Pressure Loss of Pseudo-Plastic Fluid Flow in Pipes*

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In order to predict both the velocity profile and the relationship between friction factor and Reynolds number for pseudo-plastic fluids flow in smooth pipes, an improved mixing-length distribution is proposed in this paper. This distribution is a modification of Van Driest's model for Newtonian fluids flow obtained by referring to the results of the other researcher's studies about Newtonian and pseudo-plastic fluids. It seems that this mixing-length distribution is suitable for estimating and calculating both the velocity profile and the pressure loss for Newtonian and pseudo-plastic fluids flow in circular smooth pipes through the analyzed results as compared with experimental results. A set of design curves of friction factor versus Reynolds number is presented.

1. Introduction

In order to make an accurate estimation of pressure loss of the pseudo-plastic fluids flow in pipes, it is necessary to know the relationship between friction factor and Reynolds number. When the theoretical results given by Dodge⁽¹⁾, Clapp⁽²⁾, Tomita⁽³⁾ and Hanks⁽⁴⁾ are compared with experimental results, it is recognized that there are disagreements between their theoretical and experimental ones in the turbulent flow region. For example, Dodge's results disagree with experimental results in high Reynolds number region as flow behavior index is reduced, since the experimental constants are obtained by use of experimental results in comparatively near-transitional Reynolds number region. Clapp's results agree with experimental ones in high Reynolds number. However, his results have not a good agreement with experimental ones in turbulent region near the critical Reynolds number, especially in Newtonian fluids. Tomita's results agree well with experimental ones when the value of flow behavior index n is almost 1, but his results disagree with the diminution of the value of n . On the other hand, the authors think that there is a theoretical problem in Hanks' analysis as described in 4.1.

The purpose of this paper is to present an improved mixing-length distribution which permits accurate predictions of both the velocity profile and the relationship between friction factor and Reynolds number for Newtonian and pseudo-plastic fluids flow in circular smooth pipes. The present model is a modification of Van Driest's model⁽⁵⁾ for Newtonian fluids flow obtained by referring to the mixing-length

model proposed by Hecht⁽⁶⁾, Hanks⁽⁴⁾ and Kim⁽⁷⁾.

2. Nomenclature

D : inner diameter of circular pipe
 K : consistency index
 l : mixing-length
 r : distance from center axis
 R : radius of circular pipe
 u : velocity
 u_{max} : maximum velocity
 u^* : friction velocity
 V_m : mean velocity
 y : distance from the wall
 ρ : density of fluid
 τ : shear stress
 τ_w : shear stress at the wall
 f : friction factor
 $L^* = l/R$
 n : flow behavior index
 NRe : Reynolds number
 $NRep$: Reynolds number defined by Eq.(6)
 $u^+ = u/u^*$
 $X = r/R$
 $y^+ = \rho u^{*2-n} y^n / K$
 $\alpha = \rho u^{*2-n} R^n / K$
 α_c : the value given by Eq.(14)

3. Analysis

For the turbulent flow of pseudo-plastic fluids, the shear stress is given as the following relation:

$$\tau = K (du/dy)^n + \rho l^2 (du/dy)^2 \quad \text{----- (1)}$$

As the shear stress distribution across a circular pipe may be taken as being linear with a value of zero at the center line of pipe, Eq.(1) may be written in the dimensionless form as follows;

$$X = \frac{1}{\alpha} \left(-\frac{du^+}{dX} \right)^n + L^{*2} \left(-\frac{du^+}{dX} \right)^2 \quad \text{----- (2)}$$

where, $X = r/R = \tau/\tau_w$, $u^+ = u/\sqrt{\tau_w/\rho} = u/u^*$,
 $\alpha = \rho u^{*2-n} R^n / K$, $L^* = l/R$

In order to solve Eq.(2) for $(-du^+/dX)$, the authors use the numerical method and assume that $(-du^+/dX)$ is given as follows;

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$$-\frac{du^*}{dx} = G(x, L^*, \alpha, n) \quad (3)$$

Integrating Eq.(3) under the condition, $u^* = 0$ at $x = 1$, the velocity profile is given as follows;

$$u = u^* \int_x^1 \left(-\frac{du^*}{dx} \right) dx' \\ = u^* \int_x^1 G(x', L^*, \alpha, n) dx' \quad (4)$$

Moreover, integrating Eq.(4) and dividing the integrated value by the area of pipe, the mean velocity is obtained as follows;

$$V_m = 2 u^* \int_0^1 x \int_x^1 G(x', L^*, \alpha, n) dx' dx \\ = u^* \int_0^1 x^2 G(x, L^*, \alpha, n) dx \quad (5)$$

The Reynolds number of pseudo-plastic fluids flow in circular pipes defined by Metzner and Reed⁽⁸⁾ is obtained by use of Eq.(5) as follows;

$$N_{Re} = \frac{\rho V_m^{2-n} D^n}{K} \left(\frac{4n}{3n+1} \right)^n \delta^{1-n} = 2^n \alpha \left(\frac{V_m}{u^*} \right)^{2-n} \left(\frac{4n}{3n+1} \right) \delta^{1-n} \\ = 8 \alpha \left(\frac{n}{3n+1} \right)^n \left\{ \int_0^1 x^2 G(x, L^*, \alpha, n) dx \right\}^{2-n} \quad (6)$$

The friction factor defined by Fanning is obtained by use of Eq.(5) as follows;

$$f = \frac{2u^{*2}}{V_m^2} = 2 \left/ \left\{ \int_0^1 x^2 G(x, L^*, \alpha, n) dx \right\}^2 \right. \quad (7)$$

4. Discussion

In order to analyze the pseudo-plastic fluids flow in pipes by the mixing-length, it is necessary to establish the mixing-length distribution. The authors think that there is no other method of obtaining the value of mixing-length but to get it by either experiment or hypothesis until the theory of turbulent flow is established perfectly. In this paper, the authors propose the mixing-length distribution for pseudo-plastic fluids flow in circular smooth pipes by referring to the other researcher's studies and analyze the velocity profile and the relationship between friction factor and Reynolds number.

4.1 Mixing-length model by previous researchers

Nikuradse⁽⁹⁾ obtained an experimental equation concerning the mixing-length distribution for Newtonian fluids flow in pipes for the range $N_{Re} \geq 10^5$ as follows;

$$\frac{l}{R} = 0.14 - 0.08 \left(1 - \frac{y}{R} \right)^2 - 0.06 \left(1 - \frac{y}{R} \right)^4 \quad (8)$$

On the other hand, Prandtl⁽⁵⁾ proposed a mixing-length formula with the dimensional analysis as follows;

$$l = k y \quad (9)$$

Moreover, the value of Prandtl constant is obtained as 0.4 from the first approximation of the Nikuradse's experimental equation.

Van Driest⁽⁵⁾ proposed a modification of Prandtl's mixing-length equation as follows;

$$l = 0.4 y \left\{ 1 - \exp \left(-\frac{y}{R} \frac{\alpha}{26} \right) \right\} \quad (10)$$

The theoretical velocity profile calculated by using Eq.(10) agrees well with the experimental results in the near wall region. However, the value of friction factor obtained by using Eq.(10) is a little larger than the value of Kármán-Nikuradse equation, especially it is remarkable in the turbulent flow region near critical Reynolds number.

Hecht⁽⁶⁾ assumed that the mixing-length was Prandtl's mixing-length model in the near wall region and was constant in the turbulent core region. However, his model has a discontinuous form on the differential coefficient of mixing-length.

On the other hand, Hanks⁽⁴⁾ proposed a modification of Van Driest's mixing-length model which took into account the effects of transitional flow characteristics as follows;

$$\frac{l}{R} = 0.36 \frac{y}{R} \left\{ 1 - \exp \left[-\frac{y}{R} \frac{\alpha^{1/n} \{ 1 - (\alpha'/\alpha)^{1/n} \}}{22 B(n)} \right] \right\} \quad (11)$$

where, α' is the value of α at the laminar-turbulent transition and is given by Hanks' transition theory⁽¹¹⁾. $B(n)$ is a function of n which accounts for the influence of the flow behavior index to the constant of damping effect factor, which is 22 in Hanks' theory, and is given by Hanks as follows;

$$B(n) = 1/n \quad (12)$$

The velocity profile computed from Hanks' modified mixing-length model has not a good agreement with experimental results in the turbulent core region. Moreover, the values of friction factor about Newtonian fluids computed from his model are a little smaller than the values of Kármán-Nikuradse equation in the turbulent flow region near the critical Reynolds number. The authors think that these disagreements are due to the difference between the values of Hanks' mixing-length and the values of Nikuradse's experimental results near the central position of pipe and due to the unsatisfactory correction for the damping effect factor in transitional flow region. Moreover, the authors think that his correction by using α' is not suitable because his transition theory for predicting the lower critical Reynolds number has a theoretical problem which was pointed out in their previous study.⁽¹²⁾

4.2 Mixing-length model

In order to analyze the pseudo-plastic fluids flow in circular smooth pipes, the mixing-length model is obtained from hypothetical conditions which are established by taking the merits and the problem of previous studies into consideration. The hypothetical conditions are as follows;

(1) The mixing-length distribution of pseudo-plastic fluids flow in pipes is almost the same as that of Newtonian fluids both in high Reynolds number region and in turbulent core region, because it is thought that the distribution is affected scarcely by K and n in those region.

(2) Prandtl's hypothesis is applicable to the mixing-length for pseudo-plastic fluids flow in the near pipe wall region.

Moreover, the value of Prandtl constant is fixed at 0.4.

(3) Both mixing-length and its differential coefficient are continuous.

(4) The mixing-length of pseudo-plastic fluids flow is affected by the existence of pipe wall as well as that of Newtonian fluids and must be corrected by using Van Driest's method.

(5) As it is thought that the damping effect factor is affected by n , it must be corrected by using Hanks' method.

(6) The authors think that it is necessary to correct the damping effect factor in the turbulent flow region near the critical Reynolds number for the same reason as Kim pointed out about Newtonian fluids.

The authors propose a mixing-length distribution according to the above mentioned hypothetical condition as follows;

$$L^+ = \frac{L}{R} = 0.4 \frac{y}{R} \exp\left(-\frac{y}{R}\right) \times \left\{ 1 - \exp\left\{ -\frac{y}{R} \frac{\alpha^{1/n} F(\alpha, \alpha_c, n)}{26.8(n)} \right\} \right\} \quad (13)$$

where, α_c is the value of α at the laminar-turbulent transition and is computed from their previous study⁽¹²⁾ as follows;

$$\alpha_c = \left\{ \frac{280(2n+1)(3n+2)}{n} \right\}^{n/2} \quad (14)$$

Furthermore, $\exp(-y/R)$ in Eq.(13) is a term to correct the differences between the condition(2) and the experimental results both in the high Reynolds number region and in the

turbulent core region. Therefore, Eq.(13) approximates Eq.(8) within 5.2 % in those regions. In order to satisfy the conditions (4), (5) and (6), a corrected term, i.e. [], is arranged in Eq.(13). It is certified through the numerical calculation in next chapter that Eq.(12) proposed by Hanks is suitable to $B(n)$ in Eq.(13). Moreover, the function $F(\alpha, \alpha_c, n)$ in Eq.(13) is a corrected term for the damping effect factor in the turbulent flow region near the critical Reynolds number. The above function is analyzed through numerical calculation in next chapter as follows; In case of $\alpha > \alpha_c$ (for transitional and turbulent flow)

$$F(\alpha, \alpha_c, n) = 1 - (\alpha_c/\alpha)^2$$

In case of $\alpha \leq \alpha_c$ (for laminar flow)

$$F(\alpha, \alpha_c, n) = 0$$

5. Comparison

5.1 Newtonian fluids flow

Fig.1 shows the relation between the experimental results obtained by Nikuradse⁽⁹⁾ and the numerical results about velocity profile which are analyzed using the mixing-length model given by Van Driest⁽⁵⁾, Hanks⁽⁴⁾ and the authors. Fig.2 in which the velocity profile is expressed in velocity defect law shows the relation between Nikuradse's results and the numerical results computed using the authors' model. It is shown in Fig.3 that the relations between the ratio of mean velocity to maximum velocity and Reynolds number are computed using the mixing-length model of Hanks⁽³⁾ and the authors and the experimental data are quoted from literatures^(9,10,11,12). The relations between friction factor and Reynolds number are shown in Fig.4 which are computed using the mixing-length model given by Van Driest, Hanks and the authors as well as Kármán-Nikuradse.

As shown in Fig.1, the authors' results about velocity profile agree better with the experimental results of Nikuradse than Van Driest's results and Hanks' results

Fig.1 The comparison between experimental results and analyzed results concerning the velocity profile of Newtonian fluids.

have not a good agreement with the experimental results in central part of pipe. As shown in Fig.2, it seems that the present results concerning the velocity defect agree better with experimental results than the equation of velocity defect law which is expressed as Eq.(16).

$$(U_{max} - U)/U^* = 5.75 \log_{10} (R/y) \quad \text{----- (16)}$$

Moreover, as shown in Fig.3, Hanks' analyzed results about the relation between the ratio of V_m to U_{max} and NRe agree well with experimental results in the laminar and transitional flow regions. However, his results have not a good agreement with experimental results because his values are a little larger than the experimental results. But, the authors' results agree well with them in the laminar, transitional and turbulent flow regions. Furthermore, as shown in Fig.4, Van Driest's results about the relation between friction factor and Reynolds number are a little larger than the values computed from Kármán-Nikuradse equation and approach the values of that equation as Reynolds number increases. Hanks' results agree with the values of that equation in high Reynolds number. However, his results in the turbulent flow region near the critical Reynolds number

Fig.2 The comparison between experimental results and analyzed results in velocity defect law.

Fig.3 The comparison between experimental results and analyzed results concerning the relation between V_m/U_{max} and NRe .

are a little smaller than the values of that equation. But, the authors' results agree well with Kármán-Nikuradse's results. Compared with experimental results, the authors' correction for damping effect factor in turbulent flow region near the critical Reynolds number seems to be more suitable than Hanks' correction. It is thought that the mixing-length distribution for Newtonian fluids flow in circular smooth pipes proposed in this paper is suitable to analyze the velocity profile and the relation between friction factor and Reynolds number.

5.2 Pseudo-plastic fluids flow

Fig.5 shows the experimental results of bentonite slurry ($\gamma_m = 1.02 \times 10^3 \text{ kg/m}^3$) about the relation between $D\Delta P/4L$ and $8V_m/D$. Test pipes are stainless steel pipes, one of them has 22.1 mm inner diameter, 5.94 m entrance length and 1.50 m test section length for measuring the pressure loss and the other has respectively the values of 27.1 mm, 5.49 m and 2.00 m. The test pipes were confirmed to be hydraulically smooth by the measurement of pressure loss of clear

Fig.4 The comparison of theoretical results concerning the relation between f and NRe .

Fig.5 The experimental results of pressure loss of bentonite slurry.

water. Flow behavior index n and consistency index K of this slurry are obtained from the data of laminar flow region as follows;

$$n = 0.867, \quad K = 8.11 \times 10^{-4} \text{ kg s}^n/\text{m}^2$$

Fig.6 shows the relation between the flow behavior index n and the values of damping effect factor A , i.e. $26B(n)$ in Eq.(13), which are obtained to fit the experimental results quoted from literature^{(3),(11),(16)} and this experimental results. Moreover, a solid line in the same figure shows the values of A which are computed from $26B(n)$ for the case that Eq.(12) given by Hanks applies to $26B(n)$. As the solid line agrees with the experimental results, the values of damping effect factor A can be computed from $26/n$. In Fig.7 are shown the experimental results by Clapp⁽²⁾ and the analyzed results by Hanks⁽⁴⁾ and the authors about the velocity profile. As shown in Fig.7, the authors' results agree surely with experimental results as well as Newtonian fluids. On the other hand, Hanks' results agree with them in the near pipe wall region as well as the authors'. However, his results have not a good agreement with experimental results in the region of central part of pipe, because his values are a little smaller than them.

The results analyzed by using Eq.(13) and the experimental results by Dodge⁽¹¹⁾, Hanks⁽¹¹⁾, Vocadlo⁽¹⁶⁾ and the authors concerning the relation between friction factor and Reynolds number are shown in Fig.8 and Fig.9. As shown in these figures, the analyzed results agree well with the experimental results in the laminar and turbulent region and agree comparatively well with them in the transitional region. Therefore,

Fig.6 The relation between values of damping effect factor A and flow behavior index n .

it seems that Eq.(15) is suitable for the correction of damping effect factor in the turbulent flow near the critical Reynolds number in which Kim pointed out that the correction was necessary. In order to compare the other researchers' studies^{(11),(2),(4)} with this study, Fig.10 shows the relation between friction factor and Reynolds number for the case $n = 0.6$. It is recognized

Fig.7 The comparison between analyzed results and experimental results concerning the velocity profile of pseudo-plastic fluids.

Fig.8 The comparison between experimental results and analyzed results concerning the relation between f_p and NR_p .

Fig.9 The comparison between experimental results and analyzed results concerning the relation between f_p and NR_p .

Fig.10 The comparison of theoretical results concerning the relation between f_p and N_{Re} for the case $n = 0.6$.

that analyzed results in the high Reynolds number region give almost the same as Clapp's results which are obtained from the velocity profile in the turbulent flow core region and the results in the turbulent flow region near the critical Reynolds number have some differences from Clapp's results. Fig.11 shows a set of design curves about pseudo-plastic fluids flow in circular smooth pipes which is computed from Eq.(13). The design curves for the case $n \leq 0.5$ are shown by dashed lines, because they are not verified by experimental results. It is thought that the mixing-length model proposed by the authors is suitable for analyzing the pseudo-plastic fluids flow in circular smooth pipes through the above comparisons.

6. Conclusion

The following results were obtained through this study in which the velocity profile and the relation between friction factor and Reynolds number for Newtonian and pseudo-plastic fluids flow in circular smooth pipes were analyzed by using a mixing-length model proposed by the authors.

(1) The modified mixing-length distribution shown in Eq.(13) is suitable for analyzing the velocity profile and the relation between friction factor and Reynolds number about Newtonian and pseudo-plastic fluids flow in circular smooth pipes. The analyzed results agree well with experimental results in the laminar and turbulent flow regions and also agree better with them than the other researchers' in the turbulent flow region near the critical Reynolds number.

(2) It is certified that Eq.(12) proposed by Hanks is suitable to $B(n)$ which is a function for taking account of the effect of n on the damping effect factor.

(3) Eq.(15) is suitable to correct the damping effect factor in the turbulent flow region near the critical Reynolds number.

Thus, it is evident through this study

Fig.11 A set of design curves of f_p versus N_{Re} concerning the pseudo-plastic fluids flow in circular smooth pipes.

that the modified mixing-length distribution proposed by the authors makes it possible to analyze the velocity profile and the relation between friction factor and Reynolds number.

As this study has not been verified by experimental results in the region of small values of flow behavior index, further examination would be necessary.

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