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Considerations of a Penalty Shifting Method for Nonlinear Programming Problems*

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Ordinary Lagrange function and penalty function methods are typical methods for solving nonlinear programming problems with constraints. The application of the ordinary Lagrange function method is in general limited to convex programming problems. The penalty function method can be applied to nonconvex programming problems, but may suffer from numerical difficulties. The penalty shifting method is superior to the above two methods, since it can be applied to nonconvex programming problems without suffering from numerical difficulties.

In this paper two geometric interpretations for the penalty shifting method are mainly given, so that its characteristics may be more clarified. First, the correction rules of two parameters and the relation between one of the parameters and the existence of a saddle point are investigated by the ordinary geometric interpretation. Secondly, the operation of the parameters is also investigated by a new geometric interpretation, and finally, the result for a nonconvex programming problem is illustrated by two geometric interpretations.

1. Introduction

Let us consider nonlinear programming problems of minimizing an objective function with inequality constraints. The penalty function method is one of the powerful numerical methods in which the penalty function is such that the constraints are combined with the objective function using the parameter called the penalty coefficient. Under some value of penalty coefficient, the penalty function is minimized, and the penalty coefficient is corrected successively. Optimal solution of the original problem can be determined sequentially as the operation is repeated. As for penalty function method, several kinds of penalty functions were proposed. Among them, exterior and interior penalty function methods (hereafter called exterior method and interior method, respectively) are well known. In these methods, it is theoretically needed to increase the penalty coefficient to infinity in order to determine the optimal solution. In numerical calculation, however, the minimization of the penalty function becomes difficult.

For equality constraint problems Powell proposed a new method, in which another new parameter besides the penalty coefficient was introduced to overcome the numerical difficulty⁽¹⁾. In the previous

paper the method of Powell was extended to an optimal control problem with equality and inequality constraints, and the numerical results were given⁽²⁾. The extended method was termed the penalty shifting method. On the other hand, when the transformation function (which was called multiplier function or augmented Lagrange function) which was proposed for equality constrained problems by Hestenes and was extended to problems with inequality constraints by Rockafellar, is used, then the ill-condition of numerical calculation does not occur either^{(3),(4)}. The augmented Lagrange function of Rockafellar is closely related to the function of the penalty shifting method⁽⁵⁾. The algorithm of the penalty shifting method has a special feature that the parameter corresponding to the penalty coefficient takes a different and unfixed value for each constraint.

Recently, the meanings of necessary condition and sufficient condition in the ordinary Lagrange function method have been clarified by a geometric interpretation⁽⁶⁾. The interpretation can be applied in the same way to the penalty function method and the multiplier method, to illustrate the difference of the characteristics between those methods intuitively^{(7),(8)}.

In this paper, the following matters are considered in order to clarify further the properties of the penalty shifting method. First, in Chapter 2, the penalty shifting method is summarized, and in a simple example the changes in the shape of the transformation function for the corrections of two parameters included in the function are illustrated with sequential values of two parameters. By this example, it can be explained intuitively that the parameter does not take a very large

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value unlike the penalty function method, that is, the optimal solution can be determined without making a sharp gradient of the function around the solution. In Chapter 3, a theoretical basis of the proposed algorithm is shown from a view-point differing from the previous paper⁽²⁾. That is, in the algorithm one parameter makes the existence of a saddle point possible, and another parameter searches the saddle point. Moreover, in Chapter 4, from a geometric view-point the correction rules of two parameters are illustrated, especially the effectiveness of this method for nonconvex function problems is shown by means of two geometric interpretations. From the above considerations various meanings for the roles of the proposed transformation function and of its parameters are derived, and the characteristics of penalty shifting method are more clarified.

2. Penalty Shifting Method

2.1 Summary of the method

A problem of minimizing the objective function $f(x)$ subject to inequality constraints, $g_i(x) \geq 0$, $i=1, \dots, m$ is termed an original problem, and its optimal solution is described as $x^* = (x_1^*, \dots, x_n^*)^T$, where the superscript T denotes a transpose. It is assumed that $f(x)$ and $g_i(x)$, $i=1, \dots, m$ are continuously differentiable.

In the penalty shifting method, using two parameters $\sigma = (\sigma_1, \dots, \sigma_m)^T$, $\theta = (\theta_1, \dots, \theta_m)^T$, the original problem is transformed into the following problem to be solved. The following transformation function is introduced.

$$\phi(x, \sigma, \theta) = f(x) + \sum_{i=1}^m \sigma_i \{ [g_i(x) + \theta_i] \times H_i[g_i(x) + \theta_i] \} \quad (1)$$

where $H(z)$ is the Heaviside step function defined as

$$H(z) = \begin{cases} 0, & \text{if } z \geq 0, \\ 1, & \text{if } z < 0. \end{cases} \quad (2)$$

Under some σ and θ , $\phi(x, \sigma, \theta)$ is minimized as to x . This problem is called a transformation problem, and its solution is described as $\hat{x}$. For a problem with inequality constraints a flow diagram of the algorithm of the penalty shifting method is shown in Fig.1. A superscript (k) in

Fig. 1 A flow diagram of the algorithm

this figure denotes an iteration number.

The solution $\hat{x}$ which minimizes a transformation function under some $\sigma > 0$ and $\theta \leq 0$ becomes a solution of the following problem.

Minimize $f(x)$ subject to the constraints

$$g_i(x) \geq v_i(\hat{x}), \quad i=1, \dots, m \quad (3)$$

where $v_i(\hat{x})$ is given as follows.

$$\begin{aligned} v_i(\hat{x}) &= 0, & \text{if } g_i(\hat{x}) > 0 \text{ and } \theta_i = 0 \\ v_i(\hat{x}) &= g_i(\hat{x}), & \text{otherwise} \end{aligned} \quad (4)$$

From the above result when σ and θ make $v_i(\hat{x})$ zero, the solution $\hat{x}$ of the transformation problem becomes the optimal solution x^* . Therefore, it is sufficient to find the parameters σ and θ which make $v_i(\hat{x})$ zero.

The parameters $\sigma > 0$ and θ are corrected by either of the following corrections.

When a convergence goes well, i.e., a convergence rate of $c^{(k)}$ is less than or equal to $(1/b)$, the correction of the loop 1 is performed, where $c^{(k)}$ is defined as $c^{(k)} = \max\{[v_i(\hat{x})]^2\}$. In the loop 1, σ is fixed, and only θ is corrected as follows.

$$\begin{aligned} \theta_i^{(k+1)} &= [\theta_i^{(k)} + g_i(\hat{x}^{(k)})] H_i[g_i(\hat{x}^{(k)}) + \theta_i^{(k)}], \\ i &= 1, \dots, m \end{aligned} \quad (5)$$

The loop 2 is used when the convergence rate of $c^{(k)}$ is slow by the correction in the loop 1. For a constraint i where the rate of convergence is rapid, the correction (5) is performed. For the other constraints i ,

$$\sigma_i^{(k+1)} = a \sigma_i^{(k)}, \quad \theta_i^{(k+1)} = \frac{1}{a} \theta_i^{(k)}, \quad a > 1 \quad (6)$$

are performed.

For the penalty shifting method the following theoretical results have already been obtained^{(1),(2)}. The parameters σ_i and θ_i , $i=1, \dots, m$ and the Lagrange multiplier λ_i , $i=1, \dots, m$ are related as $\lambda_i^* = -2\sigma_i^* \theta_i^*$, $i=1, \dots, m$ where superscript $*$ denotes an optimal value. $c^{(k)}$ in Fig.1 which indicates the norm of constraint being the least satisfactory converges to zero when the present algorithm is used. Under proper conditions it is proved that $\{f(\hat{x}^{(k)})\}$ and $\{\hat{x}^{(k)}\}$ converge to $f(x^*)$ and x^* , respectively. Moreover, the convergence rate is improved as σ increases.

2.2 Numerical considerations

The present method is called the penalty shifting method, because a range of

Fig. 2 The trajectory of the sequential solution and a contour of the penalty function T in the exterior penalty method.

active constraints is shifted. A range which receives a penalty, i.e., $g_i(x) + \theta_i \leq 0$ is expanded as a sequential solution x approaches a boundary of a constraint from outside of the feasible region. Therefore, the shape of the function $\phi(x, \sigma, \theta)$ is prevented from becoming a deep and narrow valley. Moreover, when $f(x)$, $g_i(x)$, $i=1, \dots, m$ are twice continuously differentiable functions, the function $\phi(x, \sigma, \theta)$ is also twice continuously differentiable on the boundaries. Therefore, the minimization of $\phi(x, \sigma, \theta)$ in the neighborhood of the boundaries becomes easy. Thus, this method does not suffer from numerical instability.

In this section let us explain the transition of the transformation function when two parameters are corrected and the difference from the existing penalty function method.

An example is to determine $x^* = (x_1^*, x_2^*)^T$ which minimizes

$$f(x) = 3(x_1 - 5)^2 + 2(x_1 - 5)(x_2 - 5) + 3(x_2 - 5)^2 \dots (7)$$

subject to the following constraints

$$g_1(x) = 4 - x_1 - x_2 \geq 0 \dots (8)$$

$$g_2(x) = x_2 - x_1 - 2 \geq 0 \dots (9)$$

The optimal solution of the problem is $x^* = (1, 3)^T$ and $f(x^*) = 76$.

This example is solved by the exterior penalty function method and the penalty shifting method, and the two methods are compared. The penalty function for the above problem is given as follows.

$$T(x, r) = f(x) + r \sum_{i=1}^2 [g_i(x) H_i(g_i(x))]^2 \dots (10)$$

In the exterior penalty method, in order to determine x^* , the penalty coefficient r must be increased to infinity theoretically, but the gradient of the contour of the penalty function increases without bound near the boundary of constraints as r increases. When the contour map of the penalty function in Fig. 2 (the case of $k=$

7, $r=10^5$ in the exterior penalty method, where the shaded portion shows the feasible region) is compared with the following contour map of the penalty shifting method, the above mentioned property is evident from the results. And since the derivatives of the penalty function become extremely large as the iteration goes on, it becomes difficult to determine the sequential solution accurately. (When the interior penalty function method is used, the same results are obtained, but omitted in this paper.)

The same example is solved by the penalty shifting method, and the result is shown in Table 1, and the contour maps of the function at $k=2 \sim 4$ are shown in Fig. 3 (a)~(c). For the initial values, $x^{(0)} = (0, 0)^T$, $\sigma^{(0)} = (1, 1)^T$ and $\theta^{(0)} = (0, 0)^T$ are used, and the values of a and b are 10 and 2, respectively. In this case the sequential solution approaches the optimal solution from outside of the feasible region.

Under the fixed σ_i , only correction of θ_i corresponds to that of the Lagrange multiplier. The change from Fig. 3(b) to (c) is one example of this case. In the case where the sequential solution approaches the optimal one from outside of the feasible region, the sequential solution agrees with the optimal one of the original problem almost without change of the contour map as $|\theta_i|$ increases.

In Fig. 3(a) ($k=2$), since it is judged that the distance from the sequential solution to the boundary of x_1 is greater than $1/2$ of that of the previous iteration, i.e., that the convergence rate is slow, the loop 2 is performed, i.e., the values of σ_i and θ_i are multiplied by 10 and $1/10$, respectively. As the result, the sequential solution in Fig. 3(b) approaches the boundary, i.e., $g_1(x) = 0$, nearer than that in Fig. 3(a). In the case where the se-

Table 1. A numerical result for the penalty shifting method

k	z_1	z_2	λ_1	λ_2	σ_1	σ_2	θ_1	θ_2	$f(z)$	$\phi(z, \sigma, \theta)$
0	0.0	0.0	0.0	0.0	1.0	1.0	0.0	0.0	200.000 0	204.000 0
1	3.500 00	4.500 00	0.0	0.0	1.0	1.0	0.0	0.0	9.000 0	26.000 0
2	2.583 33	4.063 33	8.000 0	2.000 0	1.0	1.0	-4.000 00	-1.000 00	24.472 2	71.166 7
3	1.458 33	3.208 33	13.333 3	3.000 0	10.0	1.0	-0.400 00	-1.500 00	59.951 4	74.391 7
4	1.118 06	2.993 06	21.333 3	3.500 0	10.0	1.0	-1.066 67	-1.750 00	72.873 7	90.260 9
5	1.040 51	2.978 01	23.555 6	3.750 0	10.0	1.0	-1.177 78	-1.875 00	75.310 2	93.375 3
6	1.007 23	2.995 86	23.925 9	3.875 0	10.0	10.0	-1.196 30	-0.187 50	75.880 6	90.661 3
7	1.000 77	2.999 74	23.987 7	3.977 3	10.0	10.0	-1.199 38	-0.198 86	75.983 5	90.780 6
8	1.000 09	3.000 00	23.997 9	3.997 9	10.0	10.0	-1.199 90	-0.199 90	75.997 6	90.797 1
9	1.000 01	3.000 00	23.999 7	3.999 8	10.0	10.0	-1.199 98	-0.199 99	75.999 6	90.799 6
10	1.000 00	3.000 00	23.999 9	3.999 9	10.0	10.0	-1.200 00	-0.200 00	75.999 9	90.799 9

(a) $k=2$

(b) $k=3$

(c) $k=4$

Fig. 3 Contours of the transformation function T in the penalty shifting method

quential solution approaches rapidly one of the boundaries, since the solution does not stay at almost the same place for a long time until approaching the other boundaries, it is expected that a rapid convergence may be obtained as a whole. If this correction is performed many times, the shape of the contour map in $g_i(x) + \theta_i \leq 0$ becomes sharp. Since the optimal solution is obtained when $\sigma=10$ in this example, the gradient of the transformation function near the optimal solution is much less than that of the exterior penalty function.

When the sequential solution approaches the optimal one from inside of the feasible region, the same results can be obtained, but omitted in this paper.

In this method since the transformation function is defined in both outside and inside of the feasible region unlike the interior penalty function method, any value can be chosen as a starting point $x^{(0)}$. Generally speaking, if $|\sigma_i^{(0)} \theta_i^{(0)}|$ for all i is chosen less than $|\sigma^* \theta^*|$, the sequential solution approaches the optimal one from outside of the feasible region; if $|\sigma_i^{(0)} \theta_i^{(0)}|$ is chosen larger than $|\sigma^* \theta^*|$, it approaches that from inside of the feasible region. This is independent of the initial value of x . Though the value of $|\sigma^* \theta^*|$ is unknown a priori, in the former case it is sufficient to take $\theta_i^{(0)}$ zero and in the latter case it is sufficient to take $|\theta_i^{(0)}|$ sufficiently large. When the sequential solution approaches the optimal solution from outside of the feasible region, the objective function $f(x)$ increases and converges to the optimal value. On the other hand, when it approaches that from inside, the objective function $f(x)$ decreases and converges to the optimal value.

3. Properties of the Algorithm

In this Chapter it is shown that in the present algorithm the role of σ is to make the existence of the saddle point possible, and θ plays a role in searching the saddle point. Now, let us consider the following function $\Phi_L(x, \sigma, \theta)$ which is made by subtracting $\sum_{i=1}^m \sigma_i \theta_i^2$ from $\Phi(x, \sigma, \theta)$ in Eq.(1). $\Phi_L(x, \sigma, \theta)$ is transformed in such away that $\Phi_L(x, \sigma, \theta)$ agrees with $f(x)$ in the optimal solution.

$$\Phi_L(x, \sigma, \theta) = f(x) + \sum_{i=1}^m \sigma_i [(g_i(x) + \theta_i) \times H_i(g_i(x) + \theta_i)]^2 - \theta_i^2 \quad \dots\dots\dots(11)$$

Of course the value of $\Phi_L(x, \sigma, \theta)$ in Eq.(11) is different from the value of $\Phi(x, \sigma, \theta)$ in Eq.(1) at each iteration, but there is no change in the proposed algorithm by using $\Phi_L(x, \sigma, \theta)$. If Eq.(11) is used, the following results which are analogous to those for the exponential transformation method can be proved⁽⁹⁾.

The properties for a saddle point problem and a dual problem, in the same way as for the ordinary Lagrange function, can be proved for the present method using the function $\Phi_L(x, \sigma, \theta)$. Based on those properties, if a saddle point exists, the solution of the original problem can be obtained by determining a saddle point. And

a saddle point can be determined by repetition of the procedure that the function in the unconstrained problem which is transformed from the original problem is minimized as to x under some σ and θ and subsequently the function in dual problem is maximized as to θ under some x which gives the minimum value. The correction of θ by a gradient method which is to increase the dual function is as follows.

$$\theta_i^{(k+1)} = \begin{cases} \theta_i^{(k)} + \alpha \frac{1}{4\sigma_i^2} \frac{\partial L_\Phi(\theta^{(k)})}{\partial \theta_i} \\ \theta_i^{(k)} + \alpha \frac{1}{4\sigma_i^2} \frac{\partial L_\Phi(\theta^{(k)})}{\partial \theta_i} < 0 \\ 0, & \theta_i^{(k)} + \alpha \frac{1}{4\sigma_i^2} \frac{\partial L_\Phi(\theta^{(k)})}{\partial \theta_i} \geq 0 \end{cases} \quad \dots\dots\dots(12)$$

where $\alpha > 0$ and $\partial L_\Phi(\theta^{(k)}) / \partial \theta_i$ is

$$\frac{\partial L_\Phi(\theta^{(k)})}{\partial \theta_i} = \begin{cases} 2\sigma_i g_i(\hat{x}^{(k)}), & g_i(\hat{x}^{(k)}) < -\theta_i^{(k)} \\ -2\sigma_i \theta_i^{(k)}, & g_i(\hat{x}^{(k)}) \geq -\theta_i^{(k)} \end{cases} \quad \dots\dots\dots(13)$$

When α is taken as $2\sigma_i$, Eq.(12) agrees with the correction (5). Therefore, the correction (5) can be considered a kind of correction which increases the dual function described above, and the loop 1 in the present algorithm can be considered a solution of dual problems. It is assumed that the Hessian of $\Phi_L(x, \sigma, \theta)$ at $(\bar{x}, \bar{\theta})$ is positive definite for x . Since the algorithm which uses the correction (12) is a gradient method, the local convergence to a saddle point can be proved. Since positive definiteness of the Hessian for $\Phi_L(x, \sigma, \theta)$ at $(\bar{x}, \bar{\theta})$ and at any $\bar{\sigma}$ is the same as the existence of a saddle point for $\Phi_L(x, \sigma, \theta)$ at $(\bar{x}, \bar{\sigma}, \bar{\theta})$, when the algorithm searches a saddle point, the existence of a saddle point is required. The correction of in the loop 2 plays a role in making the existence of a saddle point possible⁽¹⁰⁾.

From the above considerations it is concluded that in the present method the correction of σ in the loop 2 makes the existence of a saddle point possible, and the correction of θ searches a saddle point. The relation between the existence of a saddle point and the correction of σ may be made more clear by means of geometric interpretations described in the next Chapter.

4. Geometric Interpretations

In this Chapter let us demonstrate mainly the procedure of corrections of two parameters by geometric interpretations. Especially, the relationship between the existence of a saddle point and the parameter will be explained by using two geometric interpretations, and the effectiveness for problems with nonconvex function will be confirmed.

4.1 Formulation by geometric representation

Let us introduce the parameter $y = (y_1, \dots, y_m)^T$ in the righthand side of the constraints in the original problem, and consider the following problem.

The objective function $f(x)$ is minimized subject to the constraints

$$g_i(x) \geq y_i, \quad i=1, \dots, m \quad \dots\dots\dots(14)$$

This problem is termed Problem A. When y

is zero, Problem A agrees with the original problem. $w(y)$ is defined as

$$w(y) = \min \{f(x) | g_i(x) \geq y_i, i=1, \dots, m\} \quad (15)$$

where $w(y)$ is considered in the domain of y where there exist some x such that $g_i(x) \geq y_i, i=1, \dots, m$. Evidently, $w(y)$ is a nondecreasing function, and it is well known that $w(y)$ is a convex function when $f(x)$ and $-g_i(x), i=1, \dots, m$ are convex functions⁽⁶⁾.

The ordinary Lagrange function is

$$L(x, \lambda) = f(x) - \sum_{i=1}^m \lambda_i g_i(x) \quad (16)$$

Then, at some $(\hat{y}, \hat{\lambda})$, $s_0(y, \hat{\lambda})$ is defined as

$$\begin{aligned} S_0(y, \hat{\lambda}) &= w(\hat{y}) - \sum_{i=1}^m \hat{\lambda}_i \hat{y}_i + \sum_{i=1}^m \hat{\lambda}_i y_i \\ &= L(\hat{x}, \hat{\lambda}) + \sum_{i=1}^m \hat{\lambda}_i y_i, \quad \hat{\lambda}_i \geq 0, \quad i=1, \dots, m \end{aligned} \quad (17)$$

When $y_0 = s_0(y, \hat{\lambda})$ supports $y_0 = w(y)$ at $[w(\hat{y}), \hat{y}]$ as shown in Fig. 4(a), $y_0 = s_0(y, \hat{\lambda})$ is called a supporting hyperplane, where $\hat{x}$ is the value of x which minimizes $L(x, \hat{\lambda})$ and y is $g(x)$. To support means that $s_0(y, \hat{\lambda})$ satisfies

$$S_0(\hat{y}, \hat{\lambda}) = w(\hat{y}), \quad S_0(y, \hat{\lambda}) \leq w(y) \quad (18)$$

at some $\lambda = \hat{\lambda}$. In Fig. 4 the set R which is shaded, i.e., which is the upper part of $y_0 = w(y)$ is defined as

$$R = \{(y_0, y) | y_0 \geq w(y)\} \quad (19)^*$$

If $y_0 = s_0(y, \hat{\lambda})$ supports $y_0 = w(y)$ at $[w(\hat{y}), \hat{y}]$, then $[\hat{x}, \hat{\lambda}]$ is a saddle point for $L(x, \lambda)$. The converse is also true. If $w(y)$ is convex, there exists $\hat{\lambda} \geq 0$ such that $y_0 = s_0(y, \hat{\lambda})$ supports $y_0 = w(y)$ at $[w(\hat{y}), \hat{y}]$.

(a) An example where $s_0(y, \hat{\lambda})$ can support $w(y)$ at $\hat{y}$ when $w(y)$ is a convex function.

(b) An example where $s_0(y, \hat{\lambda})$ cannot support $w(y)$ at $\hat{y}$ when $w(y)$ is a nonconvex function.

Fig. 4 A supporting hyperplane $s_0(y, \hat{\lambda})$

The ordinary Lagrange function method can be applied only the problems in which $w(y)$ is convex near the solution⁽⁶⁾. If $w(y)$ is a nonconvex function, $s_0(y, \hat{\lambda})$ defined by Eq. (17) cannot support $w(y)$, and another type of function is needed in order to support a nonconvex $w(y)$ (see Fig. 4 (a) and (b); s_1 and s_2 in (b) are explained later.).

A geometric interpretation which is extended from that for the ordinary Lagrange function $L(x, \lambda)$ is performed for the penalty shifting function given by Eq. (11). In this case the following $s(y, \hat{\sigma}, \hat{\theta})$ corresponding to $s_0(y, \hat{\lambda})$ is considered⁽¹¹⁾.

$$\begin{aligned} S(y, \hat{\sigma}, \hat{\theta}) &= w(\hat{y}) + \sum_{i=1}^m \hat{\sigma}_i [(y_i + \hat{\theta}_i) H_i(y_i + \hat{\theta}_i)]^2 - \hat{\theta}_i^2 \\ &\quad - \sum_{i=1}^m \hat{\sigma}_i [(y_i + \hat{\theta}_i) H_i(y_i + \hat{\theta}_i)]^2 - \hat{\theta}_i^2 \\ &= \phi_L(\hat{x}, \hat{\sigma}, \hat{\theta}) - \sum_{i=1}^m \hat{\sigma}_i [(y_i + \hat{\theta}_i) \\ &\quad \times H_i(y_i + \hat{\theta}_i)]^2 - \hat{\theta}_i^2 \end{aligned} \quad (20)$$

When $y_0 = s(y, \hat{\sigma}, \hat{\theta})$ supports $y_0 = w(y)$ at $[w(\hat{y}), \hat{y}]$, i.e., $[f(\hat{x}), g(\hat{x})]$, then $y_0 = s(y, \hat{\sigma}, \hat{\theta})$ is termed a supporting hypersurface. In this case to support means that for some $\sigma = \hat{\sigma}$ and $\theta = \hat{\theta}$, $s(y, \sigma, \theta)$ satisfies

$$S(\hat{y}, \hat{\sigma}, \hat{\theta}) = w(\hat{y}), \quad S(y, \hat{\sigma}, \hat{\theta}) \leq w(y) \quad (21)$$

Using the above supporting hypersurface, the following results can be obtained. When $y_0 = s(y, \hat{\sigma}, \hat{\theta})$ supports $y_0 = w(y)$ at $[w(\hat{y}), \hat{y}]$, then $(\hat{x}, \hat{\sigma}, \hat{\theta})$ is a saddle point for $\phi_L(x, \sigma, \theta)$. The converse is also true.

4.2 Interpretation of the procedure of correction in parameters

The procedure of corrections in parameters in Chapter 2 is interpreted geometrically.

In order to understand the correction in the loop 1, let us assume that θ_j is corrected for only j th element of the constraint function $g(x)$. Fig. 5(a) shows the case where the sequential solution approaches from outside of the constraints in the original problem. Hypersurface $\mathcal{L}_1$ shows the supporting one at the k th iteration in the present method. By the correction of Eq. (5) and the minimization of $\phi_L(x, \sigma, \theta)$, the supporting hypersurface $\mathcal{L}_2$ is obtained at the $(k+1)$ th iteration. (Since $|\theta_j^{(k+1)}| > |\theta_j^{(k)}|$ holds, the supporting hypersurface moves to the right.) Even if θ_j is corrected, the shape of the supporting hypersurface is not changed, and the supporting point moves only from a to b . For the case of the nonconvex function shown in Fig. 5(a), by repeating this correction the optimal solution which is shown by point c in Fig. 5(a) can be obtained for finite values of both σ and θ .

Next, let us consider the correction

* When the minimization of $f(x)$ is determined in inside of $g_i(x) \geq 0, i=1, \dots, m$, $w(y)$ takes a constraint value in $y_i \leq g_i(x^*)$, $i=1, \dots, m$. In the case almost the same treatment can be done. Hereafter, it is assumed that all constraints are active, and the discussions are made by using graph of $y_0 = w(y)$. In this case it is noted that $[w(\hat{y}), \hat{y}]$ becomes $[f(\hat{x}), g(\hat{x})]$.

in the loop 2. It is assumed that both σ_j and θ_j of the j th element are simultaneously corrected, and that the supporting hypersurface of the k th iteration is l_4 in Fig.5(b). By the correction that $\sigma_j^{(k)}$ is increased and $|\theta_j^{(k)}|$ is decreased, first, at the $(k+1)$ th iteration the supporting hypersurface moves to the hypersurface l_5 . The top d of the hypersurface l_4 moves to smaller value of y_j until e , since $|\theta_j^{(k+1)}|$ is smaller than $|\theta_j^{(k)}|$. Moreover, since the value of σ_j for the hypersurface l_5 is greater than that for the hypersurface l_4 , the curvature for the hypersurface l_5 is greater than that for hypersurface l_4 . Next, when $\phi_L(x, \sigma^{(k+1)}, \theta^{(k+1)})$ is minimized for the values of $\sigma_j^{(k+1)}$ and $\theta_j^{(k+1)}$, the hypersurface l_5 changes to the hypersurface l_6 . In this case the supporting point moves from f to g . Even if the curvature of the initial hypersurface is rather small, the point c which gives the optimal solution of the original problem can be obtained by repeating the correction in the loop 2. Generally speaking, degree of openness of the supporting hypersurface approaches a right angle at the top of the hypersurface (for instance, d and e) as σ_j increases, and it is easy to support $w(y)$ whose nonconvexity is great. The detail of the relation between the magnitude of σ and supportability is described in the next section. By using the loops 1 and 2 simultaneously, this method can be applied to nonconvex function problems, and the optimal solution can be obtained with finite values of parameters σ and θ .

(a) θ_j is only corrected.

(b) Both σ_j and θ_j of the j th element are simultaneously corrected.

Fig. 5 Movement of the supporting hypersurface when $w(y)$ is a nonconvex function.

4.3 Interpretation for the existence of a saddle point (part 1)

When $w(y)$ is a convex function, $y_0 = w(y)$ can be supported by a supporting hypersurface $y_0 = s(y, \sigma, \theta)$ for any $\sigma > 0$. Hence, let us consider the case where $w(y)$ is a nonconvex function. Now, it is assumed that $y_0 = s(y, \bar{\sigma}, \bar{\theta})$ for $\sigma = \bar{\sigma}$ does not support $y_0 = w(y)$ at $[w(y), y]$ [see the hypersurface s_1 in Fig.4(b)]. In this case there exists $\bar{\sigma}$ such that $y_0 = w(y)$ can be supported by the hypersurface $s(y, \sigma, \theta)$ for any σ which satisfies $\bar{\sigma} < \sigma < \bar{\sigma}$. This fact can be proved as follows. Evidently from the definition (20), $s(y, \sigma, \theta)$ is a non-increasing function as σ increases, i.e., the curvature increases as σ increases. And when the values of $\theta_i, i=1, \dots, m$ are chosen the magnitude of infinity,

$$\begin{aligned} s(y, \bar{\sigma}, \bar{\theta}) &= -\infty \text{ for } y_i < \bar{y}_i \\ s(y, \bar{\sigma}, \bar{\theta}) &= w(y) \text{ for } y_i \geq \bar{y}_i \end{aligned} \quad (22)$$

holds. Therefore, even if $y_0 = s(y, \sigma, \theta)$ using σ cannot support, $y_0 = s(y, \sigma, \theta)$ using some $\bar{\sigma}$ which is greater than σ can support. Generally, it is sufficient that the value of $\bar{\sigma}$ be finite. (see the supporting hypersurfaces s_1 and s_2 in Fig.4(b).)

From the above results, the following property for the role of σ can be proved. Whether the function $w(y)$ is convex or nonconvex, it is possible to choose σ such that the hypersurface $y_0 = s(y, \sigma, \theta)$ supports $y_0 = w(y)$ at $[w(y), y]$.

4.4 Interpretation for the existence of a saddle point (part 2)

Let us interpret geometrically the existence of a saddle point by an approach which differs from that described in the previous section. $\phi_L(x, \sigma, \theta)$ given by Eq. (11) is rewritten as follows.

$$\begin{aligned} \phi_L(x, \sigma, \theta) &= \left[f(x) + \sum_{i=1}^m \sigma_i \{ g_i(x) H_i[g_i(x) + \theta_i] \}^2 \right] \\ &+ \left[\sum_{i=1}^m \sigma_i \{ [2g_i(x)\theta_i + \theta_i^2] H_i[g_i(x) + \theta_i] - \theta_i^2 \} \right] \dots (23) \end{aligned}$$

Let us consider the following problem instead of Problem A. $F[f(x), y, \sigma, \theta]$ is minimized subject to constraints

$$g_i(x) \geq y_i, \quad i=1, \dots, m \quad (24)$$

where

$$F[f(x), y, \sigma, \theta] = f(x) + \sum_{i=1}^m \sigma_i \{ y_i H_i[y_i + \theta_i] \}^2 \quad (25)$$

This problem is denoted as Problem B. Moreover, $v(y, \hat{\sigma}, \hat{\theta})$ and $u(y, \hat{\sigma}, \hat{\theta})$ which correspond to $w(y)$ and $s(y, \hat{\sigma}, \hat{\theta})$, respectively, in the previous interpretation are defined as

$$v(y, \hat{\sigma}, \hat{\theta}) = \min_{i=1, \dots, m} \{ F[f(x), y, \hat{\sigma}, \hat{\theta}] | g_i(x) \geq y_i \} \quad (26)$$

$$\begin{aligned} u(y, \hat{\sigma}, \hat{\theta}) &= v(y, \hat{\sigma}, \hat{\theta}) + \sum_{i=1}^m \hat{\sigma}_i \{ [2\hat{y}_i \hat{\theta}_i + \hat{\theta}_i^2] \\ &\times H_i[\hat{y}_i + \hat{\theta}_i] - \hat{\theta}_i^2 \} - \sum_{i=1}^m \hat{\sigma}_i \{ [2\hat{y}_i \hat{\theta}_i + \hat{\theta}_i^2] \\ &\times H_i[\hat{y}_i + \hat{\theta}_i] - \hat{\theta}_i^2 \} \dots (27) \end{aligned}$$

Then, it means to satisfy the following equations that $y_0 = u(y, \hat{\sigma}, \hat{\theta})$ supports $y_0 = (y, \hat{\sigma}, \hat{\theta})$ at $[v(y, \hat{\sigma}, \hat{\theta}), y]$.

$$u(y, \hat{\sigma}, \hat{\theta}) = v(y, \hat{\sigma}, \hat{\theta}), \quad u(y, \hat{\sigma}, \hat{\theta}) \leq v(y, \hat{\sigma}, \hat{\theta}) \quad (28)$$

These functions are shown in Fig.6. $u(y, \sigma, \theta)$ is constructed from two hyperplanes

where R' is defined as

$$R' = \{(y, \sigma) | y_0 \geq v(y, \hat{\sigma}, \hat{\theta})\} \quad (29)$$

In this case the same properties as described previously are proved. If $y_0 = u(y, \hat{\sigma}, \hat{\theta})$ supports $y_0 = v(y, \hat{\sigma}, \hat{\theta})$ at $[v(y, \hat{\sigma}, \hat{\theta}), \hat{y}]$, then $[x, \hat{\sigma}, \hat{\theta}]$ is a saddle point for $\phi(x, \sigma, \theta)$. The converse is also true. Since Problem B is considered and $F[f(x), y, \sigma, \theta]$ is treated as objective function in the same manner as $f(x)$, the above result is the same relationship as that in the ordinary Lagrange function.

Now, similarly to the case in section 4.3, let us consider the role of σ for the both cases where the function $w(y)$ is convex and nonconvex. First, the range that $y_i < -\hat{\theta}_i$ is considered. When the function $w(y)$ is convex, the function $v(y, \hat{\sigma}, \hat{\theta})$ is also convex for any $\hat{\sigma} > 0$. On the other hand, it is assumed that the function $w(y)$ is nonconvex. When $y_i < -\hat{\theta}_i$, $w(y)$ and $v(y, \sigma, \theta)$ are related as

$$v(y, \sigma, \theta) = w(y) + \sum_{i=1}^m \sigma_i y_i^2,$$

and even if $w(y)$ is nonconvex, $v(y, \hat{\sigma}, \hat{\theta})$ becomes a convex function as σ increases properly. Moreover, when the function $v(y, \hat{\sigma}, \hat{\theta})$ is convex, there exists $\hat{\beta}_i = -2\hat{\sigma}_i \hat{\theta}_i$ such that $y_0 = u(y, \hat{\sigma}, \hat{\theta})$ supports $y_0 = v(y, \hat{\sigma}, \hat{\theta})$ at $[v(y, \hat{\sigma}, \hat{\theta}), \hat{y}]$. This is the same result as that for the ordinary Lagrange function (see Figs. 4(a) and 6). If it is possible to support when $y_i < -\hat{\theta}_i$, then whether $w(y)$ is convex or nonconvex, it is possible to support for proper magnitude of σ when $y_i \geq -\hat{\theta}_i$. From the above consideration the following conclusion is obtained. Whether the function $w(y)$ is convex or nonconvex, if σ is chosen properly, it is possible that $y_0 = u(y, \hat{\sigma}, \hat{\theta})$ supports $y_0 = v(y, \hat{\sigma}, \hat{\theta})$ at $[v(y, \hat{\sigma}, \hat{\theta}), \hat{y}]$.

It means that the above mentioned property is the same as one which is related with the role of σ in the ordinary geometric interpretation described in the section 4.3. However, the interpretations differ from each other. The interpretation in the previous section is as follows; A term including σ is added to the supporting function, that is, Eq. (20) is made such that a term including σ is added to Eq. (17), and the nonconvex function $w(y)$ can be supported. On the other hand, the interpretation in this section is as follows. The term including σ is added to

Fig. 6 An example where $u(y, \hat{\sigma}, \hat{\theta})$ supports $v(y, \hat{\sigma}, \hat{\theta})$.

the supported function, that is, $F[f(x), y, \sigma, \theta]$ is made such that a term including σ is added to $f(x)$, and the supported function becomes a convex function which is supported by the supporting hyperplane in the same way as in the ordinary Lagrange function.

4.5 Numerical example

A simple example with a nonconvex function is solved by the penalty shifting method, and the results are illustrated by means of the above two geometric interpretations. An objective function

$$f(x) = \begin{cases} \frac{1}{25}x^2 + \left(\frac{5\sqrt{6}}{24} - \frac{8}{5}\right)x + \frac{35\sqrt{6}}{6} \\ + 6, & (x \geq 20) \\ 5\sqrt{x+4} - 10, & (20 \geq x \geq -3) \\ \frac{1}{25}x^2 + \frac{137}{50}x + \frac{143}{50}, & (-3 \geq x) \end{cases} \quad (30)$$

is minimized subject to a constraint $g(x) = x \geq 0$. The optimal solution x^* is zero, and $f(x^*)$ is zero. $f(x)$ in $x > -137/4$ has the same shape as $w(y)$ in Fig. 7(a), since this is one dimensional nonlinear programming problem. When the initial values of x, σ, θ are chosen as $-15, 0.001, -1200$, respectively, in the beginning of the iterations the sequential solution approaches the optimal one from outside of the feasible region. Since it is judged that the convergence is slow when only θ is corrected σ is increased to 1. Then, the sequential solution can be determined in the nonconvex region of the objective function, and it converges to the optimal solution at $k=6$.

Figures 7 and 8 illustrate the cases where the support is made in the regions of convex and nonconvex, respectively. In both figures, (a) illustrates the ordinary interpretation, and (b) illustrates the geometric interpretation described in the section 4.4. (c) shows the shape of the function ϕ which is minimized, that is, Eq. (1) in Chapter 2.

First, let us explain the results shown in Fig. 7. This is the case where $k=1, \sigma=0.001$. In Fig. 7(a), since the function $w(y)$ is convex in the neighborhood of $x^* = -4.2$ though σ is small, $w(y)$ can be supported locally by $y_0 = s(y, \sigma, \theta)$ of which curvature is small. As is clear from an outline of the shape near the supporting point, however, $y_0 = s(y, \sigma, \theta)$ cannot support $w(y)$ in the nonconvex region. In Fig. 7(b) the convex part of $y_0 = v(y, \sigma, \theta)$ is supported by a straight line $y_0 = u(y, \sigma, \theta)$. However, $y_0 = u(y, \sigma, \theta)$ cannot support the nonconvex part of $y_0 = v(y, \sigma, \theta)$. This is the same as the geometric interpretation for the ordinary Lagrange function. The shape of ϕ near the supporting point -4.2 is shown in Fig. 7(c). From this figure it is known that the local minimum of the objective function is given at $x = -4.2$. The value of the objective function is still smaller in the nonconvex region.

Next, Fig. 8 shows the results for $\sigma=1$ when $k=4$. $w(y)$ in Fig. 8(a) is the same shape as that in Fig. 7(a). In this figure the neighborhood of the supporting point

0.6 in the nonconvex region is shown. Since the curvature of $y_0=s(y,\sigma,\theta)$ increases as σ increases, the nonconvex part which cannot be supported by $y_0=s(y,\sigma,\theta)$ in Fig.7(a) can be supported. In Fig.8 (b), $y_0=u(y,\sigma,\theta)$ is a straight line analogously to the case of the ordinary Lagrange function. And in this case, the nonconvex $v(y,\sigma,\theta)$ in Fig.7(b) becomes convex for $y < 1.2$ as shown in Fig.8(b) when σ increases. Therefore, $v(y,\sigma,\theta)$ can be supported by a straight line $y_0=u(y,\sigma,\theta)$. It is known that to increase σ corresponds to the fact that the function ϕ has a minimum point in nonconvex part of the objective function.

5. Conclusions

Penalty shifting method for nonlinear programming problems was considered from various viewpoints, especially for the roles of two parameters. First, along with a summarized explanation of the proposed method it was intuitively illustrated that the present method could determine the optimal solution without suffering from the numerical difficulty. Next, it was described in detail how the algorithm made the existence of a saddle point possible and searched the saddle point. Moreover, the characteristics of the penalty shifting method were clarified by means of two geometric interpretations, and its effectiveness for the nonconvex programming problems was shown. Especially, geometrical support in the 2nd geomet-

ric interpretation proposed in this paper is similar to that in the ordinary Lagrange multiplier method. Therefore, various kinds of results can be explained in relation to the ordinary Lagrange function method, and in this respect the 2nd interpretation has an advantage in that it makes it easy to understand the role of parameters.

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(a) The geometric interpretation (part 1) (b) The geometric interpretation (part 2) (c) The value of ϕ

Fig. 7 The case where the support is made in the region of convex.

(a) The geometric interpretation (part 1) (b) The geometric interpretation (part 2) (c) The value of ϕ

Fig. 8 The case where the support is made in the nonconvex region.