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Free Vibration of Circular Plate Elastically Supported at Some Points*

By

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This paper studies the free vibration of a circular plate whose deflection, rotation and torsion are elastically supported at some points located inside the plate or on the circumference. Regarding the reactive forces and moments at all points as unknown harmonic exciting loads, the stationary response of the plate to these loads is derived. The natural frequencies and the mode shapes of the plate are calculated by determining the unknown quantities with use of constraint conditions at the supports.

Applying the method to a circular plate supported at some equi-spaced points of equal radii, the natural frequencies and the mode shapes are calculated numerically. From the result of this study, it is concluded that the natural frequencies of the plate become larger with an increase in the stiffness and the number of supports, but they do not vary monotonously according to the location of the supports.

1. Introduction

Circular plates supported at some points are used in many structures such as hatch plate and optical mirror⁽¹⁾. The free vibration of rectangular plates supported at some points has been studied in many papers⁽²⁾. However, a limited number of papers are available on the free vibration of circular plates except for a plate simply supported at three points on the circumference⁽³⁾, a plate simply supported at three points of arbitrary radius⁽⁴⁾⁽⁵⁾ and a plate whose lateral deflection is elastically supported at points on the circumference⁽⁶⁾.

This paper presents a theoretical method to analyze the free vibration of a circular plate whose deflection, rotation and torsion are elastically supported at some points inside the plate or on the circumference. For this purpose, regarding the reaction forces, the bending moments and the torsional moments acting at the supports as unknown harmonic loads, the stationary response of the plate to these loads is obtained. The homogeneous linear equations concerning the unknown forces and moments are given in a matrix form with use of constraint conditions at the supports. By obtaining the eigenvalues and eigenvectors of the matrix equation, the natural frequencies and the mode shapes of the plate are determined.

The present method is applicable to the analysis of the free vibration of a circular plate without regard to the location, the number and the constraint conditions of the supports. In this paper, the frequency equations are derived for various symmetrical and antisymmetrical vibrations of a circular plate supported at several equi-spaced points of equal radii and the natural frequencies and the mode shapes of the plate are calculated numerically.

2. Fundamental equations and boundary conditions

Figure 1 shows a circular plate elastically supported at several points P_n ($n = 1, 2, \dots, N$). Denoting the radius of the plate by a , polar coordinates (r, φ) are taken in the neutral surface of the plate as shown in the figure. Regarding the unknown reaction acting at the supports as an unknown harmonic external force $Q(r, \varphi) \times \sin \omega t$ distributing on the plate, the equa-

Fig.1 Circular plate elastically supported at several points

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tion of motion for the lateral deflection of a thin elastic plate with uniform thickness h is given by

$$D\nabla^2 W - \rho h \omega^2 W = Q(r, \varphi) \quad \dots\dots\dots (1)$$

where $W(r, \varphi)$ is the deflection amplitude of the plate, ρ is the mass per unit volume and $\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial \varphi^2$ denotes the Laplacian operator. D is the flexural rigidity of the plate expressed by $D = Eh^3/12(1-\nu^2)$ (E : Young's modulus, ν : Poisson's ratio). In the case of a free circular plate without supports, the bending moment M_r and the shearing force V_r are equal to zero on the circumference and therefore

$$M_r = -D \left\{ \frac{\partial^2 W}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial W}{\partial r} + \frac{1}{r^2} \frac{\partial^2 W}{\partial \varphi^2} \right) \right\} = 0 \quad \dots\dots (2)$$

$$V_r = -D \left\{ \frac{\partial}{\partial r} \nabla^2 W + (1-\nu) \frac{1}{r} \frac{\partial}{\partial \varphi} \left(\frac{1}{r} \frac{\partial^2 W}{\partial r \partial \varphi} - \frac{1}{r^2} \frac{\partial W}{\partial \varphi} \right) \right\} = 0 \quad \dots\dots\dots (3)$$

must be satisfied at $r=a$. And also the displacement of the plate must be finite at the center.

The deflection of the plate satisfying Eq.(1) and boundary conditions (2) and (3) is written as

$$W(r, \varphi) = \int_0^{2\pi} \int_0^a Q(\eta, \theta) G(r, \varphi; \eta, \theta) \eta d\eta d\theta \quad \dots\dots (4)$$

with use of Green's function of the plate

$$G(r, \varphi; \eta, \theta) = \frac{a^2}{\pi D} \left[-\frac{1}{\lambda^4} - \frac{4}{\lambda^4} \left(\frac{r}{a} \right) \left(\frac{\eta}{a} \right) \cos(\varphi - \theta) + \sum_{m=0}^{\infty} \sum_{s=1}^{\infty} \frac{1}{\lambda_{ms}^4 - \lambda^4} \frac{1}{\sqrt{L(\lambda_{ms})}} \left\{ J_m(\lambda_{ms} \frac{r}{a}) + g(\lambda_{ms}) I_m(\lambda_{ms} \frac{r}{a}) \right\} \frac{1}{\sqrt{L(\lambda_{ms})}} \left\{ J_m(\lambda_{ms} \frac{\eta}{a}) + g(\lambda_{ms}) I_m(\lambda_{ms} \frac{\eta}{a}) \right\} \frac{\epsilon_m}{2} \cos m(\varphi - \theta) \right] \quad (\epsilon_0=1, \epsilon_1=\epsilon_2=\dots=2) \quad \dots\dots\dots (5)$$

where $\lambda^4 = \rho h a^4 \omega^2 / D$ is a dimensionless parameter concerning the excitation frequency ω , and λ_{ms} is a parameter corresponding to the natural frequency ω_{ms} of a free circular plate without supports, and $J_m(\lambda_{ms} r/a)$ and $I_m(\lambda_{ms} r/a)$ denote the Bessel functions and the modified Bessel functions of the first kind, respectively. The parameter λ_{ms} and the coefficient $g(\lambda_{ms})$ in Eq.(5) are determined by the calculation of the equation

$$\left\{ \begin{array}{l} \{ z^2/(1-\nu) - m(m-1) \} J_m(z) - z J_{m+1}(z) \\ m \{ z^2/(1-\nu) + m(m-1) \} J_m(z) - z \{ z^2/(1-\nu) + m^2 \} J_{m+1}(z) \end{array} \right\} \left\{ \begin{array}{l} \{ -z^2/(1-\nu) - m(m-1) \} I_m(z) + z I_{m+1}(z) \\ m \{ -z^2/(1-\nu) + m(m-1) \} I_m(z) - z \{ z^2/(1-\nu) - m^2 \} I_{m+1}(z) \end{array} \right\} \left\{ \begin{array}{l} 1 \\ g(z) \end{array} \right\} = 0 \quad (z = \lambda_{ms}) \quad \dots\dots\dots (6)$$

derived from Eqs.(2) and (3). $L(\lambda_{ms})$ is given by

$$L(z) = \frac{1}{2} \left\{ J_m^2(z) + J_{m+1}^2(z) \right\} - \frac{m}{z} J_m(z) J_{m+1}(z) + g(z) \left\{ J_m(z) I_{m+1}(z) + J_{m+1}(z) I_m(z) \right\} + g^2(z) \left[\frac{1}{2} \{ I_m^2(z) - I_{m+1}^2(z) \} - \frac{m}{z} I_m(z) I_{m+1}(z) \right] \quad (z = \lambda_{ms}) \quad \dots\dots\dots (7)$$

In Eq.(5), the symbol $\Sigma' \Sigma'$ with the superscript ' denotes a double series excluding the terms $(m, s) = (0, 1)$, $(1, 1)$, and the first and the second terms in the bracket [] correspond to the translation and the rotation of a free plate, which can be obtained by calculating the limiting values as λ_{01} and λ_{11} approach zero, respectively.

Denoting the unknown reaction acting at a support $P_n(r_n, \varphi_n)$ by R_n , the external force $Q(r, \varphi)$ is written as $Q(r, \varphi) = R_n \delta(r - r_n) \delta[\varphi - \varphi_n]$ with use of Dirac's delta function and therefore Eq.(4) is expressed by $W(r, \varphi) = R_n G(r, \varphi; r_n, \varphi_n)$. When the rotation and the torsion of the plate are also constrained in addition to the lateral deflection, unknown moments $M_{r, n}$, $M_{\varphi, n}$, and $M_{r\varphi, n}$ act at the support. In this case, the deflection of the plate is derived by calculating the response of the plate to couples of the forces that produce these moments, as follows:

$$W(r, \varphi) = \sum_{n=1}^N \left\{ R_n G(r, \varphi; r_n, \varphi_n) + M_{r, n} \frac{\partial}{\partial r_n} G(r, \varphi; r_n, \varphi_n) + M_{\varphi, n} \frac{\partial}{\partial r_n \partial \varphi_n} G(r, \varphi; r_n, \varphi_n) + M_{r\varphi, n} \frac{\partial^2}{\partial r_n \partial \varphi_n} G(r, \varphi; r_n, \varphi_n) \right\} \quad \dots\dots\dots (8)$$

When a circular plate is elastically supported at a point P_n , the following conditions between deflection and reaction, rotation and bending moment, and torsion and torsional moment must be satisfied there

$$R_n = -k_n W(r_n, \varphi_n), \quad M_{r, n} = -k_{r, n} \frac{\partial}{\partial r_n} W(r_n, \varphi_n), \quad M_{\varphi, n} = -k_{\varphi, n} \frac{\partial}{\partial r_n \partial \varphi_n} W(r_n, \varphi_n), \quad M_{r\varphi, n} = -k_{r\varphi, n} \frac{\partial^2}{\partial r_n \partial \varphi_n} W(r_n, \varphi_n) \quad \dots\dots\dots (9)$$

where k_n ; $k_{r, n}$, $k_{\varphi, n}$ and $k_{r\varphi, n}$ denote the spring stiffnesses for the lateral deflection, the rotation in the radial and circumferential directions, and the torsion, respectively. When $k_n = \infty$ and $k_{r, n} = k_{\varphi, n} = k_{r\varphi, n} = 0$, the plate is simply supported at the point, and when all the stiffnesses are equal to infinity, the plate is clamped at the point. Equations (9) present the constraint conditions of the vibrating system. Substituting Eq.(8) into Eqs. (9), the following equation is derived.

$$\begin{pmatrix}
 \frac{1}{k_n} \delta_{nl} + G_{nl}(\lambda) & \frac{\partial}{\partial r_l} G_{nl}(\lambda) & -\frac{\partial}{r_l \partial \varphi_l} G_{nl}(\lambda) & -\frac{\partial^2}{\partial r_l \partial \varphi_l} G_{nl}(\lambda) \\
 \frac{1}{k_{r,n}} \delta_{nl} + \frac{\partial^2}{\partial r_n \partial r_l} G_{nl}(\lambda) & \frac{\partial^2}{\partial r_n \partial \varphi_l} G_{nl}(\lambda) & \frac{\partial^2}{\partial r_n \partial r_l \partial \varphi_l} G_{nl}(\lambda) & -\frac{\partial^3}{\partial r_n \partial r_l \partial \varphi_l} G_{nl}(\lambda) \\
 \text{sym} & & &
 \end{pmatrix}
 \begin{pmatrix}
 R_l \\
 M_{r,l} \\
 M_{\varphi,l} \\
 M_{r\varphi,l}
 \end{pmatrix} = 0$$

(n, l = 1, 2, \dots, N) \dots\dots\dots (10)

where $G_{nl}(\lambda)$ expresses $G(r_n, \varphi_n; r_l, \varphi_l)$ and δ_{nl} denotes Kronecker's delta. The physical meaning of the torsional stiffness $k_{r\varphi, n}$ of the support is not clear practically, and if the stiffness is not considered, it is sufficient to treat the equation which is obtained by omitting the rows and columns concerning the torsional stiffness in Eq.(10) and the equations described later. When the plate is simply supported at the points, Eq.(10) is written in the simple form of

$$[G_{nl}(\lambda)] R_l = 0, \quad M_{r,l} = M_{\varphi,l} = M_{r\varphi,l} = 0 \dots\dots\dots (11)$$

The natural frequency ω of the system is obtained by calculating the eigenvalues λ of Eq.(10) and the mode shapes are determined by obtaining the eigenvectors $\{R_l, M_{r,l}, M_{\varphi,l}, M_{r\varphi,l}\}^T$. When the eigenvalues of a free circular plate have been calculated, the natural frequencies and the mode shapes can be obtained accurately up to a considerably higher order by this method.

3. Circular plate elastically supported at several equi-spaced points of equal radii

We consider the free vibration of a circular plate which is elastically supported at

Table 1 Circular plate elastically supported at a point

S-type	A-type

$[F_1(1) + G_1(\lambda)] \begin{Bmatrix} \xi a R \\ M_r \end{Bmatrix} = 0 \dots\dots\dots (12)$	$[F_2(1) + G_2(\lambda)] \begin{Bmatrix} M_\varphi \\ M_{r\varphi} \end{Bmatrix} = 0 \dots\dots\dots (13)$
$m = 0, 1, 2, \dots$	
$M_\varphi = M_{r\varphi} = 0$	$R = M_r = 0$

Table 2 Circular plate elastically supported at two points

SS'-type	SA'-type	AS'-type	AA'-type

$[F_1(2) + G_1(\lambda)] \begin{Bmatrix} \xi a R \\ M_r \end{Bmatrix} = 0 \dots\dots\dots (14)$		$[F_2(2) + G_2(\lambda)] \begin{Bmatrix} M_\varphi \\ M_{r\varphi} \end{Bmatrix} = 0 \dots\dots\dots (15)$	
$m = 0, 2, 4, \dots$	$m = 1, 3, 5, \dots$	$m = 1, 3, 5, \dots$	$m = 2, 4, 6, \dots$
$M_\varphi = M_{r\varphi} = 0$		$R = M_r = 0$	

Table 3 Circular plate elastically supported at three points

SSS-type	AAA-type	S-type	A-type

$[F_1(3) + G_1(\lambda)] \begin{Bmatrix} \xi a R \\ M_r \end{Bmatrix} = 0 \dots\dots\dots (16)$		$[F_2(3) + G_2(\lambda)] \begin{Bmatrix} \xi a R' \\ M_{r'} \\ c M_{\varphi'} \\ c M_{r\varphi'} \end{Bmatrix} = 0 \dots\dots\dots (18)$	
$m = 0, 3, 6, \dots$	$m = 3, 6, 9, \dots$	$m = 1, 2, 4, 5, 7, \dots$	
$M_\varphi = M_{r\varphi} = 0$	$R = M_r = 0$	$-R = 2R', -M_r = 2M_{r'}, -M_{r\varphi} = 2M_{r\varphi'}, -M_{\varphi} = 2M_{\varphi'}$	$R = M_r = 0, -M_{\varphi} = 2M_{\varphi'}, -M_{r\varphi} = 2M_{r\varphi'}$

Table 4 Circular plate elastically supported at four points

SSS'-type	SSA'A'-type	AAS'S'-type	AAA'A'-type	SA-type	AS-type
$[F_1(4) + G_1(\lambda)] \begin{Bmatrix} \xi a R \\ M_r \end{Bmatrix} = 0 \dots (19)$		$[F_2(4) + G_2(\lambda)] \begin{Bmatrix} M_\varphi \\ M_{r\varphi} \end{Bmatrix} = 0 \dots (20)$		$[(1/2)F_3(4) + G_3(4, \lambda)] \begin{Bmatrix} \xi a R^* (R')^{**} \\ M_r (M_r') \\ -M_{r\varphi} (M_{r\varphi}') \end{Bmatrix} = 0 \dots (21)$	$\gamma_4 = 1$
$m = 0, 4, 8, \dots$	$m = 2, 6, 10, \dots$	$m = 2, 6, 10, \dots$	$m = 4, 8, 12, \dots$	$m = 1, 3, 5, \dots$	
$M_\varphi = M_{r\varphi} = 0$		$R = M_r = 0$		$R' = M_r' = 0, M_\varphi = M_{r\varphi} = 0$	$R = M_r = 0, M_{\varphi'} = M_{r\varphi'} = 0$

*SA-type vibration, **AS-type vibration

1, 2, 3 and 4 points located at a constant interval ($2\pi/N$) of equal radius ($r = \xi a$) under the same supporting conditions. For these cases, the frequency equations are shown in Tables 1~4, where the following quantities are used.

$$F_1(N) = \frac{1}{N} \begin{bmatrix} 1/\kappa & 0 \\ 0 & 1/\kappa_r \end{bmatrix}, \quad F_2(N) = \frac{1}{N} \begin{bmatrix} 1/\kappa_\varphi & 0 \\ 0 & 1/\kappa_{r\varphi} \end{bmatrix}, \quad F_3(N) = \begin{bmatrix} 1/\kappa & 0 \\ 1/\kappa_r & \gamma_N/\kappa_\varphi \\ 0 & \gamma_N/\kappa_{r\varphi} \end{bmatrix} \dots (22)$$

and

$$G_1(\lambda) = \sum_{m=1}^{\infty} \frac{1}{\lambda_{m1}^4 - \lambda^4} \begin{bmatrix} G^2(\lambda_{m1}\xi) & G(\lambda_{m1}\xi)H(\lambda_{m1}\xi) \\ \text{sym} & H^2(\lambda_{m1}\xi) \end{bmatrix}, \quad G_2(\lambda) = \sum_{m=1}^{\infty} \frac{1}{\lambda_{m1}^4 - \lambda^4} \begin{bmatrix} m^2 G^2(\lambda_{m1}\xi) & m^2 G(\lambda_{m1}\xi)H(\lambda_{m1}\xi) \\ \text{sym} & m^2 H^2(\lambda_{m1}\xi) \end{bmatrix}$$

$$G_3(N, \lambda) = \sum_{m=1}^{\infty} \frac{1}{\lambda_{m1}^4 - \lambda^4} \begin{bmatrix} c_m G^2(\lambda_{m1}\xi) & c_m G(\lambda_{m1}\xi)H(\lambda_{m1}\xi) & m s_m G^2(\lambda_{m1}\xi) & m s_m G(\lambda_{m1}\xi)H(\lambda_{m1}\xi) \\ c_m H^2(\lambda_{m1}\xi) & m s_m G(\lambda_{m1}\xi)H(\lambda_{m1}\xi) & m s_m H^2(\lambda_{m1}\xi) & \gamma_N m^2 c_m G(\lambda_{m1}\xi)H(\lambda_{m1}\xi) \\ \text{sym} & \gamma_N m^2 c_m G^2(\lambda_{m1}\xi) & \gamma_N m^2 c_m H^2(\lambda_{m1}\xi) & \gamma_N m^2 c_m H^2(\lambda_{m1}\xi) \end{bmatrix} \dots (23)$$

$$c_m = 1 - \cos(2m\pi/N), \quad s_m = \sin(2m\pi/N)$$

where

$$G(\lambda_{m1}\xi) = \{J_m(\lambda_{m1}\xi) + g(\lambda_{m1})I_m(\lambda_{m1}\xi)\} / \sqrt{L(\lambda_{m1})}$$

$$H(\lambda_{m1}\xi) = \{m[J_m(\lambda_{m1}\xi) + g(\lambda_{m1})I_m(\lambda_{m1}\xi)] - \lambda_{m1}\xi\{J_{m+1}(\lambda_{m1}\xi) - g(\lambda_{m1})I_{m+1}(\lambda_{m1}\xi)\}\} / \sqrt{L(\lambda_{m1})} \dots (24)$$

$\kappa = k_n a^2 / \pi D$; $\kappa_r = k_r, n / \pi \xi^2 D$, $\kappa_\varphi = k_\varphi, n / \pi \xi^2 D$ and $\kappa_{r\varphi} = k_{r\varphi}, n / \pi \xi^2 D$ denote dimensionless spring stiffnesses, where the subscript n is omitted since each stiffness is the same at all supports.

In the case of a plate supported at equi-spaced points, the vibrations of several types arise according to the number of supports and hence Eq.(10) can be divided into some sets of equations corresponding to these types as shown in the tables. The S-type vibration is a symmetrical vibration about the diameter passing through a support (broken line) and A-type vibration is an antisymmetrical one about the diameter corresponding to a nodal line (solid line). The S'- and A'-type vibrations are symmetrical and antisymmetrical ones about the diameter which divides equally the angle between the diameters passing through the adjacent supports, respectively. For each type vibration, the number m takes only the integers specified in the tables. The eigenvalues of the S- and A-type vibrations (Eq.(18) in Table 3), and the SA- and AS-type vibrations (Eq.(21) in Table 4) are calculated with use of the same equation and the same integers m presented in each column of the tables and therefore they take the same values (degeneration of frequencies).

In the case of a plate simply supported at the points, it is not necessary to calculate the A-type vibration of the plate supported at a point and the AS'- and AA'-type vibrations of the plate supported at two points, etc., because these mode shapes are the same as those of a free plate without supports.

4. Numerical calculation

We consider the free vibration of a circular plate supported at several equi-spaced points of equal radii under the same conditions. In the numerical calculation, the number of terms (m, s) of the infinite series in Eq.(23) may be set at an appropriate finite number by considering the convergence and accuracy of the solution.

Tables 5 and 6 present the convergence characteristics of the eigenvalues of a circular plate simply supported at three points on the circumference and a plate clamped at a point located at $\xi = 1/2$, respectively. As shown in Table 5, in the case of a plate simply supported at three points, the use of 33 terms of the series ($\lambda_{m1} < 25$) is sufficient to obtain convergent values for the SSS-type vibration, but 60 terms ($\lambda_{m1} < 25$) must be used for the

Table 5 Convergence characteristics of eigenvalues of circular plate simply supported at three points ($\nu=0.33$, $\xi=1$)

number of terms	λ	
	SSS-1	SSS-2
3 ($\lambda_{min} < 5$)	1.885	3.166
7 (10)	1.847	3.159
23 (20)	1.832	3.157
33 (25)	1.830	3.157
48 (30)	1.830	3.157

number of terms	λ	
	S-1(A-1)	S-2(A-2)
4 ($\lambda_{min} < 5$)	1.832	3.795
12 (10)	1.817	3.593
39 (20)	1.814	3.553
60 (25)	1.813	3.548
83 (30)	1.813	3.545

Table 6 Convergence characteristics of eigenvalues of circular plate clamped at a point ($\nu=0.33$, $\xi=1/2$)

number of terms	λ			
	S-1	S-3	A-1	A-3
62 ($\lambda_{min} < 20$)	1.231	3.151	1.519	3.672
217 (40)	1.175	3.088	1.439	3.598
448 (60)	1.148	3.058	1.402	3.577
650 (80)	1.134	3.043	1.379	3.568
768 (100)	1.129	3.037	1.372	3.565

S-type vibration. As seen in Table 6, in the case of a plate clamped at a point, it is necessary to use hundreds of terms for obtaining convergent values.

Figure 2 shows the frequencies and the mode shapes of a plate simply supported at a point of $\xi=1/2$ or at a point on the circumference. As seen in the figure, the location of the support changes remarkably the natural frequencies and the mode shapes of the plate. It should be noticed that when the support is inside the plate, there exists a nodal line passing through the point, even though it may be small.

Figure 3 shows the frequencies and the mode shapes of the plate clamped at a support located $\xi=1/2$. It is observed from the comparison of this figure with Fig.2(a) that the frequencies of the S-type vibration of the plate clamped at a support are a little higher than those of the plate simply supported at a point and the mode shapes are different from each other. The torsional stiffness k_{rv} is considered here, but when the stiffness is not considered, the frequencies of the A-type vibration become a little smaller as written in the parentheses.

Fig.2 Mode shapes of circular plate simply supported at a point ($\nu=0.33$)

Fig.3 Mode shapes of circular plate clamped at a point ($\nu=0.33$, $\xi=1/2$)

Fig.4 Mode shapes of circular plate simply supported at three points ($\nu=0.33$)

Figure 4 shows the frequencies and the mode shapes of the plate simply supported at three points located at $\xi=1/2$ or on the circumference, respectively. When the supports are inside the plate, there exist nodal lines passing through the supports as shown in Fig.2(a).

The natural frequencies and the mode shapes of the plate are influenced by the constraint conditions of the supports. Figure 5 shows the effect of the stiffness of the supports on the frequencies of a plate elastically supported at three points on the circumference. In the figure, the natural frequencies $\lambda_{01}, \lambda_{02}, \dots$ of the SSS-type vibration and $\lambda_{s1}, \lambda_{s2}, \dots$ of the AAA-type vibration of a free circular

plate are shown by broken lines, and the mode shapes are presented on the left side.

The natural frequencies of the plate increase monotonously within the range between the adjacent broken lines corresponding to the frequencies of a free plate, with an increase of the stiffness of supports. However, the rate of the increase of frequencies has different characteristics depending upon the mode of vibration.

Nakazawa⁽⁹⁾ obtained the natural frequencies of a plate simply supported at 3, 4, and 6 points on the circumference with use of a method in which the reactions of the supports are expanded into a Fourier series on the whole circumference. In the

Fig.5 Eigenvalues of circular plate elastically supported at three points on the circumference ($\nu=0.33$)

Fig.6 Eigenvalues of circular plate simply supported at four points ($\nu=0.33$)

Table 7 Eigenvalues of circular plate simply supported at four points on the circumference ($\nu=0.30$)

	SA-1	SA-2	SSS'S'-1	SSS'S'-2	SSA'A'-1
Nakazawa ⁽⁹⁾	2.592	4.164	2.047	3.620	4.543
Present	2.592	4.165	2.047	3.621	4.550

case of a plate simply supported at four points, the eigenvalues obtained by the present method are compared with Nakazawa's results in Table 7, where they are in good agreement.

Figure 6 shows the effect of the location of the supports on the frequencies of the plate simply supported at four points. As seen in the figure, the natural frequencies of each type of vibrations are within the adjacent frequencies of a free plate (broken line), and do not vary monotonously with an increase of the radius of the supports. The abscissa of the point marked with $\circ$ in Fig.6 gives the radius of a nodal circle of a vibrating free plate, and when the supports are located on the nodal circle, the frequencies of the plate treated here come to coincide with those of a free circular plate.

Figure 7 shows the effect of the number of supports on the natural frequencies of the plate simply supported or clamped at several points on the circumference. The mode shapes of the plate simply supported or clamped on the whole circumference are presented on the right side of the figure, and the ordinates of the center of each circle show the natural frequencies corresponding to their mode shapes. The frequencies increase gradually with an increase of the number of points except in some cases and when the number of supports is over 16~20, the frequencies approach those of the plate simply supported or clamped on the whole circumference as seen in the figure. The points marked with $\bullet$ show the frequencies λ_{01} , λ_{11} , λ_{21} , ... of a free plate which correspond to the mode shapes similar to those presented on the right side. The eigenvalues of the fourth order of a plate supported at one or two points are smaller than those of a free plate. The reason is that a little higher frequencies than λ_{02} do not exist and hence the frequencies near λ_{02} are adopted. When a plate is supported at many points, the mode shapes are similar to those of a plate supported on the

Fig. 7 Eigenvalues of circular plate simply supported or clamped at some points on the circumference ($\nu=0.33$)

whole circumference, but as the number of supports decreases, the mode shapes considerably change. When the torsional stiffness is not considered, the constraint of the plate is weakened as the result and hence the frequencies of the second and third order become a little small, as shown by chain lines in Fig. 7(b). On the contrary, the vibrations of the first, fourth order, etc. which are symmetrical about the diameter passing through the supports are not affected by the torsional stiffness.

5. Conclusions

Free vibration of a circular plate elastically supported at some points was analyzed theoretically and the following conclusions were obtained.

- (1) Regarding the reaction forces, bending moments and torsional moments acting at the points as unknown harmonic

forces and moments, the stationary response of the plate to these loads was obtained.

- (2) Frequency equations of a circular plate elastically supported at N points were derived with use of the constraint conditions of the supports. Applying the theory to a circular plate supported at several equi-spaced points of equal radii, the natural frequencies and the mode shapes of the plate were calculated numerically.
- (3) The frequencies of the plate increased monotonously with an increase of the stiffness of the supports, without regard to the location and the number of supports.
- (4) The natural frequencies and the mode shapes of the plate varied considerably with the location of supports, but the variation of the frequencies was not monotonous.
- (5) With an increase of the number of supports, the natural frequencies of the plate with similar mode shapes became large and approached those of the plate continuously supported on the whole circle of the same radius as the supports.

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