Systematic Analyses of Velocity, Acceleration and Force of Planar Multilink Mechanisms*

By Hiroaki FUNABASHI**, Kiyoshi OGAWA***
and Toshiyuki HARA****

The velocities and accelerations of planar multilink mechanisms have been systematically analysed by means of several transformation functions which describe the relative motions of links in the basic open chains and the conditions of combinations of the open chains.

Forces caused by loads, inertial forces and frictions have been also systematically analysed by means of two functions which describe the equivalent inertial forces of links and the internal forces of pairs.

1. Introduction

As there exist numerous kinds of multi-link mechanisms^{(1)~(3)} and their analyses by means of their individual displacement equations are quite difficult, systematic and simple methods of analysis applicable to all kinds of mechanisms have been strongly demanded. L. Markus⁽⁴⁾, V. Brát⁽⁵⁾ and their colleagues have tried to develop systematic methods of displacement analyses. However, their methods need initial conditions which must be determined by constructions or some other complicated methods, and they are ineffective on the conjugate mechanisms made by geometric inversions of links because they are approximate methods.

Thus, the authors have previously discussed systematic analyses⁽⁶⁾ of displacements and errors of planar multilink mechanisms without displacement equations or initial conditions. In the present paper, the authors will develop these analyses further to enable systematic analyses of the velocities, accelerations and forces of planar multilink mechanisms.

2. Transformation functions of velocities and accelerations

2.1 Transformation functions of basic open chains

Three open chains⁽⁶⁾ shown in Fig.1 shall be adopted as the basic open chains of planar multilink mechanisms with revolute pairs. In the figure, the marks O indicate pairing points or coupler points; l_i , θ_i and β_i express the lengths of links and the displacement angles of the i -th open chain in a mechanism. β_i and β_{i+1} are constant angles describing the positions of pairing or coupler points. O_i , x_i , y_i are orthogonal coordinate systems whose origins

coincide with the pairing points O_i on the links l_{i+1} .

Transformation function AI In Fig.1 (a), the next relation holds between the angle θ_i and the coordinates (x_i, y_i) .

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} l_{i1} + l_{i2} \cos \theta_i \\ l_{i2} \sin \theta_i \end{pmatrix} \dots\dots\dots (1)$$

Differentiating the above equation with time, we get the following equation.

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \ddot{x}_i \\ \ddot{y}_i \end{pmatrix} = \begin{pmatrix} l_{i2} \sin \theta_i \\ l_{i2} \cos \theta_i \\ -l_{i2} \sin \theta_i \dot{\theta}_i \\ l_{i2} \cos \theta_i \dot{\theta}_i \\ -l_{i2} (\cos \theta_i \dot{\theta}_i^2 + \sin \theta_i \ddot{\theta}_i) \\ -l_{i2} (\sin \theta_i \dot{\theta}_i^2 - \cos \theta_i \ddot{\theta}_i) \end{pmatrix} \dots\dots\dots (2)$$

Because the right side of Eq.(2) is a function which transforms the angular displacement, velocity and acceleration, $(\theta_i, \dot{\theta}_i, \ddot{\theta}_i)$, of the link l_{i2} relative to the link l_{i1} into the relative displacement, velocity and acceleration, $(x_i, y_i, \dot{x}_i, \dot{y}_i, \ddot{x}_i, \ddot{y}_i)$, of the pair on the link l_{i2} , we call it the transformation function AI.

Transformation functions AII, AIII, AIV, AV The following equations may be similarly derived by differentiating the

Fig.1 Basic open chains

* Received 8th December, 1976.

** Associate Professor, Tokyo Institute of Technology, O-okayama, Meguro-ku, Tokyo.

*** Professor, Tokyo Institute of Technology.

**** Mechanical Engineer, ISUZU Motors, Ltd.

equations which describe the relations among the coordinates of pairs (x_{11}, y_{11}) , (x_{12}, y_{12}) , (x_{13}, y_{13}) and the angular displacement, θ_{11} , θ_{12} , of the chain in Fig.1(b).

$$\begin{pmatrix} \cos \theta_{12} \\ \dot{\theta}_{12} \\ \ddot{\theta}_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2l_{11}l_{12}}(x_{12}^2 + y_{12}^2 - l_{11}^2 - l_{12}^2) \\ -\frac{\operatorname{cosec} \theta_{12}}{l_{11}l_{12}}(x_1 \dot{x}_{12} + y_{12} \dot{y}_{12}) \\ \frac{\cos \theta_{12} \operatorname{cosec}^2 \theta_{12}}{l_{11}l_{12}}(x_{12} \dot{x}_{12} + y_{12} \dot{y}_{12}) \dot{\theta}_{12} - \frac{\operatorname{cosec} \theta_{12}}{l_{11}l_{12}}(\dot{x}_{12}^2 + \dot{y}_{12}^2 + x_{12} \ddot{x}_{12} + y_{12} \ddot{y}_{12}) \end{pmatrix} \quad (3)$$

$$\begin{pmatrix} \cos \theta_{11} \\ \sin \theta_{11} \\ \dot{\theta}_{11} \\ \ddot{\theta}_{11} \end{pmatrix} = \begin{pmatrix} \frac{1}{x_{12}^2 + y_{12}^2}(-(l_{11} + l_{12} \cos \theta_{12})x_{12} + l_{12}y_{12} \sin \theta_{12}) \\ \frac{1}{x_{12}^2 + y_{12}^2}((l_{11} + l_{12} \cos \theta_{12})y_{12} + l_{12}x_{12} \sin \theta_{12}) \\ -\frac{\operatorname{cosec} \theta_{11}}{(x_{12}^2 + y_{12}^2)^2}[(x_{12}^2 - y_{12}^2)(l_{11} + l_{12} \cos \theta_{12}) - 2l_{12}x_{12}y_{12} \sin \theta_{12}] \dot{x}_{12} \\ + [2x_{12}y_{12}(l_{11} + l_{12} \cos \theta_{12}) + l_{12}(x_{12}^2 - y_{12}^2) \sin \theta_{12}] \dot{y}_{12} \\ + l_{12}(x_{12}^2 + y_{12}^2)(x_{12} \sin \theta_{12} + y_{12} \cos \theta_{12}) \dot{\theta}_{12} \\ -\frac{\operatorname{cosec} \theta_{11}}{(x_{12}^2 + y_{12}^2)^2} [2(x_{12}(l_{11} + l_{12} \cos \theta_{12}) - l_{12}y_{12} \sin \theta_{12})(\dot{x}_{12}^2 + \dot{y}_{12}^2) \\ + 2l_{12}(x_{12}^2 + y_{12}^2)(\dot{x}_{12} \sin \theta_{12} + \dot{y}_{12} \cos \theta_{12}) \dot{\theta}_{12} + 2l_{12}(x_{12}^2 \\ + y_{12}^2)(x_{12} \cos \theta_{12} - y_{12} \sin \theta_{12}) \dot{\theta}_{12}^2 + (x_{12} \sin \theta_{12} + y_{12} \cos \theta_{12}) \ddot{\theta}_{12} \\ + ((x_{12}^2 - y_{12}^2)(l_{11} + l_{12} \cos \theta_{12}) - 2l_{12}x_{12}y_{12} \sin \theta_{12}) \dot{x}_{12} \\ + [2x_{12}y_{12}(l_{11} + l_{12} \cos \theta_{12}) + l_{12}(x_{12}^2 + y_{12}^2) \sin \theta_{12}] \dot{y}_{12} \\ - \frac{4}{x_{12}^2 + y_{12}^2}(x_{12} \dot{x}_{12} + y_{12} \dot{y}_{12}) \dot{\theta}_{12} - \cot \theta_{11} \dot{\theta}_{11}^2 \end{pmatrix} \quad (4)$$

$$\begin{pmatrix} x_{11} \\ y_{11} \\ \dot{x}_{11} \\ \dot{y}_{11} \\ \ddot{x}_{11} \\ \ddot{y}_{11} \end{pmatrix} = \begin{pmatrix} -l_{12} \cos(\theta_{11} - \beta_{11}) \\ l_{12} \sin(\theta_{11} - \beta_{11}) \\ l_{12} \sin(\theta_{11} - \beta_{11}) \dot{\theta}_{11} \\ l_{12} \cos(\theta_{11} - \beta_{11}) \dot{\theta}_{11} \\ l_{12} \cos(\theta_{11} - \beta_{11}) \dot{\theta}_{11}^2 + l_{12} \sin(\theta_{11} - \beta_{11}) \ddot{\theta}_{11} \\ -l_{12} \sin(\theta_{11} - \beta_{11}) \dot{\theta}_{11}^2 + l_{12} \cos(\theta_{11} - \beta_{11}) \ddot{\theta}_{11} \end{pmatrix} \quad (5)$$

$$\begin{pmatrix} x_{13} \\ y_{13} \\ \dot{x}_{13} \\ \dot{y}_{13} \\ \ddot{x}_{13} \\ \ddot{y}_{13} \end{pmatrix} = \begin{pmatrix} -l_{11} \cos \theta_{11} - l_{14} \cos(\theta_{11} + \theta_{12} - \beta_{12}) \\ l_{11} \sin \theta_{11} + l_{14} \sin(\theta_{11} + \theta_{12} - \beta_{12}) \\ (l_{11} \sin \theta_{11} + l_{14} \sin(\theta_{11} + \theta_{12} - \beta_{12})) \dot{\theta}_{11} + l_{14} \sin(\theta_{11} + \theta_{12} - \beta_{12}) \dot{\theta}_{12} \\ (l_{11} \cos \theta_{11} + l_{14} \cos(\theta_{11} + \theta_{12} - \beta_{12})) \dot{\theta}_{11} + l_{14} \cos(\theta_{11} + \theta_{12} - \beta_{12}) \dot{\theta}_{12} \\ l_{11} \cos \theta_{11} \dot{\theta}_{11}^2 + l_{14} \cos(\theta_{11} + \theta_{12} - \beta_{12})(\dot{\theta}_{11} + \dot{\theta}_{12})^2 + (l_{11} \sin \theta_{11} \\ + l_{14} \sin(\theta_{11} + \theta_{12} - \beta_{12})) \dot{\theta}_{11} \dot{\theta}_{12} + l_{14} \sin(\theta_{11} + \theta_{12} - \beta_{12}) \dot{\theta}_{12}^2 \\ -l_{11} \sin \theta_{11} \dot{\theta}_{11}^2 - l_{14} \sin(\theta_{11} + \theta_{12} - \beta_{12})(\dot{\theta}_{11} + \dot{\theta}_{12})^2 + (l_{11} \cos \theta_{11} \\ + l_{14} \cos(\theta_{11} + \theta_{12} - \beta_{12})) \dot{\theta}_{11} \dot{\theta}_{12} + l_{14} \cos(\theta_{11} + \theta_{12} - \beta_{12}) \dot{\theta}_{12}^2 \end{pmatrix} \quad (6)$$

$\sin \theta_{12}$ in Eqs.(3)~(6) is given by

$$\sin \theta_{12} = \pm \sqrt{1 - \cos^2 \theta_{12}}$$

and $\cos \theta_{12}$ is given by the first equation of Eq.(3).

Equations (3) and (4) transform $(x_{12}, y_{12}, \dot{x}_{12}, \dot{y}_{12}, \ddot{x}_{12}, \ddot{y}_{12})$ into $(\cos \theta_{12}, \dot{\theta}_{12}, \ddot{\theta}_{12})$ and $(\cos \theta_{11}, \dot{\theta}_{11}, \ddot{\theta}_{11})$ respectively, while Eqs.(5) and (6) transform $(\theta_{11}, \dot{\theta}_{11}, \ddot{\theta}_{11})$ and $(\theta_{12}, \dot{\theta}_{12}, \ddot{\theta}_{12})$ into $(x_{11}, y_{11}, \dot{x}_{11}, \dot{y}_{11}, \ddot{x}_{11}, \ddot{y}_{11})$ and $(x_{13}, y_{13}, \dot{x}_{13}, \dot{y}_{13}, \ddot{x}_{13}, \ddot{y}_{13})$ respectively. Thus, we call the right sides of Eqs.(3), (4), (5) and (6) the transformation functions AII, AIII, AIV and AV respectively.

Transformation function AM The following relation may hold in the five-link chain shown in Fig.1(c).

$$(x_4 - x_{11})^2 + (y_4 - y_{11})^2 = l_{16}^2$$

Describing (x_{11}, y_{11}) as a function of θ_1 by means of the function AV and substituting them into the above equation we obtain

$$\cos^6 \theta_1 + A_7 \cos^5 \theta_1 + A_6 \cos^4 \theta_1 + A_5 \cos^3 \theta_1 + A_4 \cos^2 \theta_1 + A_3 \cos \theta_1 + A_2 \cos^2 \theta_1 + A_1 \cos \theta_1 + A_0 = 0 \quad (7)$$

where $A_j (j=0,1,2,\dots,7)$ are functions of x_4 and y_4 .

Solving Eq.(7), θ_1 may be expressed as a function of (x_4, y_4) . Thus, the function AM which transforms $(x_4, y_4, \dot{x}_4, \dot{y}_4, \ddot{x}_4, \ddot{y}_4)$ into $(\theta_1, \dot{\theta}_1, \ddot{\theta}_1)$ may be obtained as follows by differentiating Eq.(7).

$$\begin{pmatrix} \cos \theta_i \\ \dot{\theta}_i \\ \ddot{\theta}_i \end{pmatrix} = \begin{pmatrix} \cos \theta_i \\ \frac{\sum_{n=0}^8 A_n \cos^n \theta_i}{\sum_{n=1}^8 n A_n \cos^{n-1} \theta_i \sin \theta_i} \\ -\cot \theta_i \dot{\theta}_i^2 + \frac{\sum_{n=0}^8 \ddot{A}_n \cos^n \theta_i}{\sum_{n=1}^8 n A_n \cos^{n-1} \theta_i \sin \theta_i} + \frac{\dot{\theta}_i^2 \sin \theta_i \sum_{n=2}^8 n(n-1) A_n \cos^{n-2} \theta_i - 2\dot{\theta}_i \sum_{n=1}^8 n A_n \cos^{n-1} \theta_i}{\sum_{n=1}^8 n A_n \cos^{n-1} \theta_i} \end{pmatrix} \quad (8)$$

where $A_0=1$ and $\sin \theta_i = \pm \sqrt{1 - \cos^2 \theta_i}$. $\cos \theta_i$ is given by Eq. (7).

2.2 Transformation functions of coordinate systems

The transformations of coordinate systems are considered here in order to describe the relative positions among open chains. In Fig. 2, $O_i-x_iy_i$ and $O_j-x_jy_j$ are the coordinate systems fixed on the i -th and j -th open chains in a mechanism respectively. Both of points O_i and Q on the x_j -axis are assumed to be beforehand described with the system $O_i-x_iy_i$.

The transformation of the coordinates (x_{Pi}, y_{Pi}) of an arbitrary point P described with the system $O_i-x_iy_i$ into its coordinates (x_{Pj}, y_{Pj}) described with the system $O_j-x_jy_j$ and that of (x_{Pj}, y_{Pj}) into (x_{Pi}, y_{Pi}) may be carried out by the following equations.

$$\begin{pmatrix} x_{Pj} \\ y_{Pj} \end{pmatrix} = \begin{pmatrix} c x_{Pi} + s y_{Pi} \\ -s x_{Pi} + c y_{Pi} \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} x_{Pi} \\ y_{Pi} \end{pmatrix} = \begin{pmatrix} x_{0i} + c x_{Pj} - s y_{Pj} \\ y_{0i} + s x_{Pj} + c y_{Pj} \end{pmatrix} \quad (10)$$

where

$$\left. \begin{aligned} c &= \cos \theta = x_{Q0}/K, \quad s = \sin \theta = y_{Q0}/K \\ K &= \sqrt{x_{Q0}^2 + y_{Q0}^2} \\ x_{P0} &= x_{Pi} - x_{0i}, \quad y_{P0} = y_{Pi} - y_{0i} \\ x_{Q0} &= x_{Qi} - x_{0i}, \quad y_{Q0} = y_{Qi} - y_{0i} \end{aligned} \right\} \quad (11)$$

The differentiation of Eqs. (9) and (10) with respect to time leads to the following equations.

$$\begin{pmatrix} \dot{x}_{Pj} \\ \dot{y}_{Pj} \\ \ddot{x}_{Pj} \\ \ddot{y}_{Pj} \end{pmatrix} = \begin{pmatrix} c \dot{x}_{P0} + s \dot{y}_{P0} \\ -s \dot{x}_{P0} + c \dot{y}_{P0} \\ c \ddot{x}_{P0} + s \ddot{y}_{P0} + \dot{c} \dot{x}_{P0} + \dot{s} \dot{y}_{P0} \\ c \ddot{x}_{P0} + s \ddot{y}_{P0} + 2(\dot{c} \dot{x}_{P0} + \dot{s} \dot{y}_{P0}) + \ddot{c} x_{P0} + \ddot{s} y_{P0} \\ -s \ddot{x}_{P0} + c \ddot{y}_{P0} \\ -2(\dot{s} \dot{x}_{P0} - \dot{c} \dot{y}_{P0}) - \ddot{s} x_{P0} + \ddot{c} y_{P0} \end{pmatrix} \quad (12)$$

$$\begin{pmatrix} x_{Pi} \\ y_{Pi} \\ \dot{x}_{Pi} \\ \dot{y}_{Pi} \\ \ddot{x}_{Pi} \\ \ddot{y}_{Pi} \end{pmatrix} = \begin{pmatrix} x_{0i} + c x_{Pj} - s y_{Pj} \\ y_{0i} + s x_{Pj} + c y_{Pj} \\ \dot{x}_{0i} + c \dot{x}_{Pj} - s \dot{y}_{Pj} + \dot{c} x_{Pj} - \dot{s} y_{Pj} \\ \dot{y}_{0i} + s \dot{x}_{Pj} + c \dot{y}_{Pj} + \dot{s} x_{Pj} + \dot{c} y_{Pj} \\ \ddot{x}_{0i} + c \ddot{x}_{Pj} - s \ddot{y}_{Pj} + 2(\dot{c} \dot{x}_{Pj} - \dot{s} \dot{y}_{Pj}) + \ddot{c} x_{Pj} - \ddot{s} y_{Pj} \\ \ddot{y}_{0i} + s \ddot{x}_{Pj} + c \ddot{y}_{Pj} + 2(\dot{s} \dot{x}_{Pj} + \dot{c} \dot{y}_{Pj}) + \ddot{s} x_{Pj} + \ddot{c} y_{Pj} \end{pmatrix} \quad (13)$$

where

$$\begin{aligned} \dot{c} &= (y_{Q0} \dot{x}_{Q0} - x_{Q0} \dot{y}_{Q0})/K^3 \\ \dot{s} &= (-x_{Q0} \dot{y}_{Q0} \dot{x}_{Q0} + x_{Q0}^2 \dot{y}_{Q0})/K^3 \end{aligned}$$

$$\left. \begin{aligned} \ddot{c} &= 3(x_{Q0} \dot{y}_{Q0} (\dot{x}_{Q0}^2 - \dot{y}_{Q0}^2) - y_{Q0} (x_{Q0}^2 \\ &\quad - y_{Q0}^2) \dot{x}_{Q0} \dot{y}_{Q0})/K^3 + (y_{Q0} \dot{x}_{Q0} \dot{y}_{Q0} - x_{Q0} \dot{y}_{Q0}^2 \\ &\quad + y_{Q0}^2 \dot{x}_{Q0} - x_{Q0} \dot{y}_{Q0} \dot{y}_{Q0})/K^3 \\ \ddot{s} &= 3(x_{Q0}^2 \dot{y}_{Q0} (\dot{x}_{Q0}^2 - \dot{y}_{Q0}^2) - x_{Q0} (x_{Q0}^2 \\ &\quad - y_{Q0}^2) \dot{x}_{Q0} \dot{y}_{Q0})/K^3 + (x_{Q0} \dot{x}_{Q0} \dot{y}_{Q0} - y_{Q0} \dot{x}_{Q0}^2 \\ &\quad - x_{Q0} \dot{y}_{Q0} \dot{x}_{Q0} + x_{Q0}^2 \dot{y}_{Q0})/K^3 \end{aligned} \right\} \quad (14)$$

Equation (12) transforms the displacement, velocity and acceleration of an arbitrary point P described with the system $O_i-x_iy_i$ into those described with the system $O_j-x_jy_j$, and Eq. (13) transforms them inversely. Thus, the right sides of Eqs. (12) and (13) shall be called the transformation functions AMIA and AMIB respectively.

3. Velocity and acceleration analyses

As mentioned in the previous paper⁽⁶⁾, the computer programs for the velocity and acceleration analyses may be constructed only by combining the subprograms which describe the transformation functions derived just above. The combination of the subprograms may be easily carried out by making use of the block diagrams whose elements are transformation functions.

Citing the four-link mechanism as the simplest example, we shall derive its block diagram. In Fig. 3(a), **b** and **d** are respectively the driver and follower and **P** is a coupler point. The mechanism can be divided into two dyads consisting of links **a-b** and **c-d**, which are the same as the dyads shown in Fig. 1(a) and (b). Therefore, the displacement, velocity and acceleration of the driver, $(\theta, \dot{\theta}, \ddot{\theta})$, are transformed by AI into those of the pair **B**, $(x_1, y_1, \dot{x}_1, \dot{y}_1, \ddot{x}_1, \ddot{y}_1)$, which are further transformed into $(\theta_{21}, \dot{\theta}_{21}, \ddot{\theta}_{21})$ and $(\theta_{21}, \dot{\theta}_{21}, \ddot{\theta}_{21})$ respectively by AII and AIII. $(\theta_{21}, \dot{\theta}_{21}, \ddot{\theta}_{21})$ and $(\theta_{22}, \dot{\theta}_{22}, \ddot{\theta}_{22})$ are furthermore transformed into $(x_{21}, y_{21}, \dot{x}_{21}, \dot{y}_{21}, \ddot{x}_{21}, \ddot{y}_{21})$ by AV. Thus, the block diagram becomes

Fig. 2 Transformation of coordinates

as shown in Fig.3(b).

Combining the subprograms of the transformation functions according to the diagram and carrying out numerical calculations, we can know the characteristics of motions of the coupler and follower. Figure 4 shows an example of the displacement, velocity and acceleration curves of the follower.

As an example of multilink mechanisms, the eight-link mechanism shown in Fig.5 shall be cited. The mechanism may be divided into five open chains. Fixing coordinate systems

$O_i-x_iy_i$ ($i=1,2,\dots,5$) on each open chain as shown in Fig.5(a), we obtain the block diagram shown in Fig.5(b).

For a set of kinematic parameters shown in Fig.6, the numerical calculation gives eight coupler curves⁽⁶⁾, one of which is shown in Fig.6. The displacement, velocity and acceleration curves become as shown in Fig.7 when the driver rotates at a constant speed of 50 rpm.

4. Force analysis

(a)

(b)

Fig.3 Description of planar four-link mechanism by transformation functions

Parameter: $a=50$, $b=20$, $c_1=50$, $d=50$ mm

Fig.4 Displacement, velocity and acceleration curves of follower (crank speed 50 rpm)

4.1 Inertial forces of links

In Fig.8, $O-XY$ is a fixed coordinate system in which X - and Y -axes are respectively horizontal and vertical, and a link l is moving on the plane $X-Y$. When the coordinates (X_G, Y_G) , (X_L, Y_L) , (X_M, Y_M) and accelerations $\ddot{G}=\ddot{X}_G+i\ddot{Y}_G$, $\ddot{L}=\ddot{X}_L+i\ddot{Y}_L$, $\ddot{M}=\ddot{X}_M+i\ddot{Y}_M$ ($i^2=-1$) of the center of gravity G and two arbitrary points L , M on the link are given, the equivalent inertial force F of the link and its point of action $E(X_E, Y_E)$ may be obtained as follows.

Let $Q(X_Q, Y_Q)$ and k be the pole of acceleration and the radius of gyration of the link l respectively, then the point E lies on the straight line QG and the distance QE is given as $QE=k^2/QG$. Thus, the next equation may be derived.

$$\left. \begin{aligned} X_E &= X_G + \frac{k^2(X_G - X_Q)}{\sqrt{(X_G - X_Q)^2 + (Y_G - Y_Q)^2}} \\ Y_E &= Y_G + \frac{k^2(Y_G - Y_Q)}{\sqrt{(X_G - X_Q)^2 + (Y_G - Y_Q)^2}} \end{aligned} \right\} \dots (15)$$

In Fig.8, points L and M are the heads of vectors $\ddot{L}$ and $\ddot{M}$ drawn from points L

(a)

(b)

Fig.5 Description of planar eight-link mechanism by transformation functions

and M respectively, and $R(X_R, Y_R)$ is the point of intersection of lines $L\bar{L}$ and $M\bar{M}$. C_1 is a circle on which points L , M and R lie and C_2 is a circle on which points $\bar{L}$, $\bar{M}$ and R lie. As the pole Q is the point of intersection of the circles C_1 and C_2 , its coordinates may become as follows.

$$\left. \begin{aligned} X_Q &= \frac{\left\{ \begin{aligned} &X_R[(X_{C1}-X_{C1})^2 - (Y_{C1}-Y_{C1})^2] \\ &+ 2(Y_{C1}-Y_{C1})(X_{C1}(Y_{C1}-Y_{C1}) \\ &+ Y_R(X_{C1}-X_{C1})) \end{aligned} \right\}}{(X_{C1}-X_{C1})^2 + (Y_{C1}-Y_{C1})^2} \\ Y_Q &= \frac{\left\{ \begin{aligned} &Y_R[(X_{C1}-X_{C1})^2 - (Y_{C1}-Y_{C1})^2] \\ &+ 2(X_{C1}-X_{C1})(Y_{C1}(X_{C1}-X_{C1}) \\ &- X_R(Y_{C1}-Y_{C1})) \end{aligned} \right\}}{(X_{C1}-X_{C1})^2 + (Y_{C1}-Y_{C1})^2} \end{aligned} \right\} \quad (16)$$

$$\left. \begin{aligned} X_R &= \frac{\bar{Y}_L \bar{X}_M X_L - \bar{X}_L \bar{Y}_M X_M - \bar{X}_L \bar{X}_M (Y_L - Y_M)}{\bar{Y}_L \bar{X}_M - \bar{X}_L \bar{Y}_M} \\ Y_R &= \frac{\bar{X}_L \bar{Y}_M Y_L - \bar{Y}_L \bar{X}_M Y_M - \bar{Y}_L \bar{Y}_M (X_L - X_M)}{\bar{X}_L \bar{Y}_M - \bar{Y}_L \bar{X}_M} \end{aligned} \right\} \quad (17)$$

Parameter: $a_1=100, a_2=70, a_3=40, b_1=20, b_2=20, b_3=30, c_1=70, c_2=50, c_3=30, d=60, e=50, f_1=60, f_2=80, f_3=30, g_1=80, g_2=30, h=80$ (mm), $\beta=30^\circ$

Fig.6 Coupler curve of eight-link mechanism

Fig.7 Displacement, velocity and acceleration curves of coupler point of eight-link mechanism (crank speed: 50 rpm)

(X_{C1}, Y_{C1}) and (X_{C2}, Y_{C2}) in Eq.(16) are the coordinates of the centers of circles C_1 and C_2 respectively and they are given as follows.

$$\left. \begin{aligned} X_{C1} &= \frac{\left\{ \begin{aligned} &\bar{X}_L \bar{Y}_M (X_L + X_R) - \bar{Y}_L \bar{X}_M (X_M + X_R) \\ &+ \bar{Y}_L \bar{Y}_M (Y_L - Y_M) \end{aligned} \right\}}{2(\bar{X}_L \bar{Y}_M - \bar{Y}_L \bar{X}_M)} \\ Y_{C1} &= \frac{\left\{ \begin{aligned} &\bar{Y}_L \bar{X}_M (Y_L + Y_R) - \bar{X}_L \bar{Y}_M (Y_M + Y_R) \\ &+ \bar{X}_L \bar{X}_M (X_L - X_M) \end{aligned} \right\}}{2(\bar{Y}_L \bar{X}_M - \bar{X}_L \bar{Y}_M)} \end{aligned} \right\} \quad (18)$$

$$\left. \begin{aligned} X_{C2} &= \frac{\left\{ \begin{aligned} &\bar{X}_L \bar{Y}_M (X_L + X_R + \rho \bar{X}_L) \\ &- \bar{Y}_L \bar{X}_M (X_M + X_R + \rho \bar{X}_M) \\ &+ \bar{Y}_L \bar{Y}_M (Y_L - Y_M + \rho \bar{Y}_L - \rho \bar{Y}_M) \end{aligned} \right\}}{2(\bar{X}_L \bar{Y}_M - \bar{Y}_L \bar{X}_M)} \\ Y_{C2} &= \frac{\left\{ \begin{aligned} &\bar{Y}_L \bar{X}_M (Y_L + Y_R + \rho \bar{Y}_L) \\ &- \bar{X}_L \bar{Y}_M (Y_M + Y_R + \rho \bar{Y}_M) \\ &+ \bar{X}_L \bar{X}_M (X_L - X_M + \rho \bar{X}_L - \rho \bar{X}_M) \end{aligned} \right\}}{2(\bar{Y}_L \bar{X}_M - \bar{X}_L \bar{Y}_M)} \end{aligned} \right\} \quad (19)$$

where ρ is a scale factor to figure the acceleration vectors.

When L or M is a point where the link l is paired to the fixed link, the pole Q coincides with L or M because $\ddot{L} \equiv 0$ or $\ddot{M} \equiv 0$. Therefore, the poles in these cases are given by the next equations instead of Eq.(16).

$$\left. \begin{aligned} \text{When } \ddot{L} \equiv 0, X_Q &= X_L \text{ and } Y_Q = Y_L \\ \text{When } \ddot{M} \equiv 0, X_Q &= X_M \text{ and } Y_Q = Y_M \end{aligned} \right\} \quad (20)$$

Let W and g be the weight of the link and the gravitational acceleration respectively, then the equivalent inertial force F_i may become as follows.

$$F_i = -(W/g)(\ddot{X}_G + i\ddot{Y}_G) \quad (21)$$

Taking the gravitational force F_G into account, the total force F_F acting on the link and its point of action $P(X_P, Y_P)$ may become as follows.

$$F_F = F_i + F_G = -(W/g)(\ddot{X}_G + i(\ddot{Y}_G + g)) \quad (22)$$

$$\left. \begin{aligned} X_F &= X_G \\ Y_F &= (\ddot{Y}_G / \ddot{X}_G)(X_G - X_E) + Y_E \end{aligned} \right\} \quad (23)$$

Fig.8 Equivalent inertial force of moving link

Therefore, the total force acting on the link and its point of action may be formally expressed by the following function whose independent variables are the coordinates and accelerations of the center of gravity and two arbitrary points on the link.

$$\begin{aligned} & (F_P, X_P, Y_P) \\ & = B(X_G, Y_G, X_L, Y_L, X_M, Y_M, \ddot{G}, \ddot{L}, \ddot{M}) \\ & \dots\dots\dots (24) \end{aligned}$$

$$\begin{aligned} \cos \theta_M &= \frac{(X_N - X_M) \sqrt{(X_N - X_M)^2 + (Y_N - Y_M)^2} - (\delta_{NeN} - \delta_{MeM})^2 - (Y_N - Y_M)(\delta_{NeN} - \delta_{MeM})}{(X_N - X_M)^2 + (Y_N - Y_M)^2} \\ \sin \theta_M &= \frac{(Y_N - Y_M) \sqrt{(X_N - X_M)^2 + (Y_N - Y_M)^2} - (\delta_{NeN} - \delta_{MeM})^2 + (X_N - X_M)(\delta_{NeN} - \delta_{MeM})}{(X_N - X_M)^2 + (Y_N - Y_M)^2} \end{aligned} \quad \dots\dots\dots (25)$$

The line of action of F_L passes through the point of intersection, $S(X_S, Y_S)$, of the lines of action of F_P and F_M , and it touches the circle of friction at L . Therefore, the argument θ_L of F_L may become as follows.

$$\begin{aligned} \cos \theta_L &= \frac{(X_S - X_L) \sqrt{(X_S - X_L)^2 + (Y_S - Y_L)^2} - e_L^2 + \delta_{LeL}(Y_S - Y_L)}{(X_S - X_L)^2 + (Y_S - Y_L)^2} \\ \sin \theta_L &= \frac{(Y_S - Y_L) \sqrt{(X_S - X_L)^2 + (Y_S - Y_L)^2} - e_L^2 - \delta_{LeL}(X_S - X_L)}{(X_S - X_L)^2 + (Y_S - Y_L)^2} \end{aligned} \quad \dots\dots\dots (26)$$

$$\begin{aligned} X_S &= \frac{X_P \tan \theta_P - X_M \tan \theta_M - Y_P - \delta_{PeP} \sqrt{1 + \tan^2 \theta_P} + Y_M + \delta_{MeM} \sqrt{1 + \tan^2 \theta_M}}{\tan \theta_P - \tan \theta_M} \\ Y_S &= \frac{(Y_P + \delta_{PeP} \sqrt{1 + \tan^2 \theta_P}) \cot \theta_P - (Y_M + \delta_{MeM} \sqrt{1 + \tan^2 \theta_M}) \cot \theta_M - X_P + X_M}{\cot \theta_P - \cot \theta_M} \end{aligned} \quad \dots\dots\dots (27)$$

In the above equations, θ_P is the argument of F_P , and δ_L , δ_M , etc. are parameters describing how the lines of action of forces touch the circles of friction.

In a three-link chain, l_{mn} , paired at points M and N , there may exist four lines

4.2 Internal forces at pairs

In Fig.9, L , M , N and P are pairing points, and links l and m are connected mutually at the point M . e_L , e_M , etc. are the radii of the circles of friction at points L , M , etc. F_L and F_M are internal forces acting at pairs L and M respectively which are caused by an external force F_P at the point P .

The argument θ_M of F_M may become as follows because the line of action of F_M is given as a line which touches the circles of friction at M and N .

which touch the circles of friction at L and M , as shown in Fig.10. The determination of the real line of action of the force depends on whether the magnitudes of angles β_M and β_N increase or decrease with respect to time. Thus, δ_M and δ_N will be defined

Fig.9 Internal forces at pairs

Fig.10 Lines of action of forces

Table 1 Parameters of a six-link mechanism

Link	Length mm	Center of gravity mm	Weight g	Moment of inertia g mm s ²
a_1	128.15	—	—	—
a_2	123.37	—	—	—
a_3	48.58	—	—	—
b	38.37	$\overline{AG}_b = 18.45$	172.0	7.64
c_1	96.93	$\overline{BG}_c = 55.00$	165.3	24.95
c_2	80.20	$\overline{CG}_c = 45.50$	—	—
c_3	38.17	—	—	—
d	89.50	$\overline{DG}_d = 44.76$	114.7	21.73
e	64.37	$\overline{FG}_e = 32.19$	160.1	13.39
f	37.58	$\overline{HG}_f = 12.62$	215.9	37.01

Fig.11 Planar six-link mechanism
(driver : b, follower : f)

as 1 or -1 respectively according as β_M and β_N increase or decrease, leading to Eqs. (25) ~ (27).

Because the summation of F_P , F_L and F_M is zero, the internal forces in Fig. 9 may become as follows.

$$\left. \begin{aligned} F_L &= \frac{\sin(\theta_P - \theta_M)}{\sin(\theta_M - \theta_L)} F_P (\cos \theta_L + i \sin \theta_L) \\ F_M &= \frac{\sin(\theta_P - \theta_L)}{\sin(\theta_L - \theta_M)} F_P (\cos \theta_M + i \sin \theta_M) \end{aligned} \right\} \dots (28)$$

Summarizing the equations derived above, we can express the internal forces at pairs as the following equation.

$$\begin{aligned} (F_L, F_M) \\ = C(F_P, X_P, Y_P, X_L, Y_L, X_M, Y_M, X_N, Y_N) \end{aligned} \dots (29)$$

Putting as $e_P = 0$ in the above equations, we can evaluate the forces at pairs M and N caused by an external force acting at an arbitrary point P on the link I, though P has been assumed as a pairing point in the above discussion.

4.3 Example

Making use of the functions B and C, Eqs. (24) and (29), together with the transformation functions derived in Chapter 3, we can analyse easily the forces of multilink mechanisms with consideration of external, inertial and frictional forces. For example, the forces at pairs caused by inertias in the six-link mechanism shown in Fig. 11 can be analysed according to the block diagram shown in Fig. 12. In the diagram, F_i , F_f , etc. are the sum of the equivalent inertial and gravitational forces acting on the links e , f , etc., F_{fe} , F_{fe} , etc. are the forces at the pair E caused by F_i , F_f , etc., and F_A , F_B , etc. are the total forces acting at the pairs A, B, etc. F_T is the force acting perpendicularly to AB, at the pair B, and the input torque caused by the inertias and gravity may become $T = b|F_T|$.

Figure 13 shows the input torque of a real mechanism whose parameters are shown in Table 1. The dot and solid curves are respectively the theoretical and experimental results at the speed of 300 rpm, and they are coincident with each other.

5. Conclusions

Considering mechanisms to be constructed with several kinds of basic open chains, we have systematically analysed the velocities and accelerations of planar multilink mechanisms with revolute pairs by means of transformation functions which describe the relative motions of links in each basic open chain and those which describe the connecting conditions of the open chains. Moreover, we have developed a systematic analysis of forces. The results are summarized as follows.

(1) In the analyses of velocities and accelerations, mechanisms are firstly divided into the basic open chains, they are secondly illustrated by the block diagrams whose elements are transformation functions, and then the computer programs can be obtained

Fig. 12 Block diagram for force analysis

Fig. 13 Input torque caused by inertial and gravitational forces

only by combining, in accordance with the diagrams, the subprograms which describe the transformation functions. These programs give not only the input and output relations but also the displacements, velocities and accelerations of arbitrary points on arbitrary links.

(2) The derivations of individual displacement equations of mechanisms are not required in the above analyses. Giving only the values of the kinematic parameters as the input data, we can evaluate the characteristics of motions of all mechanisms which may exist on account of geometric inversions of links; all of the possible motions can be separately evaluated in correspondence with the conjugate positions.

(3) We have also derived two functions which describe the equivalent inertial forces and the internal forces of links. Combining these functions with the transformation functions of velocities and accelerations, we can analyse systematically the forces of multilink mechanisms with consideration of inertial, gravitational and frictional forces.

References

- (1) Hain, K., *Maschinenmarkt*, Bd. 75, Nr. 11 (1969), s. 170.
- (2) Shaw, W.A., *Developments in Theoretical and Applied Mechanics*, Vol. 2 (1965), p. 467, Pergamon Press.
- (3) Woo, L.S., *Trans. ASME, Ser. B*, Vol. 89, No. 1 (1967-2), p. 159.
- (4) Markus, L. and Tomas, J., *J. Mechanisms*, Vol. 3 (1968), p. 171.
- (5) Brát, V. and Lederer, P., *J. Mechanism and Machine Theory*, Vol. 8 (1973), p. 457.
- (6) Funabashi, H., et al., *Bull. JSME*, Vol. 20, No. 141 (1977-3), p. 343.