The Indentation of a Thick Elastic Plate by Rigid Punches in the Arbitrary Shape*

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In this paper, we measured the deformation of the free surface of a thick elastic plate indented by rigid, circular or arbitrary polygon-shaped punches, and showed experimentally that the surface deformation by the punch in the polygonal shape is in good agreement with the result for a circular punch by the use of the conformal mapping technique.

1. Introduction

The problem of an elastic solid indented by a rigid, flat-ended circular⁽¹⁾ or annular punches⁽²⁾⁻⁽⁶⁾ has been fully discussed. The solution for the punch in the arbitrary shape, however, has not been presented because of the mathematical difficulty. The experimental method of stress analysis used for the latter case may be photoelasticity or strain gauge. In this method, there seems to be some difficulties in accuracy or measurement. The surface deformation of an elastic solid, however, can be measured with comparative ease. It may be significant to show the qualitative or quantitative characteristics in the surface deformation of the solid indented by the punch in the arbitrary shape to obtain the foreknowledge on the contact problem. Ann⁽⁷⁾ has measured the surface deformation of an elastic plate indented by a rigid sphere with a dial indicator and compared the result with the deformation obtained theoretically. The method used for his study may have some difficulties due to the equipment size in measuring accurately the deformation near the edge of the punch which is of great interest. In this paper, we take the surface deformation of a plate in gypsum and measured the deformation with an optical displacement measuring equipment of untouched type. In this method, the surface deformation can be measured easily even if the shape of the punch is complicated. Ascertaining the accuracy of the measurements by the comparison of the experimental result for the circular punch

with the theoretical one, we measured the deformations in the cases of using rectangular, equilateral triangular and equilateral hexagonal punches. It is shown experimentally that the deformations for the punches in the arbitrary shape are in good agreement with the result for the circular punch by use of the conformal mapping technique.

2. Experimental Procedure

The experimental equipment is shown in Fig.1. A synthetic SBR rubber of $395 \times 395 \times 43$ mm is used as a thick elastic plate. Since the punch is below 50-mm-dia, the effect of the plate edges on the deformation is negligible small; the plate may be regarded as an infinite thick plate. A steel ball is set on the upper face of the punch to remove the effect of bending loads, and the punch is gradually loaded to the deformation ϵ_0 . The deformation is measured with two dial gauges on the upper face of the punch. To abate the friction, grease is applied on the surface of rubber and a liquid cleaner on the contact area of the punch: the cleaner also facilitates the separation of gypsum from rubber. Gypsum is poured into a paper frame on the surface of the plate indented by a rigid punch. After gypsum is congelated, it is taken out with the punch stuck. Then the deformation of gypsum from the surface of the punch is measured with an optical

Fig.1 Experimental equipment

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measuring instrument (Nissho Seimitsu Kogaku SS) which can be read to $1/100$ mm. Fig.2 shows the measuring equipment used for the study. The displacement w_c is obtained by subtracting the above value from ε_0 . In this method, the deformation near the punch edge also can be measured with high accuracy.

Fig.3 shows the variation of the load P with the depth of the indentation, ε_0 , by the circular punch. It is shown from the figures that P is proportional to the deformation ε_0 within a comparatively wide range. The rubber, within the range, can be regarded as a thick elastic plate. In this experiment, ε_0 is fixed at 1 mm in consideration of the equipment accuracy.

Fig.4 shows the punches used for the study, which are made of mild steel. Since the upper face of the punch, as shown in Fig.2, is used for measuring ε_0 , both faces of the punch are finished by the surface grinder. The sectional area of each punch is nearly equal to about 19.6 cm^2 , because the surface deformation varies with the thickness of the plate even when the punch shapes are similar. As the load P is approximately invariable in each punch, each average contact stress seems to be nearly constant. The aspect ratios of the rectangular punches are determined from the mapping function, which is discussed later in this paper. Holes or grooves are cut in the side of each punch to fasten gypsum tight. Although gypsum sometimes falls off partially near the punch edge because of the steepness of the surface, there is no difficulty of measurement.

3. Experimental Results and Discussions

Fig.5 shows the radial distribution of displacements on the surface of the plate indented by the circular punch: the full and dashed lines denote the experimental result and the theoretical result (Poisson's ratio $\nu=0.45$), respectively. Although much attention is paid to removing the bending load upon the punch in the experiment, a slight effect of bending load is inevitable. The experimental result is therefore, taken as the average value of the displacements crossing each other in four directions. The experimental value (the full line) almost corresponds to the theoretical value (the dashed line). Because the surface displacement is symmetric, the iso-displacement curves for $w_c/\varepsilon_0 = 1.0, 0.6, 0.5, 0.4$ and 0.3 are the concentric circles of $r/r_0 = 1.0, 1.075, 1.14, 1.22$ and 1.35 , respectively. In a similar way the deformation for each punch is measured and the curves for the displacements are plotted. In the case of a rectangular punch, for example, the deformation is measured in each direction as shown by the full lines in Fig.4(c), and the experimental result is obtained as the average value of the deformations in two symmetrical directions with regard to the origin O .

Fig.6, for example, shows the dis-

Fig.2 Experimental setup for measurement

Fig.3 Properties of SBR plate

placement distribution for the rectangular punch I. The deformations (full lines) decrease rapidly near the punch edges, and gradually for an increasing r/r_0 . In general the displacement w_c/ε_0 in the contact region on the plate indented by the punch in the arbitrary shape is equal to 1.0 and varies with r/r_0 on the free surface of the plate. From this fact, we can consider that the displacement in a unit circle on the mapped plane is also equal to $w_c/\varepsilon_0 = 1.0$ when the contact region in the arbitrary shape is transferred in the unit circle by conformal mapping. It follows therefrom that the region interior to the unit circle on the mapped plane can be regarded as the contacting region of the circular punch and the displacements for w_c/ε_0 in these regions correspond strictly to each other. Therefore, it can be expected that there may be a similar correspondence between the two on the free surface. The conformal mapping by Schwartz-Christoffel can be applied when the non-contacting region of

Fig. 8 Iso-displacement curves for rectangular punch I

Fig. 12 Iso-displacement curves for equilateral-hexagonal punch

Fig. 9 Iso-displacement curves for rectangular punch II

Fig. 10 Iso-displacement curves for square punch

Fig. 11 Iso-displacement curves for equilateral-triangular punch

values are determined from the shape of the polygonal punch.

The curves, shown by the full line in Fig. 8, correspond respectively to the case of the iso-displacements $w_z/\varepsilon_0 = 0.6, 0.5, 0.4$ and 0.3 for the rectangular punch 1. Moreover, the boundary of the punch can be regarded as an iso-displacement curve for $w_z/\varepsilon_0 = 1.0$. On the other hand, the displacement distribution of the circular punch can be regarded as the distribution in the ρ -direction on the mapping plane shown in Fig. 7. That is, the curve on z -plane corresponding to the points on the unit circle, $\rho = 1$, represents the boundary of the rectangular punch 1. The iso-displacement curve for $w_z/\varepsilon_0 = 0.6$ is the case of a circular punch is a circle of $\rho = 1.075$. Substituting the value $\rho = 1.075$ into Eq. (1) and determining points on the z -plane with variation of angle θ from 0 to 2π , we obtain a broken curve for $w_z/\varepsilon_0 = 0.6$ as shown in Fig. 8. Thus the other broken curves for $w_z/\varepsilon_0 = 0.5, 0.4, 0.3$ are obtained by use of the experimental result for a circular punch. Fig. 9 to 12 show the iso-displacement in the cases of the other punches. In any shape of the punch, it is shown from those figures that the full line curves agree closely with the broken curves. It is very difficult to prove the results mathematically. It is, however, clear that the corresponding relation is established between the full line curves and the broken ones in any of the six punches of which the shapes are selected only for the easiness of their making. Therefore, if the surface displacement given by the deformation of the plate indented by circular punch is expressed by $w_z = f(r)$, the displacement by the polygonal punch with the same sectional area may be given by $w_z = f(r)$ at a point $x + iy [= w(\rho e^{i\theta})]$. Additionally, the study proves that there is not such a relation when the punches have not the same sectional area. This may be due to the thickness of the plate.

4. Conclusions

We measured the surface deformation of a synthetic rubber used as a thick elastic plate when the plate was indented

Fig. 4 Sectional shapes of punches

Fig. 5 Displacement distribution for a circular punch

Fig. 7 Schwartz-Christoffel's conformal mapping

Fig. 6 Variation of w_z ; rectangular punch I

the polygonal punch is corresponding to that of a circular punch.

As shown in Fig. 7, if the z -plane ($z=x+iy$) coincides with the surface of the plate indented by an arbitrary punch and the ζ -plane ($\zeta=\rho e^{i\theta}$) does with the surface by the circular punch, the points on the unit circle corresponding to the apices of the polygonal punches. $A_1, A_2, \dots, A_n$ are represented by $a_1, a_2, \dots, a_n$. The mapping function which maps the z -plane exterior to a polygon shown by the shaded line onto the ζ -plane exterior to the unit circle shown by the shaded line is given by

$$z = \omega(\zeta) = C(\zeta + e_1/\zeta + e_2/\zeta^2 + e_3/\zeta^3 + \dots) \quad (1)$$

where the coefficients C and e_n are constant as are given in Table 1 and these

Table 1 Coefficients C and e_n in mapping functions

	square punch	rectangular punch I	rectangular punch II		equilateral-triangular punch		equilateral-hexagonal punch
C	1.17482516	1.05237596	0.74590004	C	0.73322151	C	0.92316617
e_1	0	0.10452646	0.5	e_2	0.33333333	e_5	0.06666666
e_3	-0.16666666	-0.16484563	-0.125	e_5	0.02222222	e_{11}	0.01010101
e_5	0	-0.01033863	-0.0375	e_8	0.00617284	e_{17}	0.00363108
e_7	0.01785714	0.01668861	-0.00334821	e_{11}	0.00251878	e_{23}	0.00178922
e_9	0	0.00419794	0.00651041	e_{14}	0.00137174	e_{29}	0.00104063

by circular or some polygonal punches. The comparison of the experimental results for the polygonal punches with the result for a circular punch is made, and the following conclusions are obtained;

(1) The surface deformation of a thick elastic plate indented by the rectangular, square, equilateral triangular and equilateral hexagonal punches almost corresponds to that mapped from the deformation for the circular punch.

(2) If the surface deformation indented by a circular punch of our unit radius is given by $w_z = f(r)$, the surface deformation for an arbitrary polygonal punch with the same sectional area as the circular one may be given approximately by $w_z = f(r)|_{r=\rho}$.

(3) Conversely speaking, the iso-displacement curves for the polygonal punches correspond to axisymmetric circles when the curves are mapped onto the exterior to the unit circle.

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