rate, the former only has been measured because of its easiness of detection in most of these experiments. On the other hand the latter has not been measured except in just a few cases with use of a hot wire anemometer⁽²⁾ or a cam mechanism⁽⁷⁾, and quantitatively precise measurement of dynamic flow rate has not been accomplished yet. This is unsatisfactory from a standpoint of the empirical research on wave transmission.

In this paper a different method of experiment is attempted in order to overcome this deficiency. Namely a light piston is placed at the inlet of an end-closed circular pipe containing oil, and its vibration is detected with an accelerometer instead of the direct measurement of flow rate. The frequency response tests

between the pressure and the piston acceleration are carried out in the range of 22 to 400 Hz. The experimental results are compared with the theoretical ones calculated by the previous wave equations. Moreover the discrepancy between the experimental and theoretical velocity profile at the inlet is discussed, by introducing another wave equations considering a radial pressure gradient.

2. Experimental investigation

2.1 Experimental apparatus and procedure

A schematic diagram of the apparatus is shown in Fig.1. The working oil (#2 turbine-oil-JIS) discharged from a hydraulic pump is caused to pulsate through the rotary sinusoidal pulse generator ①, of which a cross-sectional drawing is shown in Fig.2. As a spool with twelve sinusoidal notches on the surface rotates in a cylinder, the areas of throttling slits on the wall vary sinusoidally, so that pulsation is generated in two outlet lines. One line is connected to the flow rate measuring block ②. A drawing of the block

Fig.1 Experimental apparatus

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is shown in Fig.3. A light aluminum piston with a diametric tolerance of several 10 microns is inserted into a honed cylinder with the same inside diameter as a test pipe. The right side of the piston is flat and faces the oil contained in the test line with a closed end. A slender rod which connects the piston to a piezoelectric accelerometer outside the oil is attached to the left side. The pulsation of the left side oil causes the piston oscillation and accordingly the pulsation in the test line. The movement of the piston is precisely followed by the accelerometer. The pressures and the temperatures of the oil were measured at both ends of the line with strain gage transducers and thermistors, respectively. The detected pressures and acceleration were filtered through a tracking analyzer and recorded by photographing their sinusoidal traces on the Braun tube of a synchroscope. The frequency was varied from 22 to 400 Hz, and the recording was done at every 10 Hz. The experimental values were obtained from

Fig.2 Rotary sinusoidal pulse generator

Fig.3 Flow rate measuring block

Fig.4 Frequency response of strain meter

the data films. In these measurements the electric system of the accelerometer had good frequency characteristics over the test range, while the strain gage transducer had necessarily a deterioration of sensitivity at high frequencies. Therefore the pressure data were corrected on the basis of the frequency characteristics of the strain meter used here, which is shown in Fig.4.

2.2 Experimental results

The experimental results for the test lines with uniform cross section and with step change of cross section are shown in Figs.5 and 6, respectively. In each figure, (a) shows the frequency response of the pressure to the acceleration at the entrance, and (b) shows that between the two pressures at both ends. Circles show the experimental values and solid lines are the theoretical ones calculated by the axisymmetrical viscous wave equations. These equations which are called "TDVF wave equations" in reference (5) are hereafter abbreviated as "ASV wave equations" and expressed as follows⁽³⁾:

(a)

(b)

Fig.5 Experimental results

$$\left. \begin{aligned} A_2(s) &= A_1(s) \cosh sF_1(z)T_l \\ &\quad - \frac{s}{\rho c F_1(z)} P_1(s) \sinh sF_1(z)T_l \\ P_2(s) &= P_1(s) \cosh sF_1(z)T_l \\ &\quad - \frac{\rho c F_1(z)}{s} A_1(s) \sinh sF_1(z)T_l \end{aligned} \right\} \dots\dots\dots (1)$$

where

$$\left. \begin{aligned} F_1(z) &= [I_0(z)/I_2(z)]^{1/2} \\ z &= a\sqrt{s/\nu}, \quad T_l = l/c \end{aligned} \right\} \dots\dots\dots (2)$$

In these equations s is the Laplace variable, l the pipe length, a the pipe radius, c the velocity of sound, and ν and ρ are the kinematic viscosity and the density of liquid, respectively. A indicates the Laplace transform of the cross-sectional average acceleration and P that of the pressure. The subscripts 1 and 2 denote a pipe entrance and a closed end, respectively.

In Fig.5(a) dotted lines show the theoretical values calculated by the Hagen-Poiseuille wave equations⁽⁵⁾. Experimental constants in these runs are given in Table 1. Moreover the crude data on amplitudes

Fig.6 Experimental results

of the pressure and the displacement evaluated from the acceleration for the case of Fig.5(a) are shown in Fig.7 for information.

In these results there are qualitative and quantitative agreements between the experimental values and the theoretical ones calculated by the ASV wave equations. Now figures (b) are thought to illustrate the discrepancy between the experimental and hypothetical conditions of oil flow. As far as these figures are concerned, however, any distinct influence of the existence of the piston cannot be found. In the empirical method used here, the oil column is intercepted by the piston and it is not the case of measuring the average acceleration of the oil continuum. Therefore, before concluding from these results that the wave impedance in hydraulic lines is well represented by the ASV wave equations, the transition flow in front of the piston should be evaluated. The evaluation is theoretically done in the following section.

3. Theoretical discussion about transition region

Under the boundary condition of the foregoing experiments, the velocity profile at the entrance determined by the ASV wave equations is far different from a piston shape. Accordingly a transition region of velocity profile must be supposed in front of the piston. In order to analyze the viscous wave flow in such regions, less approximated wave equations are required. As the equations to meet the requirement, only those by Gerlach⁽⁶⁾ are available. To discuss the problem, fairly simple equations are proposed here.

3.1 Basic equations

The pressure gradient in the radial direction as well as in the axial one must be taken into account to explain the tran-

Fig.7 Data examples

Table 1 Experimental constants

		ν cm ² /s	ρ kg/m ³	l_1 m	l_2 m	a_1 cm	a_2 cm	c m/s
5	(a)	2.13	90.2	4.26	0	0.82		1430
	(b)	2.28						
6	(a)	2.34	90.3	2.16	2.02	0.82	0.64	
	(b)	2.24						

sition phenomena. The conventional procedure to simplify Navier-Stokes' equations can be applied in this case, too. In consequence the equation of motion in the axial direction and that of continuity are reduced to the following equations which are the same as the ASV basic equations.

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad (3)$$

$$\frac{1}{K} \frac{\partial p}{\partial t} + \frac{\partial v}{\partial r} + \frac{v}{r} + \frac{\partial u}{\partial x} = 0 \quad (4)$$

where p is pressure, u and v are the axial and radial velocities, x and r are the axial and radial coordinates. t is time, and ρ , μ and K are the density, the viscosity and the bulk modulus of liquid, respectively. On the other hand the general form of the equation of axisymmetrical radial motion is given as follows:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial x^2} \right) \quad (5)$$

In Eq.(5), neglecting the non-linear terms on the left-hand side, and substituting Eq.(4) into the right-hand side yields

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 v}{\partial r^2} - \frac{\partial^2 u}{\partial r \partial x} - \frac{1}{K} \frac{\partial^2 p}{\partial r \partial t} \right) \quad (6)$$

Moreover since

$$\frac{\nu}{K} \frac{\partial^2 p}{\partial r \partial t} \ll \frac{1}{\rho} \frac{\partial p}{\partial r}, \quad \frac{\partial u}{\partial r} \gg \frac{\partial v}{\partial x} \quad (7)$$

Eq.(6) reduces to the following equation.

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial r} - \mu \frac{\partial^2 u}{\partial r \partial x} \quad (8)$$

Now Eqs.(3), (4) and (8) constitute the basic differential equations to express the axisymmetrical unsteady flow of compressible liquid in the transition region. Taking the Laplace transforms of these equations with zero initial condition gives

$$\rho s U = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} \right) \quad (9)$$

$$\frac{s}{K} P + \frac{\partial V}{\partial r} + \frac{V}{r} + \frac{\partial U}{\partial x} = 0 \quad (10)$$

$$\rho s V = -\frac{\partial P}{\partial r} - \mu \frac{\partial^2 U}{\partial r \partial x} \quad (11)$$

where U , V and P are the Laplace transforms of u , v and p , respectively. Eliminating U and V from Eqs.(9), (10) and (11) gives

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} - \frac{s^2}{c^2} P = -\frac{\partial^2 P}{\partial x^2} \quad (12)$$

Solving Eq.(12) by the method of separation of variables yields

$$P = J_0 \left(r \sqrt{\gamma^2 - \frac{s^2}{c^2}} \right) [A_1 e^{-\gamma x} + A_2 e^{\gamma x}] \quad (13)$$

where A_1 and A_2 are constants of integration and γ is the separation constant. Substituting Eq.(13) into Eq.(9) gives

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} - \frac{s}{\nu} U = \frac{\gamma}{\rho \nu} J_0 \left(r \sqrt{\gamma^2 - \frac{s^2}{c^2}} \right) [A_2 e^{\gamma x} - A_1 e^{-\gamma x}] \quad (14)$$

Solving Eq.(14), applying the boundary condition $U|_{r=a}=0$ and rearranging yields

$$U = \frac{J_0(r\beta)}{J_0(a\beta)} - \frac{I_0((r/a)z)}{I_0(z)} [Ae^{-\gamma x} - Be^{\gamma x}] \quad (15)$$

where

$$\beta = \sqrt{\gamma^2 - \frac{s^2}{c^2}} \quad (16)$$

Replacing the constants A_1 and A_2 with A and B in Eq.(13) gives

$$P = \frac{\beta^2 + (z/a)^2}{\gamma(\rho \nu)} \frac{J_0(r\beta)}{J_0(a\beta)} [Ae^{-\gamma x} + Be^{\gamma x}] \quad (17)$$

Substituting Eqs.(15) and (17) into Eq.(11) gives

$$V = \frac{\beta}{z\gamma} \left(z^2 - \frac{a^2 s^2}{c^2} \right) \frac{J_1(r\beta)}{J_0(a\beta)} - \frac{a\gamma}{z} \frac{I_1((r/a)z)}{I_0(z)} [Ae^{-\gamma x} + Be^{\gamma x}] \quad (18)$$

Applying the boundary condition $V|_{r=a}=0$ yields the following characteristic equation:

$$\frac{z\beta J_1(z\beta)}{J_0(z\beta)} \left(z^2 - \frac{a^2 s^2}{c^2} \right) = \frac{z I_1(z)}{I_0(z)} z r^2 \quad (19)$$

where

$$z_\beta = a\beta, \quad z_r = a\gamma, \quad z_r^2 = z_\beta^2 + \frac{a^2 s^2}{c^2} \quad (20)$$

Consequently U and P are generally expressed as follows:

$$U = \sum_{n=1}^{\infty} \left[\frac{J_0((r/a)z_{\beta n})}{J_0(z_{\beta n})} - \frac{I_0((r/a)z)}{I_0(z)} \right] [A_n \exp\{-(x/a)z_{r n}\} - B_n \exp\{(x/a)z_{r n}\}] \quad (21)$$

$$P = \sum_{n=1}^{\infty} \rho s a \frac{z_{\beta n}^2 + z^2}{z^2 z_{r n}} \frac{J_0((r/a)z_{\beta n})}{J_0(z_{\beta n})} [A_n \exp\{-(x/a)z_{r n}\} + B_n \exp\{(x/a)z_{r n}\}] \quad (22)$$

where $z_{r n}$ and $z_{\beta n}$ are the n -th eigenvalues of Eq.(19).

3.2 Frequency characteristics

For frequency characteristics analysis, the Laplace variable s is replaced with an imaginary frequency $j\omega$ in the characteristic equation (19). Then it is seen from Eq.(19) that the eigenvalues are uniquely determined by the two non-dimensional parameters $Q = a\omega/c$ and ac/ν . At present it seems very difficult to analytically solve this equation which has an infinite number of complex roots. By use of the numerical procedure described in Appendix, however, the eigenvalues can be exactly computed. As an example the computed values of $z_{r n}$ versus Q for $ac/\nu = 100$ are shown in Fig.8, in which $z_{r n}$'s are numbered in order of smallness of the magnitudes for small Q from 10^{-4} to 10^{-2} that is usual in actual oil line pulsation. As known by Eqs.(19) and (20), appearance of an infinite slope for n above 3 means that $z_r = 0$ becomes an eigenvalue when Q coincides with the zeros of $J_1(z_\beta)$. By the way r in this analysis corresponds to the propagation operator $(s/c)F_1(z)$ in the ASV wave equations. It is found from the results of computation that $z_{r 1}$ almost agrees with $jQF_1(z)$ for small Q , while the difference between them considerably increases according as Q increases. As an example the real part of $z_{r 1}$ is compared with those of $jQF_1(z)$ and Gerlach's γ_{or} for $ac/\nu = 100$ in Fig.9.

3.3 Theoretical consideration

The basic equations (21) and (22) are applied to the evaluation of the wave flow generated by piston oscillation. The eigenvalues for the experimental conditions were similarly computed with the constants given previously. For the case of Fig.5(a), $ac/\nu = 5.374 \times 10^4$ and the range of Q is from 7.206×10^{-4} to 1.441×10^{-2} . Examples of computed values of $z_{r n}$ and $z_{\beta n}$ are given for n from 1 to 8 in table 2. It has been found that $z_{r 1}$ is almost equal to $jQF_1(z)$ of the ASV wave equations. Then the first terms of Eqs.(21) and (22)

$$U = \left[\frac{J_0((r/a)z_{\beta 1})}{J_0(z_{\beta 1})} - \frac{I_0((r/a)z)}{I_0(z)} \right] [A_1 \exp\{-(x/a)z_{r 1}\} - B_1 \exp\{(x/a)z_{r 1}\}] \quad (23)$$

$$P = j\rho c Q \frac{z_{\beta 1}^2 + z^2}{z^2 z_{r 1}} \frac{J_0((r/a)z_{\beta 1})}{J_0(z_{\beta 1})} [A_1 \exp\{-(x/a)z_{r 1}\} + B_1 \exp\{(x/a)z_{r 1}\}] \quad (24)$$

are compared with the ASV wave equations rearranged in similar form:

$$U = \left[1 - \frac{I_0((r/a)z)}{I_0(z)} \right] [A_1 e^{-(x/a)jQF_1(z)} - B_1 e^{(x/a)jQF_1(z)}] \quad (25)$$

$$P = j\rho c Q \frac{1}{jQF_1(z)} [A_1 e^{-(x/a)jQF_1(z)} + B_1 e^{(x/a)jQF_1(z)}] \quad (26)$$

As seen from Table 2, $z_{\beta 1}$ is as small as 10^{-3} , so that the following approximations are possible with good accuracy:

$$\frac{J_0((r/a)z_{\beta 1})}{J_0(z_{\beta 1})} \cong 1, \quad \frac{z_{\beta 1}^2 + z^2}{z^2 z_{r 1}} \cong \frac{1}{z_{r 1}} \quad (27)$$

Under these relations Eqs.(23) and (24) agree with Eqs.(25) and (26), respectively. Namely the fundamental modes of the wave equations (21) and (22) introduced in this paper

Table 2 Eigenvalues under experimental conditions

$ac/\nu = 5.37 \times 10^4 \quad c = 1430 \text{ m/s} \quad \nu = 2.18 \text{ cm}^2/\text{s} \quad a = 0.82 \text{ cm}$					
		100 Hz ($\Omega = 3.60 \times 10^{-3}$)		400 Hz ($\Omega = 1.44 \times 10^{-2}$)	
		Re	Im	Re	Im
$jQF_1(z)$		2.025 30E-04	3.785 22E-03	3.852 05E-04	1.477 78E-02
$z_{r,n}$	$n = 1$	2.025 30E-04	3.785 22E-03	3.852 06E-04	1.477 78E-02
	2	4.032 45E+00	-2.103 36E-01	3.929 66E+00	-1.017 97E-01
	3	7.408 81E+00	-3.579 40E-01	7.198 06E+00	-1.835 90E-01
	4	1.077 76E+01	-4.456 73E-01	1.044 46E+01	-2.593 36E-01
	5	9.839 34E+00	-9.839 34E+00	1.967 87E+01	-1.967 87E+01
	6	1.412 04E+01	-4.636 24E-01	1.693 49E+01	-3.807 47E-01
	7	1.741 50E+01	-4.397 56E-01	2.018 08E+01	-4.200 67E-01
	8	2.066 63E+01	-4.027 10E-01	2.342 33E+01	-4.432 83E-01
$z_{\beta,n}$	$n = 1$	5.951 18E-04	1.288 18E-03	1.577 76E-03	3.607 95E-03
	2	4.032 45E+00	-2.103 36E-01	3.929 69E+00	-1.017 96E-01
	3	7.408 81E+00	-3.579 40E-01	7.198 08E+00	-1.835 89E-01
	4	1.077 76E+01	-4.456 73E-01	1.044 46E+01	-2.593 36E-01
	5	9.839 34E+00	-9.839 34E+00	1.967 87E+01	-1.967 87E+01
	6	1.412 04E+01	-4.636 24E-01	1.693 49E+01	-3.807 47E-01
	7	1.741 50E+01	-4.397 56E-01	2.018 08E+01	-4.200 67E-01
	8	2.066 63E+01	-4.027 10E-01	2.342 33E+01	-4.432 83E-01

are practically identical with the ASV wave equations within the extent of these experimental conditions.

In the next place applying the closed end condition $U|_{x=l}=0$ to Eq.(21) yields

$$A_n \exp[-(l/a)z_{r_n}] - B_n \exp[(l/a)z_{r_n}] = 0 \quad (28)$$

Using Eq.(28), Eqs.(21) and (22) are rewritten as follows:

$$U = \sum_{n=1}^{\infty} A_n \left[\frac{J_0((r/a)z_{\beta n})}{J_0(z_{\beta n})} - \frac{I_0((r/a)z)}{I_0(z)} \right] [\exp\{-(x/a)z_{r_n}\} - \exp\{-(2l-x)/a; z_{r_n}\}] \quad (29)$$

$$P = \sum_{n=1}^{\infty} j\rho c Q A_n \frac{z_{\beta n}^2 + z^2}{z^2 z_{r_n}} \frac{J_0((r/a)z_{\beta n})}{J_0(z_{\beta n})} [\exp\{-(x/a)z_{r_n}\} - \exp\{-(2l-x)/a; z_{r_n}\}] \quad (30)$$

in which the coefficients A_n 's are determined by the velocity profile for $x=0$. As known from the numerical evaluations such as $e^{-R_0(z_{r_2})} \cong 2/100$, the only fundamental modes are dominant in the infinite series of Eqs.(29) and (30) even at a distance of one radius from the entrance, and the higher modes have no influence a few radii downstream.

Moreover integrating Eq.(29) over the cross section gives the average velocity as follows:

$$\bar{U} = -2Q^2 \sum_{n=1}^{\infty} A_n \left(1 + \frac{z_{\beta n}^2}{z^2} \right) \frac{J_1(z_{\beta n})}{z_{\beta n} z_{r_n}^2 J_0(z_{\beta n})} [\exp\{-(x/a)z_{r_n}\} - \exp\{-(2l-x)/a; z_{r_n}\}] \quad (31)$$

The piston rate is given by setting $x=0$ in Eq.(31). Exclusive of A_n , the fundamental term is as large as $|1/z_{r_1}^2|$ that is nearly equal to 10^{-1} from Table 2, while the higher terms are as small as $|1/z_{r_n}^2 z_{\beta n}|$ that is equal to 10^{-2} at most. These evaluations mean that the flow rate for $x=0$, that is the piston rate, depends solely on the fundamental term and accordingly agrees with the flow rate given by the ASV wave equations.

After all it is deduced from the above discussion that in this experiment the flow pattern in the test pipe can be regarded as the one expressed by the ASV wave equations with the average inlet velocity equal to the piston rate beyond a distance of about one radius, and the interposition of the piston does not exert

a substantial effect on the experimental investigations.

4. Conclusions

The frequency response tests to investigate the acoustic impedance of pulsating oil pipes have been carried out by means of a light piston inserted at a pipe inlet. The experimental results are compared with the theoretical ones given by the ASV wave equations. Another wave equations considering the radial pressure gradient have been introduced to evaluate the effect of the interposition of the piston. In consequence the following conclusions have been obtained.

(1) The empirical method to interpose a piston in oil column is available for the quantitative measurement of dynamic flow rate.

(2) The deviation of the actual flow pattern from the hypothetical one caused by the existence of the piston is negligible for the present purpose.

(3) The ASV wave equations can qualitatively and quantitatively represent the impedance characteristics of hydraulic lines.

(4) According to the wave equations developed in this paper, the dynamic

(a)

(b)

Fig.8 Eigenvalues

Fig.9 Comparison of fundamental modes

pressure and flow rate are expressed in an infinite series of propagation modes. Under the usual pulsating circumstances occurring in hydraulic systems, the fundamental mode almost agrees with the ASV wave equations, while the higher modes rapidly decrease with an increasing distance in the axial direction.

Appendix

Numerical procedure to calculate zeros of a complex function

Let z be a complex variable and $G(z)$ be a single-valued meromorphic function. Then the following relation known as "argument principle" holds⁽⁹⁾:

$$\frac{1}{2\pi j} \oint_C \frac{G'(z)}{G(z)} dz = m - n \quad (32)$$

where C is a simple closed curve in the z -plane and m , n are the number of zeros and poles of $G(z)$ within C . As

$$\int_C \frac{G'(z)}{G(z)} dz = \log G(z) \quad (33)$$

the integral on the left of Eq.(32) is identical with the increase of argument of $G(z)$ when z describes the closed curve C . Hence if the number of poles of $G(z)$ and their positions in the z -plane are already known, the closed trail generated by $G(z)$ when z completely describes C in the positive sense will reveal the number of zeros within C . Taking a circle of radius R about the origin as C and repeating the process with various R make it possible to

Fig. 10

lay one zero in an arbitrarily narrow ring determined by two concentric circles. Moreover a closed path formed by the two circles and two radii as shown in Fig.10 gives the sufficiently small neighbourhood of the zero, so that by means of Newton-Raphson method

$$z_{i+1} = z_i - \frac{G(z_i)}{G'(z_i)} \quad (34)$$

the zero can be found to any desired degree of accuracy.

In the present case let

$$G(z) = \frac{z J_1(z)}{J_0(z)} - \frac{I_0(z)}{I_1(z)} \frac{z^2 + Q^2}{z^2} - 1 \quad (35)$$

Then because the poles of $G(z)$ are $z=0$ and those obtained from the zeros of $J_0(z)$ which are exactly known, the above procedure can be applied. This procedure requires a high degree of accuracy on the determination of $G(z)$, $G'(z)$. In the present computation, a FORTRAN program of double precision complex function subroutine of $z J_1(z)/J_0(z)$ was made and zeros of $G(z)$ have been computed to ten figures of accuracy.

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