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Vibration of a Shaft Passing through Several Critical Speeds*

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The transient vibration of a shaft passing through its several critical speeds at a uniform acceleration rate was investigated. The shaft is assumed to have a uniform section and be supported at both ends by springs and dampers. First, the free vibration of the system was analyzed to get the relationship between the damping given at the ends of the shaft and the modal damping ratio. Next, after evaluating the modal eccentricities and the modal damping ratios of the system, we calculated the maximum amplitudes of the shaft passing through the first and the second critical speeds at various acceleration rates using the approximate equations, which had been obtained from the analysis of the transient vibration of a system with one degree of freedom. Further, these calculated results were compared with the experimental ones.

1. Introduction

With an increasing demand for higher performance of highspeed rotating machinery the rated speed of these machines is often designed to be above the first critical speed and in some cases above the second or the third critical speed. Rotors of these machines have to pass through these critical speeds during start-up and run-down operation. This gives rise to a problem of determining the maximum stress or amplitude as well as clarifying the vibrational characteristics of the transient motion of the rotor.

The transient motion of a rotating shaft with a single disk (a system with one degree of freedom) was treated by many authors⁽¹⁾⁻⁽³⁾ assuming that the shaft passed through a critical speed at a uniform acceleration (or deceleration) rate. One of the present authors has made an investigation on this problem and experimentally clarified the transient vibrational phenomena⁽⁴⁾. He has also derived the approximate equations of the transient motion and a formula which gives the maximum amplitude as a function of an acceleration parameter and a damping ratio.^{(5),(6)}

As for systems with multi or infinite degree of freedom, however, the problem has not been studied yet. The transient vibration of these systems which occurs during passage through higher critical speeds seems to present the same phenomenon as that of a system with one degree of freedom. But when we applied the results of a system with one degree of freedom to these systems, for example, when we estimate the maximum amplitude, there appear the following difficulties. The first is that the relationship between the given bearing damping and the modal damping ratio is not well known in these systems. The second is that the practical

method to evaluate the modal unbalance or the modal eccentricity of these systems is not established.

In this paper, the free vibration of a uniform rotating shaft, both ends of which are supported by springs and dampers, is analyzed theoretically and an approximate equation is derived to show the relation between the damping and the modal damping ratio. Next various methods to evaluate the modal eccentricities are compared. After determining the modal damping ratios and the modal eccentricities of the system, the maximum amplitudes of the transient vibration under various acceleration rates are calculated by using the equations obtained from the analysis of a system with one degree of freedom. These calculated results are compared with the experimental ones.

2. Notations

c : damping coefficient
 D : modal damping ratio
 h : angular acceleration rad/s^2
 $i = \sqrt{-1}$
 k_s : stiffness of a support spring
 $k^4 = -(A\rho/EI)s^2 = -a^2s^2$
 s : characteristic value
 q : damped natural angular frequency
 X : complex amplitude
 $\alpha = k_s/(EI)$
 $\alpha^* = \alpha l^3$
 $\beta = (\alpha + \delta s)/k^3 = (\alpha^* - i\delta^*\gamma^2)/\gamma^3$
 $\beta_0 = \gamma_0^3/\alpha^*$
 γ : non-dimensional characteristic value $= kl$
 γ_0 : non-dimensional undamped characteristic value
 $\delta = c/(EI)$
 $\delta^* = \delta l^3 \sqrt{EI/A\rho}$
 Ω : acceleration parameter $= q/\sqrt{h}$
 ρ : mass of a shaft per unit length
 EI : flexural rigidity of a shaft

3. Modal damping ratio and natural mode of vibration

The system to be analyzed is shown in Fig.1. It consists of a straight uniform rotating shaft supported at both ends by springs and dampers. Here we discuss the

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modal damping ratio, which plays a very important role in the transient vibration, and the natural mode of the system.

3.1 Characteristic equation and mode shape

The origin of the coordinates system is located at the left end of the shaft. The directions of the x and y axes are shown in Fig.1 and the direction of the z axis is perpendicular to the x - y plane. Assuming that the springs and the dampers at the shaft ends are isotropic, the whirling motion of the shaft describes a circular orbit. Then it is enough to analyze the motion of the y direction only. The differential equation of motion in the y direction of a small element between the coordinates x and $x+dx$ may be written as

$$EI \frac{\partial^4 y}{\partial x^4} + A\rho \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

Assuming the solution of Eq.(1) in the following form

$$y = R(Xe^{st}) \quad (2)$$

and substituting it into Eq.(1), we obtain

$$\frac{d^4 X}{dx^4} - k^4 X = 0 \quad (3)$$

where

$$k^4 = -\frac{A\rho}{EI} s^2 \quad (4)$$

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ -e^{i\tau} & -e^{-i\tau} & e^\tau & e^{-\tau} \\ \beta-i & \beta+i & \beta+1 & \beta-1 \\ (\beta+i)e^{i\tau} & (\beta-i)e^{-i\tau} & (\beta-1)e^\tau & (\beta+1)e^{-\tau} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = 0 \quad (7)$$

Equation (7) can be rewritten as

$$\frac{1}{\Gamma} \begin{bmatrix} -1 & -1 & 1 & 1 \\ 0 & e^{i\tau} - e^{-i\tau} & -e^{i\tau} + e^{-\tau} & -e^{i\tau} + e^{-\tau} \\ 0 & 0 & (2\beta+1+i)e^{i\tau} - (2\beta+1-i)e^{-i\tau} - 2ie^\tau & (2\beta-1+i)e^{i\tau} - (2\beta-1-i)e^{-i\tau} - 2ie^{-\tau} \\ 0 & 0 & 0 & (e^{i\tau} - e^{-i\tau})\Delta \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = 0 \quad (8)$$

where

$$\Delta = -8i - (2\beta-1+i)^2 e^{(1+i)\tau} - (2\beta+1-i)^2 e^{-(1+i)\tau} + (2\beta-1-i)^2 e^{(1-i)\tau} + (2\beta+1+i)^2 e^{-(1-i)\tau} \quad (9)$$

$$\Gamma = (e^{i\tau} - e^{-i\tau})^2 ((2\beta+1+i)e^{i\tau} - (2\beta+1-i)e^{-i\tau} - 2ie^\tau)$$

Then the characteristic equation becomes $\Delta=0$. The ratios among $C_1 \sim C_4$ can be determined by Eq.(8) as follows

$$\left. \begin{aligned} \frac{c_3}{c_4} &= -\frac{(2\beta-1+i)e^{i\tau} - (2\beta-1-i)e^{-i\tau} - 2ie^{-\tau}}{(2\beta+1+i)e^{i\tau} - (2\beta+1-i)e^{-i\tau} - 2ie^\tau}, & \frac{c_2}{c_4} &= -\frac{(-e^{i\tau} + e^\tau)c_3/c_4 + (-e^{i\tau} + e^{-\tau})}{e^{i\tau} - e^{-i\tau}} \\ \frac{c_1}{c_4} &= -\frac{c_2}{c_4} + \frac{c_3}{c_4} + 1 \end{aligned} \right\} \quad (10)$$

Then the mode shape can be written as

$$X = \frac{c_1}{c_4} e^{ikx} + \frac{c_2}{c_4} e^{-ikx} + \frac{c_3}{c_4} e^{kx} + e^{-kx} \quad (11)$$

3.2 Undamped natural angular frequency and mode shape

Let us consider the natural frequencies and the mode shape of a system which is not affected by the damping force. We put $c=0$ and $s=i\omega$ in the notation table and obtain the equations $\delta=\delta^*=0$, $k^4=(A\rho/EI)\omega_n^2$, $\gamma_0=k\ell$, $\beta=\alpha^*/\gamma_0^3=1/\beta_0$. In the characteristic equation $\Delta=0$, in which Δ is represented by Eq.(9), we put $\gamma=\gamma_0$ and separate the equation into real and imaginary parts considering that both γ_0 and β are real numbers. Then the real part of the equation vanishes, and the imaginary part can be written as follows using the relation $\beta=1/\beta_0$.

$$\beta_0^2(1 - \cos \gamma_0 \cosh \gamma_0) - 2\beta_0(\cosh \gamma_0 \sin \gamma_0 - \cos \gamma_0 \sinh \gamma_0) + 2 \sin \gamma_0 \sinh \gamma_0 = 0 \quad (12)$$

Equation (12) is the characteristic equation in this case. Solving Eq.(12), we obtain

$$\beta_0 = \frac{\cosh \gamma_0 \sin \gamma_0 - \cos \gamma_0 \sinh \gamma_0 \pm (\sin \gamma_0 - \sinh \gamma_0)}{1 - \cos \gamma_0 \cosh \gamma_0} \quad (13)$$

In Eq.(13), if a value of γ_0 is given, two values of $\beta_0 (= \gamma_0^3/\alpha^*)$ are obtained.

Fig. 1 Rotor model

The solution of Eq.(3) can be written as follows:

$$X = c_1 e^{ikx} + c_2 e^{-ikx} + c_3 e^{kx} + c_4 e^{-kx} \quad (5)$$

The boundary conditions at both ends are

$$\left. \begin{aligned} (X'')_{x=0} &= 0, \\ (X''')_{x=0} &= -\frac{k_s}{EI}(X)_{x=0} - \frac{c_s}{EI}(X')_{x=0}, \\ (X'')_{x=l} &= 0, \\ (X''')_{x=l} &= \frac{k_s}{EI}(X)_{x=l} + \frac{c_s}{EI}(X')_{x=l} \end{aligned} \right\} \quad (6)$$

where the notation ' represents the first derivative with respect to x . Substituting Eq.(5) into Eq.(6) and using the notations $\alpha=k_s/EI$, $\delta=c_s/EI$, $\beta=(\alpha+\delta s)/k^3$, $\gamma=k\ell$, we obtain

The positive sign is taken for the natural frequencies of the odd order and the negative sign for the even order. When β_0 becomes negative, there is no natural frequency. The calculated relationship between γ_0 and α^* is shown in Fig.2. Figure 2 shows that γ_0 approaches the non-dimensional natural frequency of a simply supported beam when α becomes larger and that of a freely supported beam when α becomes smaller.

The natural frequencies of a uniform shaft which is supported by springs with various stiffnesses ($\infty \sim 0$ kgf/mm) are calculated using the shaft dimensions of the experimental apparatus ($\ell=1\text{m}$, $d=18\text{mm}$). The calculated results are given in Table 1.

From Eqs.(10) and (11), the natural mode of vibration corresponding to the natural frequencies of Eq.(13) can be represented as follows,

$$X = \beta_0 (\cos kx + \cosh kx) + (\sin x - \sinh kx) + \frac{\beta_0 (-\cos \gamma_0 + \cosh \gamma_0) - (\sin \gamma_0 + \sinh \gamma_0)}{\sin \gamma_0 - \sinh \gamma_0} (\sin kx + \sinh kx) \quad (14)$$

Figures 3(a) and 3(b) show the mode shapes in case of $k_s=1000$ kgf/mm and 10 kgf/mm respectively.

Fig.2 Variation of natural angular frequencies due to support spring stiffness

3.3 Damped natural frequency and modal damping ratio

It is very difficult to obtain an exact solution of the characteristic equation $\Delta=0$. Here we introduce the approximation $e^{a\tau} = e^{a\tau_0} e^{a(\tau-\tau_0)} \approx e^{a\tau_0} [1 + a(\tau-\tau_0)]$ and solve the equation $\Delta=0$. This gives the following equation,

$$\gamma - \gamma_0 = \frac{2\beta^2 (\sin \gamma_0 \sinh \gamma_0) - 2\beta (\sin \gamma_0 \cosh \gamma_0 - \cos \gamma_0 \sinh \gamma_0) + 1 - \cos \gamma_0 \cosh \gamma_0}{2\beta^2 (\sin \gamma_0 \cosh \gamma_0 + \cos \gamma_0 \sinh \gamma_0) - 4\beta \sin \gamma_0 \sinh \gamma_0 - \cos \gamma_0 \sinh \gamma_0 + \sin \gamma_0 \cosh \gamma_0} \quad (15)$$

In Eq.(15), $\beta = (\alpha^* - i\delta^* \gamma^2) / \gamma^3$ is a function of the unknown variable γ and we further assume

$$\beta \approx (\alpha^* / \gamma^3) - i(\delta^* / \gamma_0) = A - iB \quad (16)$$

where $A=1/\beta_0$ and $B=\delta^*/\gamma_0$. Substituting Eq.(16) into Eq.(15) and applying Eq.(12) to the numerator we obtain

$$\gamma - \gamma_0 = \frac{2B^2 \sin \gamma_0 \sinh \gamma_0 + (4AB \sin \gamma_0 \sinh \gamma_0 - 2B (\sin \gamma_0 \cosh \gamma_0 - \cos \gamma_0 \sinh \gamma_0))i}{2(A^2 - B^2) (\sin \gamma_0 \cosh \gamma_0 + \cos \gamma_0 \sinh \gamma_0) - 4A \sin \gamma_0 \sinh \gamma_0 - \cos \gamma_0 \sinh \gamma_0 + \sin \gamma_0 \cosh \gamma_0 + (-4AB (\sin \gamma_0 \cosh \gamma_0 + \cos \gamma_0 \sinh \gamma_0) + 4B \sin \gamma_0 \sinh \gamma_0)i} \quad (17)$$

Equation(17) represents a deviation of the non-dimensional characteristic value from the non-dimensional undamped one. The right-hand side of Eq.(17) can be calculated if the shaft dimension, the stiffness of the spring, the damping coefficient of the damper and the order of the natural frequency under consideration are given.

The relationships among the damped natural angular frequency q , the modal damping ratio D and the non-dimensional characteristic value γ are as follows. From Eq.(4) and the notation table, we can write $\gamma^2 = i a \ell^2 s$. Now we put $\gamma = \gamma_f - i\gamma_d$ and $s = -n - iq$. Then the equations $q = (\gamma_f^2 - \gamma_d^2) / a \ell^2$ and $n = 2\gamma_f \gamma_d / a \ell^2$ are obtained. q is the damped natural angular frequency and the modal damping ratio D is defined by the equation $D = n/q = 2\gamma_f \gamma_d / (\gamma_f^2 - \gamma_d^2)$.

Table 1 Variation of the natural angular frequencies ($\ell=1\text{m}$, $d=18\text{mm}$)

natural frequency k_s kgf/mm	1st	2nd	3rd	4th
∞	36.19	144.8	325.7	579
1000	36.0	144	319	559
500	36.0	142	313	537
100	35.4	133	265	404
10	31.1	77.4	132	249
1	14.6	26.3	87.4	229
0	0	0	82.0	226

rps

Fig.3 Natural modes of vibration without damping

Both q and D change with the magnitude of the damping coefficient c under the constant spring stiffness. These changes are shown in Figs. 4(a)~(c) and 5(a)~(c). These figures show that the damped natural angular frequency increases as the damping coefficient becomes larger and approaches the natural angular frequency of a simply supported beam but these changes do not hold valid for q_5 in case of $k_s=500$ kgf/mm. In case of $k_s=10$ kgf/mm, both q_1 and q_3 increase but q_2, q_4 and q_5 decrease with an increase of the damping. As a result, q_1 and q_2 come close and approach the first natural angular frequency of a simply supported beam (227.3 rad/s). The same may be said of the relationship between q_3 and q_4 , and they approach the second one (909.2 rad/s).

In Figs. 5(a)~(c), the modal damping ratios increase linearly with the damping and take the maximum value at certain values of c and then decrease. This means that there is the most effective damping coefficient, which makes the modal damping ratio maximum, for a given spring stiffness. In other words it is not always effective to strengthen the damping force for vibration reduction. Figures 5(a)~(c) also show that for a given damping coefficient c the higher the order of the natural vibration, the larger the modal damping ratio in case of $k_s=1000$ kgf/mm and 500 kgf/mm but when k_s is equal to 10 kgf/mm the second modal damping ratio D_2 is the largest followed by

D_3, D_4, D_5 and D_1 in that order at $c=0.01$ kgf.s/mm. Thus it does not always hold good that the higher the order of the natural vibration, the larger the modal damping ratio.

In order to ascertain the validity of the approximate equation (17) we compare the results obtained by the above analysis with the ones which are obtained by the responses of the steady state vibration due to unbalance force. The vibrational responses are calculated by the transfer matrix method. The rotating shaft which has the same dimensions as Figs. 4 and 5 ($l=1$ m, $d=18$ mm) is divided into 40 elements and the unbalance (10 gmm) is attached at $x=100$ mm and both the spring rigidity and the damping coefficient are changed in each case. As an example, the response at $x=200$ mm when $k_s=500$ kgf/mm and $c=0.1$ kgf.s/mm is shown in Fig. 6.

The damped natural angular frequency is considered to be nearly equal to the angular frequency where the maximum amplitude occurs, and the modal damping ratio can be calculated from the Q factor of the response curve as $D_1=0.000192, D_2=0.00302$ and $D_3=0.0250$ from Fig. 6. For various pairs of k_s and c , both the damped natural angular frequencies and the modal damping ratios are calculated and the results are plotted in Figs. 4 and 5. In the figures $\times$ represents both the first natural angular frequency and the first modal damping ratio, and $\circ$ and Δ represent the second and the third respectively. These figures show that the plotted values have a relatively good a-

Fig. 4 Variation of damped natural angular frequencies due to damping coefficient ($l=1$ m, $d=18$ mm)

Fig. 5 Variation of modal damping ratios due to damping coefficient ($l=1$ m, $d=18$ mm)

greement with the curves obtained from Eq. (17). In case of $k_s = 10 \text{ kgf/mm}$, there are few plotted values because the mode shapes are too complex to obtain the damped natural angular frequencies and the modal damping ratios exactly. The characteristic curves of D_3 , D_4 and D_5 in Fig.5(c) are different from the other curves shown in Fig.5, which may result from unsatisfactory approximation in this region.

3.4 Damped natural mode of vibration

Damped natural modes of vibration are calculated from Eqs. (10) and (11) using the values of γ and β which are obtained from Eqs. (16) and (17). Figure 7 shows the changes of the damped natural mode due to the damping coefficient. Figure 7(a) is for the first mode in case of $k_s = 1000 \text{ kgf/mm}$ and Figs. 7(b) and (c) are for the third and the fourth respectively in case of $k_s = 10 \text{ kgf/mm}$. In each figure an inclined straight line shows the bearing center line, and a thicker solid line shows the mode shape. The rotating shaft vibrating with a damped natural mode bends not only in one plane but

Fig.6 Steady state response

also in the other plane perpendicular to the former. This means that if the shaft vibrates in one plane, the time when a certain position of the shaft reaches the maximum deflection is different from that when other position reaches the maximum. Figures 7(b) and (c) show that the third and the fourth modes turn into the second natural mode of a simply supported beam due to the increase of the support damping. These results correspond to the changes of q_3 and q_4 in Fig.4(c).

Fig.7 Variation of damped natural modes of vibration

4. Experiments

We carried out two kinds of experiments using the apparatus as shown in Fig.8. The one is a steady state vibration test, which was done in order to decide the critical speeds, the mode shapes and the modal damping ratios of the system and to find out a practical method to evaluate the modal eccentricities of the rotor. The other is a transient vibration test, which was done in order to measure the maximum amplitudes of the rotating shaft passing through the first, the second and the third critical speeds and to compare the results with the analytical ones which had been obtained by the transient vibration analysis of a system with one degree of freedom.

4.1 Experimental apparatus and experimental method

Figure 8 shows the experimental apparatus. The rotating steel shaft has a diameter of 18 mm and a span of 1 m. Both ends of the shaft are supported by ball bearings of angular contact type through layer dampers. The bearing thrust preload is about 10 kgf.

Fig.8 Experimental apparatus

* ter of 18 mm and a span of 1 m. Both ends of the shaft are supported by ball bearings of angular contact type through layer dampers. The bearing thrust preload is about 10 kgf.

The first critical speed is about 38 rps and the third is about 320 rps. The rotating shaft is driven by a high frequency motor (the maximum speed is 420 rps at 250 Volts) whose electric power is supplied by a motor-generator set. The control panel of the driving system has several on-off switches, various rheostats to set the maximum and the minimum rotational speeds of the high frequency motor and timers to set the acceleration rate, that is, the speed range of the high frequency motor is divided into 60 steps and the time interval of one step is specified by the timers.

In the steady state vibration tests, the maximum and the minimum rotational speeds of the high frequency motor were set at 20 rps and 350 rps respectively and the time interval of one step was set at 10 s. The mean angular acceleration of the shaft is $2\pi \times 0.6 \text{ rad/s}^2$.

In the transient vibration tests, the time interval was set at 5 s, 1s and 0.5 s. And when a higher acceleration rate was needed, the motor coupling of the motor-generator set was run at high speed (about 350 rps) followed by a sudden drive of the high frequency motor. In this case the acceleration rate can be changed by the field rheostat of the generator and the maximum angular acceleration obtained is $2\pi \times 300 \text{ rad/s}^2$.

After ascertaining the system symmetry, the vibrations of the shaft were measured at the positions of 500, 650, 800 and 950 mm from the right-side bearing in Fig.8 by using displacement pickups and the rotational mark was picked up near the left side bearing by a photo transistor.

Fig.10 Experimental natural modes of vibration

Fig.9 Experimental steady state response

4.2 The results of the steady state vibration tests

(1) Critical speed and modal damping ratios

Figure 9 shows an example of the response of the steady state vibration test. The first, the second and the third critical speeds were 38.6, 144 and 320 rps respectively. The experimental first critical speed was higher than the calculated one which had been obtained under simple support condition. The above discrepancy mainly comes from the thrust preload of the angular contact ball bearing which constrains the shaft bending. The second and the third critical speeds were lower than the calculated ones of the simply supported beam due to the effect of spring support. Considering these things the support spring stiffness is estimated to be about 500 kgf/mm.

Figure 10 shows the natural modes of vibration at each critical speed describing half of the mode shapes in one plane, and $\times$, $\circ$ and Δ represent the measured deflection at the first, the second and the third critical speeds respectively. The thick solid lines represent the mode shapes in case of simple support condition and the thin solid lines in case of $k_s=500 \text{ kgf/mm}$. The measured deflections near the bearings are larger than that of the theoretical prediction, so the bearing stiffness seems to be smaller.

The modal damping ratios obtained from Fig.9 are $D_1=0.024$, $D_2=0.018$ and $D_3=0.022$, but it is difficult to estimate the damping coefficient at shaft end using Fig.5(b). It is partially because the response curves in Fig.9 are not smooth. Then in the following discussion we use the experimental values of the modal damping ratios.

(2) Modal eccentricity

In the present experiments, the shaft run out caused by the plastic deformation yields unbalance in the rotor. Table 2 shows the shaft run out and the distribution of the eccentricities which was measured at twelve angular positions on the nine x positions by a fine dialgauge.

Table 2 Shaft run out and distribution of eccentricities (angular position 1 stands for the z direction and 10 for the y direction)

angular position \ x mm	100	200	300	400	500	600	700	800	900
1	52	93	79	82	59	53	37	72	13.5
2	50	91	76	82	59	53	37	73	13.5
3	44	84	68	73	52	51	33	69	12.5
4	38	73	54	57	39	40	25	63	7
5	33	67	45	45	26	30	17	56	5
6	30	62	37	38	17	18	9	52	1
7	29	59	34	34	12.5	15	6.5	50	0
8	32	62	37	34	12.5	14.5	6	49	0
9	38	69	49	45	20	20	9	53	1.5
10	45	82	62	57	33	28	12	59	5.5
11	50	90	73	67	43	40	29	66	9
12	51	93	77	76	51	49	34	67	11
e_z	12.2	17.2	23.2	24.4	24.1	18.9	15.3	11.5	6.8
e_y	3.3	3.7	3.3	-0.9	-3.4	-5.8	-4.1	-2.5	-1.5

μm

We can calculate the modal eccentricities from the measured run out using the relationship $[X_{ij}]e_j = e_i$, where e_i denotes the eccentricity (run out) of the i -th position, e_j the j -th modal eccentricity and X_{ij} the displacement of the i -th position of the j -th mode.

The three methods are tried in order to calculate the modal eccentricities up to the third mode. The method (I) uses the four measured eccentricities (at $x=200, 400, 600$ and 800 mm) and the four natural modes up to the fourth critical speed of the simple support condition. The method (II) uses all the measured eccentricities and the nine natural modes up to the ninth. The method (III) uses the same measured values as the method (I) and the four natural modes up to the fourth critical speed in case of $k_s = 500$ kgf/mm.

Table 3

(a) Modal eccentricities

	(I)	(II)	(III)
ϵ_1	23.2	23.9	23.3
ϵ_2	3.5	5.4	4.9
ϵ_3	0.6	2.2	2.1

 μm

(b) Maximum amplitudes

	(I)	(II)	(III)	experimental value
R_1	483	498	485	430
R_2	97	149	135	130
R_3	14	50	46	90

 μm

The first, the second and the third modal eccentricities calculated by the three methods are compared in Table 3(a). The maximum amplitude at the i -th critical speed R_i can be calculated by the relationship $R_i = e_i / 2D_i$, where e_i is shown in Table 3(a) and D_i is the i -th modal damping ratio and these calculated results are shown in Table 3(b) with the experimental values.

In Table 3(b) the calculated maximum amplitude at the first critical speed is larger than the experimental value. It is mainly because D_1 is underestimated for the existence of another peak of vibration near the first critical speed as shown in Fig. 9. The calculated maximum amplitude at the third critical speed is smaller than the experimental value. It is because D_3 is overestimated for the peak of vibration at the third critical speed is crushed. From these considerations the methods (II) and (III) give relatively good results for the evaluation of modal eccentricities.

4.3 Transient vibration tests

The transient vibration tests were carried out. A typical example of the oscillogram is shown in Fig. 11. The characteristics of the shaft vibration, the over all value of the vibration and the rotational speed are recorded in the figure. Figure 2 shows an oscillogram of the transient vibration with a fairly fast acceleration. The angular acceleration is about $2\pi \times 280$ rad/s² and the free vibration with the first natural angular frequency which occurs during passage through the first critical speed continues and affects the transient vibration through the second critical speed.

Fig.11 Typical example of acceleration test

Fig.12 Example of acceleration test with fairly fast acceleration

The experimental results are plotted in Fig.13. The vertical axis represents the ratio of the maximum amplitude of the transient vibration to that of the steady state vibration and the horizontal axis the acceleration parameter Ω . Three other horizontal axes $h_1/2\pi$, $h_2/2\pi$ and $h_3/2\pi$ represent the acceleration rate with which the shaft passes through each critical speed. $\times$ denotes the experimental value during passage through the first critical speed and $\circ$ through the second critical speed. As for the third critical speed, the maximum amplitudes scarcely decrease even at the angular acceleration $2\pi \times 150 \text{ rad/s}^2$ ($\Omega=70$).

The curves in Fig.13 show the analytical results obtained from the approximate equation of Reference(6), where the transient vibration of a system with one degree of freedom is analyzed.

The reproducibility of the tests is as follows. The maximum amplitudes in a series of the steady state vibration tests vary in the range of $190 \sim 220 \text{ } \mu\text{m}$ ($D_1=0.015 \sim 0.021$) at the first critical speed and $145 \sim 150 \text{ } \mu\text{m}$ ($D_2=0.02 \sim 0.022$) at the second critical speed.

Figure 13 shows that the plotted values agree well with the calculated curves. This means that the maximum amplitude of the transient vibration of a uniform rotating shaft can be estimated by the approximate equation in Reference(6) if the modal damping ratio and the modal eccentricity are known.

Fig.13 Variation of maximum amplitude ratio due to acceleration rate (comparison between experiments and calculations)

5. Conclusions

A system treated is a uniform straight rotating shaft supported by springs and dampers at its both ends. Conclusions obtained in the present paper may be summarized as follows.

- (1) An approximate equation (17), which gives the damped natural angular frequency and the modal damping ratio, is derived.
- (2) When the spring stiffness is relatively large, the damped natural angular frequencies increase with an increase in damping and reach the natural angular frequencies of the simple support condition. When the spring stiffness is relatively small, some of the damped natural angular frequencies increase with an increase in damping and others of them decrease. According to this the natural modes of vibration change.
- (3) The modal damping ratio increases proportionally to the damping coefficient and reaches the maximum value at a certain damping coefficient and then decrease linearly.

This means that there exists the proper damping coefficient which makes the modal damping ratio maximum for a given stiffness of the support spring. When the stiffness of the support spring is relatively large the modal damping ratio of the higher order is larger than that of the lower one. But this relation does not hold valid when the spring stiffness is not large.

(4) The maximum amplitude which occurs during passage through the first and the second critical speeds at a uniform acceleration rate can be estimated by the approximate equation if the modal damping ratio and the modal eccentricity are known. The equation is described in Reference (6) and is derived from the analysis of the transient vibration of a system with one degree of freedom.

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