621.82.033:534.11

Paper No. 164-7

Stability Characteristics of Spherical Spiral Groove Bearings

(1st Report, Theoretical Analysis)*

By Yuichi SATO**, Akiyoshi TAMURA***

The stability characteristics of spherical spiral groove bearings lubricated with incompressible liquid are investigated theoretically. Solving a differential equation for a bearing which have an infinite number of grooves, we obtain the pressure distribution induced by small vibrations of a rigid rotor about a static equilibrium position. From the pressure distribution we derive a stability criterion for radially unloaded bearings, and clarify the dependence of the stability on bearing parameters and on axial displacements of a rotor. The bearing parameters considered are the spiral groove angle α , the relative groove depth H_r , the groove ratio δ , and rotational speeds of grooved member ω_1 and smoothed member ω_2 .

1. Introduction

Spherical spiral groove bearings (SSGB) have both axial and radial load capacities simultaneously. Besides there is another significant advantage that SSGB can rotate stably even under radially unloaded conditions. Recently, Bootsma has derived a differential equation for smoothed pressure distribution, assuming that the number of grooves approaches infinity, and get lubricant film forces in integral forms. Since there are many bearing parameters on which the characteristics of SSGB depend, it is very difficult to clarify the performance with his solutions.

In this paper, assuming small vibrations of a rotor about a equilibrium position, we solve the differential equation approximately and derive lubricant film forces. Using them, we analyze the stability characteristics theoretically.

2. Nomenclature

- b_{ij} : damping coefficients ($i, j=x, y$)
 $B_{ij} = \frac{c_r^3 \omega}{12 \mu \omega r_0^4} b_{ij}$
 c_r : mean radial clearance
 f_r, f_ϕ : radial and tangential components of lubricant film force
 f_z : axial load capacity
 $F_j = \frac{c_r^3}{12 \mu \omega r_0^4} f_j$ ($j=r, \phi, z$)
 h : local film thickness in ridge region (Fig. 2)
 h_m : local film thickness in groove region (Fig. 2)
 h_g : depth of groove only (Fig. 2)
 $H = h/c_r$, $H_g = h_g/c_r$
 $H_m = H + H_g$, $H_r = H_g + H_g$, $H_T = H_0 + H_g$

k_{ij} : elastic coefficients ($i, j=x, y$)

$$K_{ij} = \frac{c_r^3}{12 \mu \omega r_0^4} k_{ij}$$

m : rotor mass (Fig. 5)

$$M = \frac{m c_r^3}{12 \mu \omega r_0^4} \omega$$

M_1 : stability threshold in the case of rotation of grooved member

M_2 : stability threshold in the case of rotation of smoothed member

O-xyz : fixed coordinate system

p : smoothed pressure

$$P = \frac{c_r^2}{12 \mu \omega r_0^2} p$$

q_θ : liquid flow in θ -direction

$$Q_\theta = q_\theta / (c_r r_0 \omega)$$

r_0 : bearing radius

t : time

x, y, z : coordinates of the center of rotor in the x, y, z directions (Fig. 5)

α : groove angle

β : angular frequency of self-excited vibration

$\beta/\omega_1, \beta/\omega_2$: dimensionless angular frequencies of self-excited vibration in the case of rotation of grooved member and smoothed member

δ : fraction of width of groove-ridge pair that is occupied by groove

$$\epsilon_r = \sqrt{x^2 + y^2} / c_r, \quad \epsilon_x = x / c_r, \quad \epsilon_y = y / c_r, \quad \epsilon_z = z / c_r$$

μ : lubricant viscosity

$$\tau = \omega t$$

ω : angular velocity of rotor

ω_1 : angular velocity of grooved member

ω_2 : angular velocity of smoothed member

$$\Omega_i = \omega_i / \omega \quad (i=1, 2)$$

Subscripts

$$(1) = d/d\tau$$

$\wedge$ indicates using H_v instead of H_0

3. Theoretical analysis

3.1. Smoothed pressure distribution

Hsing⁽²⁾ has already derived the differential equation for smoothed pressure distribution in terms of generalized coordinates for bodies of rotational symmetry. Since in further analysis, a spherical bear-

* Received 22nd September, 1976.

** Student of Graduate Course, Tokyo Institute of Technology, Meguro-ku Tokyo.

*** Professor, Tokyo Institute of Technology.

ing will be treated, we rewrite the equation in terms of spherical coordinates (Fig. 1) as Eq. (1), which is expressed in dimensionless form. The shape of groove is rectangular as shown in Fig. 2, and grooves are cut on the whole surface of a bearing.

$$\begin{aligned} & \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\eta_1 \sin \theta \frac{\partial P}{\partial \theta} + \eta_2 \frac{\partial P}{\partial \varphi} \right] \\ & + \lambda_e \eta_4 (\Omega_2 - \Omega_1) \sin^2 \theta \left\{ \right. \\ & + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \varphi} \left[\eta_2 \sin \theta \frac{\partial P}{\partial \theta} + \eta_3 \frac{\partial P}{\partial \varphi} \right] \\ & + \lambda_e \eta_4 (\Omega_2 - \Omega_1) \sin^2 \theta \left. \right\} \\ & - \frac{1}{2} (\Omega_2 + \Omega_1) \frac{\partial H}{\partial \varphi} - \frac{dH}{d\tau} = 0 \quad \dots\dots\dots (1) \end{aligned}$$

where

$$\left. \begin{aligned} H &= H_0 - \varepsilon_r \cos \varphi \sin \theta \\ H_0 &= 1 - \varepsilon_r \cos \theta \end{aligned} \right\} \dots\dots\dots (2)$$

$$\left. \begin{aligned} \lambda_r &= \frac{1}{2} H_r (1 - \delta) \delta \sin^2 \alpha \\ \lambda_e &= \frac{1}{2} H_r (1 - \delta) \delta \sin \alpha \cos \alpha \end{aligned} \right\} \dots\dots\dots (3)$$

and η_i ($i=1,2,3,4$) is given in Appendix 1. In Eq. (1), for the case of rotation of the grooved member, $\Omega_1=1$ and $\Omega_2=0$ ($\omega_1=\omega$ and $\omega_2=0$) and for the case, the smoothed member, $\Omega_1=0$ and $\Omega_2=1$ ($\omega_1=0$ and $\omega_2=\omega$). The last term of the left hand side of Eq. (1) is an expression for the velocity of a rotor and is responsible for the squeeze film action. Assuming that a rotor vibrates only in the direction that is perpendicular to the z-axis, the term becomes

$$dH/d\tau = -\varepsilon_r^{(1)} \cos \varphi \sin \theta - \varepsilon_r \phi^{(1)} \sin \varphi \sin \theta \quad \dots\dots\dots (4)$$

Fig. 1

Fig. 2

Under radially unloaded conditions, the shaft axis coincides with the z-axis at the static equilibrium state (i.e. $\varepsilon_r=0$). Thus we assume that the pressure P can be expanded in a series in ε_r , $\varepsilon_r^{(1)}$ and $\varepsilon_r \phi^{(1)}$ and consider the terms of zero-order and the first-order of them. Then P is expressed in the following form:

$$P(\theta, \varphi) = P_0(\theta) + \varepsilon_r P_1(\theta, \varphi) + \varepsilon_r^{(1)} P_2(\theta, \varphi) + \varepsilon_r \phi^{(1)} P_3(\theta, \varphi) \dots\dots\dots (5)$$

Substituting Eq. (2) into η_i ($i=1,2,3,4$), and collecting terms of zero-order and order ε_r , η_i can be written as

$$\eta_i = \eta_{i0} + \eta_{i1} \varepsilon_r \cos \varphi \sin \theta \quad (i=1,2,3,4) \dots\dots\dots (6)$$

where η_{i0} and η_{i1} are given in Appendix 2.

By substituting Eqs. (4), (5), and (6) into Eq. (1) and collecting terms of order zero, order ε_r , order $\varepsilon_r^{(1)}$, and order $\varepsilon_r \phi^{(1)}$, we obtain the differential equations (7), (13) (14) and (15). The equation for P_0 can be written as

$$\begin{aligned} \varepsilon_r^{(0)} : \sin \theta \frac{\partial}{\partial \theta} \left[\eta_{10} \sin \theta \frac{\partial P_0}{\partial \theta} \right] \\ + \lambda_e \eta_{40} (\Omega_2 - \Omega_1) \sin^2 \theta \left\{ \right. \end{aligned} \quad \dots\dots\dots (7)$$

Since η_{10} and η_{40} are the functions of θ in the case of $\varepsilon_r \neq 0$, it is very difficult to get the exact solution. Then we make an approximation to H_0 with H_v , which is defined by the following equation:

$$H_v = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} H_0 \sin \theta d\theta d\varphi = 1 - \frac{1}{2} \varepsilon_r \quad \dots\dots\dots (8)$$

Replacing H_0 with H_v means making an approximation to local film thickness with the film thickness at $\theta = \pi/3$. Integrating Eq. (7) twice with respect to θ , and applying the boundary conditions:

$$\begin{aligned} P_0 &= 0 : \theta = \pi/2 \\ \int_0^{2\pi} Q_0 d\varphi &= 0 \end{aligned}$$

we obtain

$$P_0 = (\tilde{\eta}_{40}/\tilde{\eta}_{10}) \lambda_e (\Omega_2 - \Omega_1) \cos \theta \quad \dots\dots\dots (9)$$

The dimensionless axial load capacity is calculated from the expression (10).

$$F_z = \int_0^{2\pi} \int_0^{\pi/2} P_0 \cos \theta \sin \theta d\theta d\varphi \quad \dots\dots\dots (10)$$

By substituting Eq. (9) into Eq. (10), F_z is obtain as follows:

$$\begin{aligned} F_z &= \frac{\pi}{6} \frac{H_r (H_r^3 - H_v^3) \delta (1 - \delta) \sin \alpha \cos \alpha}{(H_r^3 - H_v^3)^2 \delta (1 - \delta) \sin^2 \alpha + H_v^3 H_r^3} \\ & \times (\Omega_2 - \Omega_1) \cos \theta_1 (3 - \cos^2 \theta_1) \quad \dots\dots\dots (11) \end{aligned}$$

Fig. 3

where we assume that grooves are cut from $\theta = \theta_1$ to $\theta = \pi/2$ and that the region from $\theta = 0$ to $\theta = \theta_1$ is ungrooved. The load capacities calculated from Eq. (11) and those calculated numerically by Reinhoudt⁽³⁾ for a bearing with $\theta_1 = 0$ are shown simultaneously in Fig. 3. Apparently, the values from Eq. (11) are greater than his results for $0 < \epsilon_1$. For example, the difference at $\epsilon_1 = 0.5$ is about 10%. If we correct $H_V = 1 - (\epsilon_1/2)$ in Eq. (8) by $H_V = 1 - C_V \epsilon_1$ (where $C_V < 1/2$), the values of calculated in Eq. (11) approach his results. In the case of $\epsilon_1 = 0$ and $\theta_1 = 0$, Eq. (11) coincides with Muijderman's results.⁽⁴⁾

As is mentioned above, three differential equations for P_1 , P_2 and P_3 are obtained by substituting Eq. (4), (5), and (6) into Eq. (1). But it is difficult to solve them in the case of $\epsilon_1 \neq 0$ because η_{10} , η_{20} , etc are the functions of θ . So we make an approximation to H_0 with H_V . Using transformation of the variable:

$$\zeta = \ln [\tan (\theta / 2)] \quad (12)$$

we get the equations for them as follows,

$$\begin{aligned} \epsilon_r: \tilde{\eta}_{10} \frac{\partial^2 P_1}{\partial \zeta^2} + 2\tilde{\eta}_{20} \frac{\partial^2 P_1}{\partial \zeta \partial \varphi} + \tilde{\eta}_{30} \frac{\partial^2 P_1}{\partial \varphi^2} \\ = -3\nu_c(\mathcal{Q}_2 - \mathcal{Q}_1) \tanh \zeta \operatorname{sech}^3 \zeta \cos \varphi \\ - \left[\nu_c(\mathcal{Q}_2 - \mathcal{Q}_1) - \frac{1}{2}(\mathcal{Q}_2 + \mathcal{Q}_1) \right] \operatorname{sech}^3 \zeta \sin \varphi \end{aligned} \quad (13)$$

$$\begin{aligned} \epsilon_r^{(1)}: \tilde{\eta}_{10} \frac{\partial^2 P_2}{\partial \zeta^2} + 2\tilde{\eta}_{20} \frac{\partial^2 P_2}{\partial \zeta \partial \varphi} + \tilde{\eta}_{30} \frac{\partial^2 P_2}{\partial \varphi^2} \\ = -\operatorname{sech}^3 \zeta \cos \varphi \end{aligned} \quad (14)$$

$$\begin{aligned} \epsilon_r^{(2)}: \tilde{\eta}_{10} \frac{\partial^2 P_3}{\partial \zeta^2} + 2\tilde{\eta}_{20} \frac{\partial^2 P_3}{\partial \zeta \partial \varphi} + \tilde{\eta}_{30} \frac{\partial^2 P_3}{\partial \varphi^2} \\ = -\operatorname{sech}^3 \zeta \sin \varphi \end{aligned} \quad (15)$$

where

$$\begin{aligned} \nu_c = \lambda_c (\tilde{\eta}_{11} \tilde{\eta}_{40} / \tilde{\eta}_{10} - \tilde{\eta}_{41}) \\ \nu_c = \lambda_c \tilde{\eta}_{21} \tilde{\eta}_{40} / \tilde{\eta}_{10} - \lambda_1 \tilde{\eta}_{41} \end{aligned} \quad (16)$$

If the right hand sides of Eqs. (13), (14), and (15) are zero, the general solutions P_i^G ($i=1, 2, 3$) are written in the form of Eq. (17), since P_1 , P_2 and P_3 are the functions with a period of 2π with respect to φ .

$$\begin{aligned} P_i^G = A_{1i} \{ \cos \varphi \cos (\sigma_r \zeta) \cosh (\sigma_m \zeta) \\ - \sin \varphi \sin (\sigma_r \zeta) \cosh (\sigma_m \zeta) \} \\ + A_{2i} \{ \sin \varphi \cos (\sigma_r \zeta) \sinh (\sigma_m \zeta) \\ + \cos \varphi \sin (\sigma_r \zeta) \sinh (\sigma_m \zeta) \} \\ + A_{3i} \{ \sin \varphi \cos (\sigma_r \zeta) \cosh (\sigma_m \zeta) \\ + \cos \varphi \sin (\sigma_r \zeta) \cosh (\sigma_m \zeta) \} \\ + A_{4i} \{ \cos \varphi \cos (\sigma_r \zeta) \sinh (\sigma_m \zeta) \\ - \sin \varphi \sin (\sigma_r \zeta) \sinh (\sigma_m \zeta) \} \quad (i=1, 2, 3) \end{aligned} \quad (17)$$

where

$$\begin{aligned} \sigma_r = -\frac{\tilde{\eta}_{20}}{\tilde{\eta}_{10}} \\ = \frac{(H_1^3 - H_V^3)^2 \tilde{\eta} (1 - \tilde{\eta}) \sin \alpha \cos \alpha}{(H_1^3 - H_V^3)^2 \tilde{\eta} (1 - \tilde{\eta}) \sin^2 \alpha + H_1^3 H_V^3} \\ \sigma_m = \frac{\sqrt{\tilde{\eta}_{10} \tilde{\eta}_{30} - \tilde{\eta}_{20}^2}}{\tilde{\eta}_{10}} \\ = \frac{\sqrt{H_1^3 H_V^3 [(H_1^3 - H_V^3)^2 \tilde{\eta} (1 - \tilde{\eta}) + H_1^3 H_V^3]}}{(H_1^3 - H_V^3)^2 \tilde{\eta} (1 - \tilde{\eta}) \sin^2 \alpha + H_1^3 H_V^3} \end{aligned} \quad (18)$$

Since the particular solutions, P_i^* ($i=1, 2, 3$), of Eqs. (13), (14), and (15) are also the functions with a period of 2π with respect to φ , they are written in the following form:

$$P_i^* = y_i(\zeta) \cos \varphi + z_i(\zeta) \sin \varphi \quad (i=1, 2, 3) \quad (20)$$

Among y_1 , y_2 , y_3 , z_1 , z_2 , and z_3 , there are certain relations as follows:

$$\begin{aligned} y_1 = - \left\{ \nu_c(\mathcal{Q}_2 - \mathcal{Q}_1) - \frac{1}{2}(\mathcal{Q}_2 + \mathcal{Q}_1) \right\} z_2 \\ - \nu_c(\mathcal{Q}_2 - \mathcal{Q}_1) (dy_2/d\zeta) \\ z_1 = \left\{ \nu_c(\mathcal{Q}_2 - \mathcal{Q}_1) - \frac{1}{2}(\mathcal{Q}_2 + \mathcal{Q}_1) \right\} y_2 \\ - \nu_c(\mathcal{Q}_2 - \mathcal{Q}_1) (dz_2/d\zeta) \\ y_3 = -z_3, \quad z_3 = y_3 \end{aligned} \quad (21)$$

Therefore, if y_2 and z_2 are known the others are obtained. But they can be written only in integral forms, i.e., it is necessary to integrate them numerically in order to obtain the values of y_1 , y_2 , etc. Introducing a formula:

$$\operatorname{sech}^3 \zeta = \sum_{n=1}^{\infty} (-1)^n \{1 - (2n+1)^2\} e^{(2n+1)\zeta} \quad (22)$$

we will obtain a good approximation whose accuracy is satisfactory without numerical integration. Since a series of formula (22) converges slowly, we apply Euler's transformation to Eq. (22) and make an approximation to $\operatorname{sech}^3 \zeta$ with first eight terms. The approximation is written as follows:

$$\operatorname{sech}^3 \zeta = \sum_{n=1}^8 E_n e^{(2n+1)\zeta} \quad (23)$$

where E_n is a constant and is given in Appendix 3. Substituting Eq. (23) into Eq. (14), y_2 and z_2 are obtained as follows:

$$\begin{aligned} y_2 = \sum_{n=1}^8 Y_n e^{(2n+1)\zeta} \\ z_2 = \sum_{n=1}^8 Z_n e^{(2n+1)\zeta} \end{aligned} \quad (24)$$

where

$$\begin{aligned} Y_n = \frac{-(\tilde{\eta}_{10}(2n+1)^2 - \tilde{\eta}_{30})}{\{\tilde{\eta}_{10}(2n+1)^2 - \tilde{\eta}_{30}\}^2 + 4\tilde{\eta}_{20}^2(2n+1)^2} E_n \\ Z_n = \frac{-2\tilde{\eta}_{20}(2n+1)}{\{\tilde{\eta}_{10}(2n+1)^2 - \tilde{\eta}_{30}\}^2 + 4\tilde{\eta}_{20}^2(2n+1)^2} E_n \end{aligned} \quad (25)$$

From the relation (21), we obtain y_i , z_i ($i=1, 2, 3$).

The general solution of Eqs. (13), (14) and (15) is written as

$$P_i = P_i^* + P_i^G \quad (i=1, 2, 3) \quad (26)$$

The integration constants A_{1i} ($\sim A_{4i}$) are determined by boundary conditions:

$$P_i = 0 \quad (i=1, 2, 3): \theta = \pi/2 (\zeta = 0)$$

$$P_i = \text{finite}, P_2 = P_3 = 0: \theta = 0 (\zeta \rightarrow -\infty)$$

Then Eq. (26) is rewritten as

$$\begin{aligned} P_i = e^{\sigma_m \zeta} \sin \varphi \{ y_i(0) \sin (\sigma_r \zeta) - z_i(0) \cos (\sigma_r \zeta) \} \\ - e^{\sigma_m \zeta} \cos \varphi \{ y_i(0) \cos (\sigma_r \zeta) + z_i(0) \sin (\sigma_r \zeta) \} \\ + y_i(\zeta) \cos \varphi + z_i(\zeta) \sin \varphi \quad (i=1, 2, 3) \end{aligned} \quad (27)$$

Integrating P_i , we can obtain the lubricating film forces. The radial component of lubricating film forces F_{ir} and the tangential component $F_{i\phi}$, as shown in Fig. 4, are calculated from the expressions:

$$\begin{aligned} F_{ir} = \int_0^{2\pi} \int_0^{\pi/2} P_i \cos \varphi \sin^2 \theta d\theta d\varphi \\ (i=1, 2, 3) \\ F_{i\phi} = \int_0^{2\pi} \int_0^{\pi/2} P_i \sin \varphi \sin^2 \theta d\theta d\varphi \\ (i=1, 2, 3) \end{aligned} \quad (28)$$

But in Eq. (28) the integration of the terms of P_i ($i=1, 2, 3$) which correspond to P_i^G , is difficult. Using Eq. (23) again, we obtain the damping and the elastic coefficients of lubricating film forces as follows:

$$\left. \begin{aligned} F_{1r} &= -F_{2\phi} \left\{ \nu_s (\mathcal{Q}_2 - \mathcal{Q}_1) - \frac{1}{2} (\mathcal{Q}_2 + \mathcal{Q}_1) \right\} \\ &\quad + \xi_s (\mathcal{Q}_2 - \mathcal{Q}_1) \\ F_{1\phi} &= F_{2r} \left\{ \nu_s (\mathcal{Q}_2 - \mathcal{Q}_1) - \frac{1}{2} (\mathcal{Q}_2 + \mathcal{Q}_1) \right\} \\ &\quad + \xi_\phi (\mathcal{Q}_2 - \mathcal{Q}_1) \end{aligned} \right\} \dots\dots\dots (29)$$

$$\left. \begin{aligned} F_{2r} &= \pi \sum_{n=1}^8 B_n Y_n - \pi (y_0 C + z_0 S) \\ F_{2\phi} &= \pi \sum_{n=1}^8 B_n Z_n + \pi (y_0 S - z_0 C) \end{aligned} \right\} \dots\dots\dots (30)$$

$$\left. \begin{aligned} F_{3r} &= -F_{2\phi} \\ F_{3\phi} &= F_{2r} \end{aligned} \right\} \dots\dots\dots (31)$$

where

$$\left. \begin{aligned} \xi_s &= \pi \nu_s \sum_{n=1}^8 (2n+1) \{ S Z_n + (C - B_n) Y_n \} \\ \xi_\phi &= -\pi \nu_s \sum_{n=1}^8 (2n+1) \{ S Y_n - (C - B_n) Z_n \} \end{aligned} \right\} \dots\dots\dots (32)$$

$$\left. \begin{aligned} C &= \sum_{n=1}^8 \frac{2n+1+\sigma_m}{(2n+1+\sigma_m)^2 + \sigma_r^2} E_n \\ S &= \sum_{n=1}^8 \frac{-\sigma_r}{(2n+1+\sigma_m)^2 + \sigma_r^2} E_n \end{aligned} \right\} \dots\dots\dots (33)$$

$$y_0 = \sum_{n=1}^8 Y_n, \quad z_0 = \sum_{n=1}^8 Z_n \dots\dots\dots (34)$$

, and B_n ($n=1 \sim 8$) are constants and are given in Appendix 3. The total radial component F_r and the total tangential component F_ϕ , which are induced by small vibrations of a rotor, are written in terms of F_{1r} , $F_{1\phi}$, etc., as in the following forms:

$$\begin{aligned} F_r &= \varepsilon_r F_{1r} + \varepsilon_r^{(1)} F_{2r} + \varepsilon_r \phi^{(1)} F_{3r} \\ &= \varepsilon_r \left\{ (\xi_s - \nu_s F_{2\phi}) (\mathcal{Q}_2 - \mathcal{Q}_1) + \frac{1}{2} F_{2\phi} (\mathcal{Q}_2 + \mathcal{Q}_1) \right\} \\ &\quad + \varepsilon_r^{(1)} F_{2r} - \varepsilon_r \phi^{(1)} F_{2\phi} \dots\dots\dots (35) \end{aligned}$$

$$\begin{aligned} F_\phi &= \varepsilon_r F_{1\phi} + \varepsilon_r^{(1)} F_{2\phi} + \varepsilon_r \phi^{(1)} F_{3\phi} \\ &= \varepsilon_r \left\{ (\xi_\phi + \nu_s F_{2r}) (\mathcal{Q}_2 - \mathcal{Q}_1) - \frac{1}{2} F_{2r} (\mathcal{Q}_2 + \mathcal{Q}_1) \right\} \\ &\quad + \varepsilon_r^{(1)} F_{2\phi} + \varepsilon_r \phi^{(1)} F_{2r} \dots\dots\dots (36) \end{aligned}$$

In Eqs. (35) and (36), the four terms of order $\varepsilon_r^{(1)}$ and $\varepsilon_r \phi^{(1)}$ are the forces induced by the velocities of a rotor. And the two terms of order ε_r are the forces induced by the displacements and they consist of two components. One that contains $(\mathcal{Q}_2 - \mathcal{Q}_1)$ is produced by the velocity difference between bearing surfaces, and is considered to be produced by "pumping effects of grooves". The other component that contains $(\mathcal{Q}_2 + \mathcal{Q}_1)$ is considered to be produced by "wedge effects". F_{3r} and $F_{3\phi}$, which are the coupling terms, are almost equal to zero. Thus substituting Eq. (31) and $F_{2\phi} = 0$ into Eqs. (35) and (36), we obtain as follows:

$$\left. \begin{aligned} F_r &= \varepsilon_r \xi_s (\mathcal{Q}_2 - \mathcal{Q}_1) + \varepsilon_r^{(1)} F_{2r} \\ F_\phi &= \varepsilon_r \left\{ (\xi_\phi + \nu_s F_{2r}) (\mathcal{Q}_2 - \mathcal{Q}_1) - \frac{1}{2} F_{2r} (\mathcal{Q}_2 + \mathcal{Q}_1) \right\} \\ &\quad + \varepsilon_r \phi^{(1)} F_{2r} \end{aligned} \right\} \dots\dots\dots (37)$$

In Eq. (37) the radial component F_r is produced by pumping effects and squeeze effects. On the other hand, the tangential component F_ϕ is produced by not only pumping and squeeze effects but also wedge effects. Therefore the values of F_ϕ in the case of rotation of smoothed member ($\mathcal{Q}_1 = 0$, $\mathcal{Q}_2 = 1$, $\alpha \rightarrow \alpha$) are different from those in the case of rotation of grooved member ($\mathcal{Q}_1 = 1$, $\mathcal{Q}_2 = 0$, $\alpha \rightarrow \pi - \alpha$). For this reason the stability characteristics differ in two cases, which will be mentioned later.

Fig. 4

Fig. 5

3.2. Stability

We will discuss the stability of such a system as shown in Fig. 5. The center of gravity of a globular rotor is located at the center of globe O' , and the rotor mass is denoted by m . The rotor is radially unloaded and rotates about an axis parallel to z -axis. The differential equations of motion in the directions perpendicular to the rotating axis, i.e., to z -axis, are written in dimensionless form as follows:

$$\left. \begin{aligned} M \varepsilon_x^{(2)} + F_r \cos \phi - F_\phi \sin \phi &= 0 \\ M \varepsilon_y^{(2)} + F_r \sin \phi + F_\phi \cos \phi &= 0 \end{aligned} \right\} \dots\dots\dots (38)$$

where there are relations between (ε_r, ϕ) and $(\varepsilon_x, \varepsilon_y)$,

$$\varepsilon_x = \varepsilon_r \cos \phi, \quad \varepsilon_y = \varepsilon_r \sin \phi \dots\dots\dots (39)$$

By substituting Eqs. (35), (36) and (39) into Eq. (38), the equations of motion are obtained as:

$$\left. \begin{aligned} M \varepsilon_x^{(2)} + B_{xx} \varepsilon_x^{(1)} + B_{xy} \varepsilon_y^{(1)} + K_{xx} \varepsilon_x + K_{xy} \varepsilon_y &= 0 \\ M \varepsilon_y^{(2)} + B_{yx} \varepsilon_x^{(1)} + B_{yy} \varepsilon_y^{(1)} + K_{yx} \varepsilon_x + K_{yy} \varepsilon_y &= 0 \end{aligned} \right\} \dots\dots\dots (40)$$

where

$$\left. \begin{aligned} B_{xx} &= B_{yy} = F_{2r}, \quad B_{xy} = -B_{yx} = -F_{2\phi} \neq 0 \\ K_{xx} &= K_{yy} = (\xi_s - \nu_s F_{2\phi}) (\mathcal{Q}_2 - \mathcal{Q}_1) \\ &\quad + \frac{1}{2} F_{2\phi} (\mathcal{Q}_2 + \mathcal{Q}_1) \doteq \xi_s (\mathcal{Q}_2 - \mathcal{Q}_1) \\ K_{xy} &= -K_{yx} = -(\xi_\phi + \nu_s F_{2r}) (\mathcal{Q}_2 - \mathcal{Q}_1) \\ &\quad + \frac{1}{2} F_{2r} (\mathcal{Q}_2 + \mathcal{Q}_1) \end{aligned} \right\} \dots\dots\dots (41)$$

We take the solutions of Eq. (40) in the form $\varepsilon_x = E_x e^{st}$ and $\varepsilon_y = E_y e^{st}$. Substituting back into Eq. (40), and considering the relations (41), we obtain a characteristic equation as,

$$\begin{aligned} M^2 s^4 + 2 B_{xx} M s^3 + (2 K_{xx} M + B_{xx}^2 + B_{xy}^2) s^2 \\ + 2 (B_{xx} K_{xx} + B_{xy} K_{xy}) s + (K_{xx}^2 + K_{xy}^2) = 0 \end{aligned} \dots\dots\dots (42)$$

Applying the Routh-Hurwitz criterion to Eq. (42), we obtain the stability criterion as,

$$M < \frac{B_{xx} (K_{xx} B_{xx} + K_{xy} B_{xy})}{K_{xy}^2} \dots\dots\dots (43)$$

When we consider $F_{2\phi} = 0$, Eq. (43) can be reduced to

$$M < \left(\frac{B_{zz}}{K_{zz}} \right)^2 K_{zz} \dots\dots\dots (44)$$

Let us now consider the angular frequency of self-excited vibration β . Taking the solution of Eq.(42) in the form $s=j(\beta/\omega)$, and substituting in Eq.(42) we obtain

$$\frac{\beta}{\omega} = \frac{K_{zz}}{B_{zz}} = \frac{1}{2} (\Omega_2 + \Omega_1) - \left(\nu_1 + \frac{\xi \phi}{F_{zz}} \right) (\Omega_2 - \Omega_1) \dots\dots\dots (45)$$

where ω is the stability threshold of angular velocity of a rotor. Using the expression (45), we may express Eq.(44) as,

$$M < K_{zz} / (\beta/\omega)^2 \dots\dots\dots (46)$$

It is found from Eq.(46) that the stability threshold is defined as a function of K_{zz} and β/ω . The calculated values will be discussed in the next section.

3.3. Calculated results

Let us consider the case of $\varepsilon_z=0$ (Fig. 6~14). The dependence of K_{zz} on the groove parameters α and δ at various values of H_g and α is shown in Figs.6 and 7. We may consider K_{zz} and radial stiffness F_{zz} as the same thing. As shown, K_{zz} reaches a maximum at $\alpha=36^\circ$, $H_g=1$ and $\delta=0.5$, and K_{zz} decreases rapidly if the groove parameters α , H_g , δ moves away from the values. Since the stability threshold is proportional to not only K_{zz} but also $1/(\beta/\omega)^2$, the threshold does not always reach a maximum at the groove parameters at which K_{zz} reaches a maximum. The value of K_{zz} in the case of rotation of smoothed member is equal to that in the case of grooved member.

In Fig.8, the dimensionless angular frequency of self-excited vibration β/ω at various values of H_g and δ is shown as functions of α . Let β/ω_1 and β/ω_2 denote the dimensionless angular frequency of self-excited vibration in the case of rotation of grooved member and that in the case of rotation of smoothed member respectively. There is a relation between them as,

$$(\beta/\omega_1) + (\beta/\omega_2) = 1 \dots\dots\dots (47)$$

As shown, β/ω_2 is smaller than β/ω_1 for $H_g > 2$, and β/ω_2 has a minimum value, whereas β/ω_1 has a maximum, for $20^\circ < \alpha < 30^\circ$. From Eq.(46) we find that the smaller the value of β/ω , the higher the stability threshold becomes. Therefore for $H_g > 2$ and $20^\circ < \alpha < 30^\circ$, the stability threshold is high when a smoothed member rotates.

The stability threshold in the case of rotation of grooved member is shown in Figs. 9 through 11. As can be seen, the groove parameters α and H_g at which the stability threshold reaches a maximum are almost equal to the groove parameters α and H_g at which K_{zz} reaches a maximum. And the stability threshold is sensitive to the changes of the groove parameters. The dependence of stability on the groove parameter δ is shown in Fig.11. In general, the stability threshold reaches a maximum where δ is a little larger than 0.5.

The stability threshold in the case of rotation of smoothed member is shown in Figs.12 through 14. As groove depth H_g becomes very large, the value of β/ω_2 will ap-

Fig. 6

Fig. 7

Fig. 8

proach a certain value. In the case of $0^\circ < \alpha < 25^\circ$, β/ω_2 approaches a negative value. Therefore in this case β/ω_2 is zero at a certain value of H_g , which is larger than 9.0, though the value of H_g depends on groove parameters α and δ . From Eq. (45) if $\beta/\omega = 0$, K_{**} is also zero. Thus, the bearing is always stable. But values of P_1 , P_2 and P_3 decrease rapidly as H_g becomes large. For example, let us consider two cases (1) $\alpha = 10^\circ$, $\delta = 0.5$, $H_g = 1$ and (2) $\alpha = 10^\circ$, $\delta = 0.5$, $H_g = 5$. The ratios of P_1/P_0 , P_2/P_0 and P_3/P_0 at the θ -coordinates, where P_1 , P_2 and P_3 reach maximums, are:

(1) $H_g = 1$	(2) $H_g = 5$
$P_1/P_0 = 10$	$P_1/P_0 = 0.25$
$P_2/P_0 = 18$	$P_2/P_0 = 0.35$
$P_3/P_0 = 18$	$P_3/P_0 = 0.35$

Fig. 9

Fig. 11

Fig. 10

Fig. 12

From above we find that the pressures produced by small vibrations are very small in comparison with P_0 in the case that H_g is large. In such a case the assumption that a rotor performs small vibrations about a static equilibrium position may not be valid since the radial bearing forces are very small. Thus we will discuss the stability of bearing the groove parameter H_g of which is less than 4. From Figs. 12 and 13 it can be seen that the stability threshold is high in the case of $20^\circ \leq \alpha \leq 30^\circ$. In Fig. 14, the dependence of stability threshold on δ at various values of H_g is shown. As can be seen, the values of δ at which the stability threshold reaches a maximum increase with increasing values of H_g .

In the extreme case, as the depth of a groove becomes very large ($H_g \rightarrow \infty$), the value of β/ω will approach a certain value which depends on the groove parameters, and the value of K_{**} will approach zero. Thus, the stability threshold falls to zero in both cases of rotation of smoothed member and grooved member.

The calculated results in the case of $\varepsilon_z = 0$ show good agreement with Bootsma's results, as can be seen in Figs. 6, 8, 10, and 13.

Next we consider the characteristics in the case of $\varepsilon_z \neq 0$. The dependence of K_{xx} , β/ω , and stability threshold on the dimensionless axial displacement ε_z is shown in Figs. 15 and 16. As the value of ε_z decreases, the stability threshold decreases in both cases of rotation of smoothed member and grooved member. In other words, the stability threshold increases as the displacement of a rotor from a bearing surface in the direction of the z-axis decreases. Thus from a viewpoint of stability, it is desirable that the displacement of a rotor in the direction of the z-axis from the bottom of a bearing be small. In Fig. 16 we note that the dependence of stability threshold on the groove depth H_g in the case of $\varepsilon_z \neq 0$ is qualitatively similar to that of $\varepsilon_z = 0$. For the values of $H_g > 2$, the stability threshold in the case of rotation of smoothed member is higher than that of grooved member.

Fig. 13

Fig. 14

Fig. 15

Fig. 16

4. Conclusions

The results of the investigation are as follows:

- (1) At $\delta = 0.5$, the stability threshold is the highest at $\alpha = 40^\circ$ and $H_g = 1$ in the case of rotation of grooved member.
- (2) For $H_g < 4$, the stability threshold is high in the range of groove angles, $20^\circ < \alpha < 30^\circ$, in the case of rotation of smoothed member.
- (3) For $H_g > 2$, the stability threshold in the case of rotation of smoothed member is higher than that in the case of rotation of grooved member. On the other hand, for $H_g < 1$, the stability threshold of grooved member is higher than that of smoothed member though the difference is small.
- (4) The smaller the axial displacement of a rotor from the bottom of a spherical spiral groove bearing, the higher the stability threshold becomes.

Appendix

[1]

$$\begin{aligned}\eta_1 &= \frac{(H_m^3 - H^3)^2 \delta (1 - \delta) \sin^2 \alpha + H^3 H_m^3}{(1 - \delta) H_m^3 + \delta H^3} \\ \eta_2 &= \frac{-(H_m^3 - H^3)^2 \delta (1 - \delta) \sin \alpha \cos \alpha}{(1 - \delta) H_m^3 + \delta H^3} \\ \eta_3 &= \frac{(H_m^3 - H^3)^2 \delta (1 - \delta) \cos^2 \alpha + H^3 H_m^3}{(1 - \delta) H_m^3 + \delta H^3} \\ \eta_4 &= \frac{H_m^3 - H^3}{(1 - \delta) H_m^3 + \delta H^3}\end{aligned}$$

[2] η_{i0} ($i=1\sim 4$) is obtained by substituting H_0 and H_T instead of H and H_m into η_i ($i=1\sim 4$) respectively.

$$\begin{aligned}\eta_{11} &= \frac{3}{\{(1 - \delta) H_T^3 + \delta H_0^3\}^2} \{-H_0^2 H_T^2 \\ &\quad \times [\delta H_0^4 + (1 - \delta) H_T^4] + \{ (H_T^3 - H_0^3) \\ &\quad \times [-(1 - \delta) H_T^2 + \delta H_0^2] + 2 H_T^2 H_0^2 \\ &\quad \times (H_T^3 - H_0^3) [(1 - \delta) H_T - \delta H_0] \} \\ &\quad \times \delta (1 - \delta) \sin^2 \alpha\} \\ \eta_{21} &= \frac{-3 (H_T^3 - H_0^3) \delta (1 - \delta) \sin \alpha \cos \alpha}{\{(1 - \delta) H_T^3 + \delta H_0^3\}^2} \\ &\quad \times \{- (H_T^3 + H_0^3) [(1 - \delta) H_T^2 - \delta H_0^2] \\ &\quad + 2 H_T^2 H_0^2 [(1 - \delta) H_T - \delta H_0]\}\end{aligned}$$

$$\begin{aligned}\eta_{31} &= \frac{3}{\{(1 - \delta) H_T^3 + \delta H_0^3\}^2} \{-H_0^2 H_T^2 \\ &\quad \times [\delta H_0^4 + (1 - \delta) H_T^4] + \{ (H_T^3 - H_0^3) \\ &\quad \times [-(1 - \delta) H_T^2 + \delta H_0^2] + 2 H_T^2 H_0^2 \\ &\quad \times (H_T^3 - H_0^3) [(1 - \delta) H_T - \delta H_0] \} \\ &\quad \times \delta (1 - \delta) \cos^2 \alpha\} \\ \eta_{41} &= \frac{3 H_0 H_0^2 H_T^2}{\{(1 - \delta) H_T^3 + \delta H_0^3\}^2}\end{aligned}$$

[3]

$$\begin{aligned}E_1 &= 255/32, E_2 = -741/32, E_3 = 1314/32, \\ E_4 &= -1630/32, E_5 = 1395/32, E_6 = -777/32, \\ E_7 &= 252/32, E_8 = -36/32. \\ B_1 &= 4 \ln 2 - 2.5, B_2 = 8.5 - 12 \ln 2, B_3 = 24 \ln 2 - \\ &16.5, B_4 = (167/6) - 40 \ln 2, B_5 = 60 \ln 2 - 41.5, B_6 = \\ &58.3 - 84 \ln 2, B_7 = 112 \ln 2 - (2327/30), B_8 = (6991/70) - 144 \ln 2.\end{aligned}$$

References

- (1) Bootsma, J., Trans. ASME, Ser. F, Vol. 97, No. 2 (1975-4), p. 236.
- (2) Hsing, F.C., Trans. ASME, Ser. F, Vol. 94, No. 1 (1972-1), p. 81.
- (3) Reinhoudt, J.P., Thesis, Technological Univ. Eindhoven, (1972).
- (4) Muijderland, E.A., Spiral Groove Bearings, (1966), p. 116, Philips Technical Library.