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Turbulent Lubrication Theory Using the Frictional Law*
(1st Report, Derivation of Turbulent Coefficients
and Lubrication Equation)

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This paper presents one of the methods for deriving a turbulent lubrication equation and a solution for the equation.

The turbulent lubrication equation in the study is developed by such a procedure that the wall shear stresses in pressure flows and shear flows are evaluated based on the friction law of fluid, and their stresses are applied to turbulent lubrication by introducing a concept of equivalent pressure flow. From the analytical solutions, the effects of pressure flows on turbulent lubrication which have been neglected hitherto are considered. The effects are remarkable in the superlaminar flow regime in transit from laminar to full turbulent flows.

1. Introduction

In recent years, many lubrication theories introducing turbulent effects have been presented with the increase in speed and scale of machines.

When a transition from laminar flows to turbulent flows occurs, the velocity distribution becomes flat due to the diffusion of fluid particles. Such a phenomenon is recognized with an increase of viscosity. Studies on turbulent lubrication have been developed to determine the turbulent coefficients including an apparent viscosity.

The turbulent coefficients have been derived by the following research workers, that is, Constantinescu⁽¹⁾ applied Prandtl's mixing length model, Ng⁽²⁾ used the eddy diffusion theory and Aoki⁽³⁾ introduced a mixing length theory based on the law of wall. In their papers, the characteristic of turbulent flows is linearized by assuming that the effect of pressure flows is smaller than that of shear flows and the turbulent coefficients depend only on shear flows, because the turbulent lubrication is very complicated. Such a procedure may be valid when shear flows are dominant, but the validity may be doubtful when the pressure flows affect turbulence.

On the other hand, Tao⁽⁴⁾ introduced a concept of bulk flow into the turbulent lubrication theory and analyzed infinitely short width bearings. Recently, Hirs⁽⁵⁾ obtained the turbulent coefficients by applying the bulk flow theory to three dimensional flows. He, however, did not fully dis-

cuss the superposition of pressure flows and shear flows, and his turbulent coefficients have some error in superlaminar flows with a small Reynolds number.

The authors formulate the relation between the mean velocity and the wall shear stress, based on the friction law of fluid, by rearranging a number of experimental data on pressure flows and shear flows, and derive the turbulent coefficients and the turbulent lubrication equation by introducing the concept of the equivalent pressure flow. The effects of pressure flows on the turbulent lubrication are also discussed for various pressure gradients.

2. Nomenclature

- B : bearing width
- C_f : friction factor
- G_x, G_z : turbulent coefficients
- Re_h : Reynolds number
- U, V, W : velocities of sliding surface respectively in x, y, z-directions
- V : velocity vector
- V_m : mean velocity vector
- ($V_m = \frac{1}{h} \int_0^h V^* dy$, where V^* is a local velocity vector in lubricant film)
- h : film thickness
- p : film pressure
- V_p^* : dimensionless pressure gradient
- n : ratio of the wall shear stress on lower wall to that on upper wall
- u, v, w : velocities of oil film respectively in x, y, z-directions
- u_m, w_m : mean velocities respectively in x, z-directions
- x, y, z : coordinates
- α_i, β : inherent constants for a range of Reynolds numbers
- ϵ : eddy viscosity
- κ : ratio of the wall shear stress

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in shear flows to that in pressure flows

μ : viscosity
 ν : kinematic viscosity
 ρ : density
 τ_w : wall shear stress vector

Subscripts

c, p : refer to shear flows and pressure flows
 0, h : refer to surfaces of $y=0$ and $y=h$
 w : refer to surface
 e : refer to equivalent pressure flow

3. Derivation of lubrication equation

3.1 Frictional law

Flows in bearings lubricated hydrodynamically shown in Fig.1 consist of both pressure flows and shear flows, then the flow mechanism in turbulence becomes very complicated.

Let us consider the characteristics of pressure flows and shear flows separately. Shear flows in sliding bearings can be recognized as ones between infinitely long parallel plates having a small clearance and a relative velocity because the clearance is much smaller than other size of bearings. Pressure flows in bearings are nearly the same as pressure flows in a rectangular pipe with a small height h and a large width B ($B \gg h$). By applying such approximations, the relationship between a factor C_f and the Reynolds number R_h in hydrodynamically lubricated bearings can be obtained from the experimental data in the above mentioned channels with a small clearance and smooth surfaces. The experimental data are shown in Fig.2, in which the quantities C_f and R_h are given as follows,

$$C_f = \frac{2\tau_w}{\rho V_{mi}^2}, \quad R_h = \frac{2hV_{mi}}{\nu}; \quad i=p, c \quad (1)$$

where the subscripts p and c refer to pressure and shear flows respectively.

The patterns of pressure and shear flows can be divided in three regimes as shown in Fig.2. The regimes are given according to the Reynolds number as

- ① laminar flow regime ($R_h \leq 10^3$)
 - ② superlaminar flow regime ($10^3 < R_h \leq 1.35 \times 10^4$)
 - ③ fully turbulent flow regime ($1.35 \times 10^4 < R_h$)
- The experimental relationship between

Fig.1 Coordinate of slider bearing

C_f and R_h in each regime can be expressed by the following power form.

$$C_f = \alpha_i R_h^\beta; \quad i=p, c \quad (2)$$

The solid and broken lines in Fig.2 drawn from formula (2) by applying the coefficients in Table 1 agree well with the experimental data, then formula (2) may be valid over a wide range of the Reynolds numbers.

By using the formulas (1) and (2), the relationship between the wall shear stresses τ_{wi} and the mean velocities V_{mi} is given as

$$\tau_{wi} = \frac{\alpha_i}{2} \left(\frac{2hV_{mi}}{\nu} \right)^\beta \rho V_{mi}^2; \quad i=p, c \quad (3)$$

where τ_{wi} and V_{mi} are scalars. From the above formula, the wall shear stresses in pressure and shear flows can be written in the same form.

Sliding bearings have two surfaces, one of which is movable and the other is stationary as shown in Fig.1. The frame of reference is fixed on the stationary surface. Rewriting the physical quantities from scalars to vectors, formula (3) is expressed with the mean velocity vector V_m and the wall shear stress vector τ_w as follows,

$$\tau_w = \frac{\alpha_i}{2} \left(\frac{2h|V_m|}{\nu} \right)^\beta \rho |V_m| V_m; \quad i=p, c \quad (4)$$

The mean velocity and shear stress vector referred to the stationary surface are V_{m0} and τ_0 respectively, and those referred to the sliding surface are V_{mh} and τ_h respectively.

Rearranging formula (4), the following expressions are obtained for each surface.

$$\tau_0 = \frac{\alpha_i}{2} \left(\frac{2h|V_{m0}|}{\nu} \right)^\beta \rho |V_{m0}| V_{m0}; \quad i=p, c \quad (5)$$

$$\tau_h = \frac{\alpha_i}{2} \left(\frac{2h|V_{mh}|}{\nu} \right)^\beta \rho |V_{mh}| V_{mh}; \quad i=p, c \quad (6)$$

The relative velocity vector V is equal to

Table 1 Experimental constants α_i, β and κ

Constants	Flow regimes		
	Laminar flow regime	Superlaminar flow regime	Full turbulent flow regime
α_p	24	0.4370	0.0791
α_c	8	0.3120	0.0637
β	-1	-0.43	-0.25
κ	3	1.40	1.24

Fig.2 Friction factor in pressure and shear flows

the difference between V_{m0} and V_{mh} , that is,

$$V_{m0} - V_{mh} = V \quad (7)$$

From formulas (6) and (7), we have

$$\tau_{ik} = \frac{\alpha_i}{2} \left(\frac{2h|V_{m0}-V|}{\nu} \right)^{\frac{1}{2}} \rho |V_{m0}-V| (V_{m0}-V);$$

$$i=p, c \quad (8)$$

3.2 Concept of equivalent pressure flow

The wall shear stress will be discussed according to Boussinesq's assumption. The wall shear stress in two dimensional flows is in general given by the following expression (10).

$$\tau_s = \mu \frac{\partial u}{\partial y} - \rho \overline{u'v'} \quad (9)$$

Introducing Boussinesq's assumption, the turbulent stress in the above equation can be written as follows,

$$-\rho \overline{u'v'} = \rho \epsilon \frac{\partial u}{\partial y} \quad (10)$$

where u' and v' are the fluctuational velocities due to the transition from laminar to turbulent flows. The quantity ϵ is the eddy viscosity which is a function of position and mean flows, and $\epsilon=0$ in laminar flows.

Rearranging after substituting Eq.(10) into Eq.(9), we have

$$\tau_s = \mu \left(1 + \frac{\epsilon}{\nu} \right) \frac{\partial u}{\partial y} \quad (11)$$

From the equilibrium of forces in the lubricant film,

$$\frac{\partial p}{\partial x} = \frac{\partial \tau_s}{\partial y} \quad (12)$$

Substituting the results obtained by integrating Eq.(11) with respect to y into Eq.(12) and integrating again, the velocity in the x -direction is given as

$$u = \frac{1}{\mu} \left(\frac{\partial p}{\partial x} I_2(y) + C_1 I_1(y) \right) + C_2 \quad (13)$$

where C_1 and C_2 are integral constants, and

$$I_1(y) = \int_0^y \frac{dy}{1+\epsilon/\nu}, \quad I_2(y) = \int_0^y \frac{y dy}{1+\epsilon/\nu} \quad (14)$$

By applying the following boundary conditions

$$u(y=0)=0, \quad u(y=h)=U \quad (15)$$

we obtain the constants C_1 and C_2 as follows,

$$C_1 = \frac{\mu U}{I_1(h)} - \frac{\partial p I_2(h)}{\partial x I_1(h)}, \quad C_2 = 0 \quad (16)$$

From Eqs.(11), (13) and (16), the turbulent shear stress is given as

$$\tau_s = h \frac{\partial p}{\partial x} \left(\frac{y}{h} - \frac{I_2(h)}{I_1(h)} \right) + \frac{\mu U}{I_1(h)} \quad (17)$$

Also, from Eq.(17), we have

$$\tau_{s0} = \tau_s(y=0) = -h \frac{I_2(h)}{I_1(h)} \frac{\partial p}{\partial x} + \frac{\mu U}{I_1(h)} \quad (18)$$

$$\tau_{sh} = \tau_s(y=h) = h \left(1 - \frac{I_2(h)}{I_1(h)} \right) \frac{\partial p}{\partial x} + \frac{\mu U}{I_1(h)} \quad (19)$$

Denoting the wall shear stress in pure shear flows by τ_{wc} and the eddy viscosity by ϵ_c , the expression for τ_{wc} is given, by setting $\partial p/\partial x=0$ in Eqs.(13) and (16), as follows,

$$\tau_{wc} = \frac{\mu U}{I_{1c}(h)}, \quad I_{1c}(h) = \int_0^h \frac{dy}{1+\epsilon_c/\nu} \quad (20)$$

Introducing τ_{wc} into Eqs.(18) and (19), we have

$$\tau_{s0} = -\xi h \frac{\partial p}{\partial x} + \eta \tau_{wc} \quad (21)$$

$$\tau_{sh} = (1-\xi) h \frac{\partial p}{\partial x} + \eta \tau_{wc} \quad (22)$$

where

$$\xi = \frac{I_2(h)}{h I_1(h)}, \quad \eta = \frac{I_{1c}(h)}{I_1(h)}$$

Since the coefficients ξ and η vary with the mean flow pattern, let us discuss them in laminar and turbulent flows which consist of three cases mentioned later.

In laminar flows, $\epsilon=0$. Therefore, we have $\xi=1/2$ and $\eta=1$ according to the definitions of ξ and η . Thus, Eqs.(21) and (22) for laminar flows are given in the following forms.

$$\tau_{s0} = -\frac{h}{2} \frac{\partial p}{\partial x} + \tau_{wc} \quad (23)$$

$$\tau_{sh} = \frac{h}{2} \frac{\partial p}{\partial x} + \tau_{wc} \quad (24)$$

When pressure flows are dominant in the turbulent mean flow, that is when $\partial p/\partial x \gg \tau_{wc}$, the shear stress distribution will be antisymmetric about $y=h/2$, and then we have from the definition of ξ

$$\xi \approx \frac{I_{2p}(h)}{h I_{1p}(h)} = \frac{1}{2}$$

Similarly, the coefficient η becomes as follows,

$$\eta \approx \frac{I_{1c}(h)}{I_{1p}(h)} = \frac{\mu U}{I_{1p}(h)} \bigg/ \frac{\mu U}{I_{1c}(h)} \approx \frac{\tau_{wp}}{\tau_{wc}} = \frac{\alpha_p}{\alpha_s} = \kappa$$

where

$$I_{1p}(h) = \int_0^h \frac{dy}{1+\epsilon_p/\nu}, \quad I_{2p}(h) = \int_0^h \frac{y dy}{1+\epsilon_p/\nu}$$

Consequently, Eqs.(21) and (22) can be approximately written as follows,

$$\tau_{s0} = -\frac{h}{2} \frac{\partial p}{\partial x} + \kappa \tau_{wc} \quad (25)$$

$$\tau_{sh} = \frac{h}{2} \frac{\partial p}{\partial x} + \kappa \tau_{wc} \quad (26)$$

When shear flows are dominant in the turbulent mean flow, that is when $\tau_{wc} \gg h \times \partial p/\partial x$, the shear stress distribution will be constant in the direction of film thickness, and the coefficients ξ and η become

$$\xi \approx \frac{I_2(h)}{h I_{1c}(h)} = \frac{1}{2}, \quad \eta \approx \frac{I_{1c}(h)}{I_{1c}(h)} = 1$$

Thus, Eqs.(21) and (22) reduce to the following equations.

$$\tau_{s0} = -\frac{h}{2} \frac{\partial p}{\partial x} + \tau_{wc} \quad (27)$$

$$\tau_{sh} = \frac{h}{2} \frac{\partial p}{\partial x} + \tau_{wc} \quad (28)$$

When both pressure and shear flows are dominant in the turbulent mean flow, the coefficients will be considered as

$$\xi \approx \frac{1}{2}, \quad \eta = \frac{I_{1c}(h)}{I_{1p,c}(h)} = \frac{\mu U}{I_{1p,c}(h)} \bigg/ \frac{\mu U}{I_{1c}(h)}$$

where $I_{1p,c}(h) = I_1(h)$.

From Fig.2 showing the friction factor, the friction factor in pressure flows is larger than that in shear flows. Then, there is the coefficient η in the following range.

$$1 < \eta < \kappa \quad (29)$$

where the quantity κ is close to unity as shown in Table 1. Thus, the wall shear stresses τ_{x0} and τ_{xh} in Eqs. (21) and (22) can be calculated by replacing ξ by $1/2$ and η by κ .

From the above discussions, the wall shear stresses in all cases of turbulent flows may be given as follows,

$$\tau_{x0} = -\frac{h}{2} \frac{\partial p}{\partial x} + \kappa \tau_{wc} \quad (30)$$

$$\tau_{xh} = \frac{h}{2} \frac{\partial p}{\partial x} + \kappa \tau_{wc} \quad (31)$$

Eliminating $\kappa \tau_{wc}$ from Eqs. (30) and (31), we have

$$\frac{\partial p}{\partial x} = \frac{\tau_{xh} - \tau_{x0}}{h} \quad (32)$$

Since the quantity κ in Eqs. (30) and (31) is a ratio of the wall shear stress in shear flows to that in pressure flows, it may be expressed by using Eq. (3) as follows,

$$\kappa = \frac{\alpha_p \left(\frac{hU}{\nu} \right)^{\frac{1}{2}} \rho \left(\frac{U}{2} \right)^2 / \alpha_s \left(\frac{hU}{\nu} \right)^{\frac{1}{2}} \rho \left(\frac{U}{2} \right)^2}{\dots \dots \dots} \quad (33)$$

The denominator in the above expression represents the wall shear stress in pure shear flows due to the sliding surface velocity U , and it is equal to κ_{wc} in Eq. (20). Therefore, we have

$$\kappa \tau_{wc} = \frac{\alpha_p \left(\frac{hU}{\nu} \right)^{\frac{1}{2}} \rho \left(\frac{U}{2} \right)^2}{\dots \dots \dots} \quad (34)$$

This is the wall shear stress in pressure flows in which the mean velocity is $U/2$. Defining that the pressure gradient is negative in the direction of U , then $\tau_{wp} = -h \partial p / \partial x / 2$.

Substituting the relation into Eqs. (30) and (31), we have

$$\tau_{x0} = -\frac{h}{2} \frac{\partial p}{\partial x} - \frac{h}{2} \frac{\partial p}{\partial x} = -\frac{h}{2} \frac{\partial p}{\partial x} \quad (35)$$

$$\tau_{xh} = \frac{h}{2} \frac{\partial p}{\partial x} - \frac{h}{2} \frac{\partial p}{\partial x} = \frac{h}{2} \frac{\partial p}{\partial x} \quad (36)$$

where

$$p_0 = p + p_e, \quad p_h = p - p_e$$

It seems from the above equations that the wall shear stresses τ_{x0} and τ_{xh} are caused by the pressure gradients $\partial p_0 / \partial x$ and $\partial p_h / \partial x$ respectively. Equations (30) and (31) for the stresses τ_{x0} and τ_{xh} mean that all flows may be treated as pressure flows by replacing shear flows with pressure flows. Therefore, it is possible to superpose real pressure flows in the film on shear flows which are equivalent to pressure flows with the same mean velocity as the mean velocity of shear flows.

Shear flows shown in Fig. 3(a) yield the wall shear stresses having the same magnitude and direction on the two surfaces as shown in Fig. 3(b). Considering pressure flows shown in Fig. 4(a) having the same boundary condition and mean velocity as those of shear flows, the shear stress is obtained as shown in Fig. 4(b). Thus, the flow with the following properties may be defined as the equivalent pressure flow.

(i) Boundary conditions, $u(y=0)=0$, $u(y=h)=U$.

(ii) The flow occurs under the pressure gradient dp_e/dx , and the sliding surface velocity U does not contribute to the flow.

(iii) $\kappa = |\tau_{wp}| / |\tau_{wc}|$ holds for the wall

shear stresses.

(iv) The flow has the same mean velocity as that of shear flows.

(v) The two wall shear stresses on the surfaces of $y=0$ and $y=h$ have the same magnitude and direction.

The equivalent pressure flows, which are different from the real pressure flows, are yielded due to the pressure gradient $-dp_e/dx$ for $0 \leq y \leq h/2$ and dp_e/dx for $h/2 \leq y \leq h$.

By applying the equivalent pressure flow, the mean flow in infinitely long width bearings is replaced by pressure flows, then the velocity and shear stress distributions can be shown in Fig. 6. The wall shear stresses τ_{x0e} and τ_{xhe} are evaluated from Eqs. (5) and (8) for $i=p$ respectively.

3.3 Extension to finite width bearings

The lubricant flow in finite width bearings is distorted by the mutual interference of flows in the directions of x and z shown in Fig. 7(a).

Extending Eq. (32) to vector equation to examine three dimensional flows, we have

$$\frac{\partial p}{\partial r} = \frac{\tau_h - \tau_0}{h} \quad (37)$$

And, replacing shear flows by the equivalent pressure flow and assuming the lubricant flow in bearings as pressure flows, then the velocity and shear stress distributions are given as shown in Fig. 8. It means that the wall shear stresses τ_{0e} and τ_{he} are approximately equal to τ_0 and τ_h in Eq. (37) respectively.

Substituting the stresses τ_{0e} and τ_{he} obtained from Eqs. (5) and (8) into Eq. (37), we have

$$\frac{\partial p}{\partial r} = \frac{\alpha_p \rho}{2h} \left\{ \left(\frac{2h|V_{m0} - V|}{\nu} \right)^{\frac{1}{2}} |V_{m0} - V| (V - V_{m0}) - \left(\frac{2h|V_{m0}|}{\nu} \right)^{\frac{1}{2}} |V_{m0}| V_{m0} \right\} \quad (38)$$

where $V_{m0} = (u_m, w_m)$, $V = (U, 0)$, $\tau_0 = (\tau_{x0}, \tau_{z0})$, $\tau_h = (\tau_{xh}, \tau_{zh})$ and $\partial p / \partial r = (\partial p / \partial x, \partial p / \partial z)$.

Fig. 3 Velocity and stress distributions in shear flow

Fig. 4 Velocity and stress distributions in equivalent pressure flow

Rewriting Eq.(38) two components in the directions of x and z , the pressure gradients are given as follows,

$$\frac{\partial p}{\partial x} = \frac{\alpha_p}{8h} \left(\frac{hU}{\nu} \right)^{\beta} \rho U^2 \left[\left(1 - \frac{2u_{mp}}{U} \right)^{\beta} \left(1 + \frac{2u_{mp}}{U} \right)^{\beta} + \left(\frac{2w_{mp}}{U} \right)^{\beta} \right] \dots (39)$$

$$\frac{\partial p}{\partial z} = -\frac{\alpha_p}{4h} \left(\frac{hU}{\nu} \right)^{\beta} \rho U^2 \frac{w_{mp}}{U} \left[\left(1 - \frac{2u_{mp}}{U} \right)^{\beta} + \left(\frac{2w_{mp}}{U} \right)^{\beta} \right] \dots (40)$$

where

$$u_{mp} = u_m - \frac{U}{2}, \quad w_{mp} = w_m \dots (41)$$

3.4 Derivation of turbulent lubrication equation

Ignoring the compressibility of fluid and assuming the steady state, the continuous equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \dots (42)$$

Integrating Eq.(42) with the boundary conditions $v(y=0)=0$ and $v(y=h)=U\partial h/\partial x$ throughout the film thickness and reversing the order of integration and differentiation, we have

$$\frac{\partial}{\partial x} \left(\int_0^h u dy \right) + \frac{\partial}{\partial z} \left(\int_0^h w dy \right) = 0 \dots (43)$$

The integration in the above equation can be evaluated by using the mean velocity given in Eq.(41).

$$\int_0^h u dy = hu_m = h \left(u_{mp} + \frac{U}{2} \right), \quad \int_0^h w dy = hw_{mp} \dots (44)$$

Then, Eq.(42) becomes

Fig.5 Velocity and stress distributions in infinite width bearing

Fig.6 Equivalent pressure flow of infinite width bearing

$$\frac{\partial}{\partial x} (hu_{mp}) + \frac{\partial}{\partial z} (hw_{mp}) = -\frac{U}{2} \frac{\partial h}{\partial x} \dots (45)$$

Now, in the mean flow introducing the equivalent pressure flow, the velocities u_{mp} and w_{mp} in Eq.(45) are based on the pressure gradients $\partial p/\partial x$ and $\partial p/\partial z$ respectively.

In the transition from laminar to turbulent flow, the velocity distribution becomes flat due to the diffusivity, and it means that the viscosity of lubricant increases apparently.

Introducing the turbulent coefficients which consider an increase of viscosity, the velocities u_{mp} and w_{mp} can be expressed as follows,

(a) Velocity distribution

(b) Shear stress distribution

Fig.7 Velocity and stress distributions of finite width bearing

$$u_{mp} = -G_x \frac{h^2}{\mu} \frac{\partial p}{\partial x}, \quad w_{mp} = -G_z \frac{h^2}{\mu} \frac{\partial p}{\partial z} \quad \dots (46)$$

where G_x and G_z are the turbulent coefficients.

Substituting Eq.(46) into Eq.(45), we have the following turbulent lubrication equation.

$$\frac{\partial}{\partial x} \left(G_x \frac{h^2}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(G_z \frac{h^2}{\mu} \frac{\partial p}{\partial z} \right) = \frac{U}{2} \frac{\partial h}{\partial x} \quad \dots (47)$$

Substituting Eq.(46) into Eqs.(39) and (40) and rearranging the results, the following equations are obtained as

$$\begin{aligned} (1+aG_x) \{ (1+aG_x)^2 + (bG_x)^2 \}^{(\beta+1)/2} \\ - (1-aG_x) \{ (1-aG_x)^2 + (bG_x)^2 \}^{(\beta+1)/2} + c = 0 \end{aligned} \quad \dots (48)$$

$$\begin{aligned} G_x \{ (1+aG_x)^2 + (bG_x)^2 \}^{(\beta+1)/2} \\ + \{ (1-aG_x)^2 + (bG_x)^2 \}^{(\beta+1)/2} + d = 0 \quad \dots (49) \end{aligned}$$

where

$$\begin{aligned} a &= \frac{2h^2}{\mu U} \frac{\partial p}{\partial x}, \quad b = \frac{2h^2}{\mu U} \frac{\partial p}{\partial z}, \\ c &= -\frac{8}{\alpha_p} R_k^{-(\beta+1)} \frac{h^2}{\mu U} \frac{\partial p}{\partial x}, \\ d &= -\frac{4}{\alpha_p} R_k^{-(\beta+1)}, \quad R_k = \frac{hU}{\nu} \end{aligned}$$

By solving the turbulent lubrication equation(47) and the equations for the turbulent coefficients (48) and (49) simultaneously, we can evaluate the coefficients G_x and G_z and the film pressure p .

Since it seems that shear flows are more dominant than pressure flows in journal bearings, $|u_{mp}| < U/2$ and $|w_{mp}| < U/2$. By use of the conditions, we can expand Eqs. (39) and (40) and can simplify the equations by taking from the first to the third order terms to take account of the correlation effect between u_{mp} and w_{mp} . Equations (48) and (49) are given approximately as

$$a_1 G_x^3 + a_2 G_x^2 G_z + a_3 G_x + a_4 = 0 \quad \dots (50)$$

$$b_1 G_x^3 + b_2 G_x^2 G_z + b_3 G_x + b_4 = 0 \quad \dots (51)$$

(a) Velocity distribution

where

$$\begin{aligned} a_1 &= \frac{\beta(\beta+1)(\beta+2)}{3} \left(\frac{h^2}{\mu U} \frac{\partial p}{\partial x} \right)^2, \\ a_2 &= \beta(\beta+1) \left(\frac{h^2}{\mu U} \frac{\partial p}{\partial x} \right), \quad a_3 = \frac{\beta+2}{2}, \\ a_4 &= -\frac{R_k^{-(\beta+1)}}{\alpha_p}, \quad b_1 = (\beta+1) \left(\frac{h^2}{\mu U} \frac{\partial p}{\partial z} \right)^2, \\ b_2 &= \beta(\beta+1) \left(\frac{h^2}{\mu U} \frac{\partial p}{\partial z} \right), \quad b_3 = \frac{1}{2}, \\ b_4 &= -\frac{R_k^{-(\beta+1)}}{\alpha_p} \end{aligned}$$

When pressure flows in the bearing clearance dominate, Eqs.(48) and (49) hold. When shear flows dominate, Eqs.(50) and (51) are used to evaluate the solution easily.

4. Discussions of turbulent coefficients

The turbulent coefficients described in the former chapter have an important significance as an index of flow in the turbulent lubrication. We shall discuss the effects of the pressure gradients and the Reynolds number on the turbulent coefficients by drawing the relations between them.

4.1 Introduction of dimensionless pressure gradients

Let us introduce the following dimensionless pressure gradients to examine the effects of the pressure gradients and the Reynolds number on the turbulent coefficients G_x and G_z .

$$\frac{\partial p^*}{\partial x} = \frac{h^3}{\mu \nu} \frac{\partial p}{\partial x}, \quad \frac{\partial p^*}{\partial z} = \frac{h^3}{\mu \nu} \frac{\partial p}{\partial z} \quad \dots (52)$$

(b) Shear stress distribution

Fig.8 Equivalent pressure flow of finite width bearing

Substituting Eq.(52) into Eqs.(48) and (49), and rearranging the results, we have

$$(1+AG_s)\{(1+AG_s)^2+(BG_s)^2\}^{(\beta+1)/2} - (1-AG_s)\{(1-AG_s)^2+(BG_s)^2\}^{(\beta+1)/2} + C=0 \quad (53)$$

$$G_s\{(1+AG_s)^2+(BG_s)^2\}^{(\beta+1)/2} + [(1-AG_s)^2+(BG_s)^2]^{(\beta+1)/2} + D=0 \quad (54)$$

where

$$A = \frac{2}{R_h} \frac{\partial p^*}{\partial x}, \quad B = \frac{2}{R_h} \frac{\partial p^*}{\partial z}, \\ C = -\frac{8}{\alpha_p} R_h^{-(\beta+2)} \frac{\partial p^*}{\partial x}, \quad D = -\frac{4}{\alpha_p} R_h^{-(\beta+1)}$$

Now, we define $|\nabla p^*|$ as

$$|\nabla p^*| = \frac{h^*}{\mu\nu} |\nabla p| = \sqrt{\left(\frac{\partial p^*}{\partial x}\right)^2 + \left(\frac{\partial p^*}{\partial z}\right)^2} \quad (55)$$

We discuss the dependencies between R_h and G_x, G_z by solving Eqs.(53) and (54) simultaneously under the combination of $\partial p^*/\partial x$ and $\partial p^*/\partial z$. As such combinations innumerable exist, generalized discussions are difficult to make. Therefore, let us consider the following two cases.

$$(i) \quad \frac{\partial p^*}{\partial x} \gg \frac{\partial p^*}{\partial z}, \quad (ii) \quad \frac{\partial p^*}{\partial x} \ll \frac{\partial p^*}{\partial z}$$

The assumption (i) holds for infinitely long width bearings. In such a case, $\partial p^*/\partial x \neq 0$ or $B \neq 0$, then we have the approximate equations from Eqs.(53) and (54) as follows,

$$(1+AG_s)|1+AG_s|^{\beta+1} - (1-AG_s)|1-AG_s|^{\beta+1} + C=0 \quad (56)$$

$$G_s\{|1+AG_s|^{\beta+1} + |1-AG_s|^{\beta+1}\} + D=0 \quad (57)$$

Elrod⁽¹¹⁾ pointed out that the ratio of the two wall shear stresses $\tau_h/\tau_o=n$ holds for a limiting range. Then, the relationship between n and G_x is given as follows,

$$\alpha_p(1-n)R_h^{\beta+2}\{(1-AG_s)|1-AG_s|^{\beta+1} + (1+AG_s)|1+AG_s|^{\beta+1}\} + 8(1+n)\partial p^*/\partial x=0 \quad (58)$$

Following Eqs.(56), (57) and (58), the turbulent coefficients based on the assumption (i) are expressed by using the parameter ∇p^* .

The assumption (ii) holds for infinitely short width bearings. In such a case, we have from $\partial p^*/\partial x \neq 0$ or $A \neq 0$

$$2G_s\{1+(BG_s)^2\}^{(\beta+1)/2} + D=0 \quad (59)$$

Assuming $\partial p^*/\partial x=0$ by the assumption (ii), Eq.(53) becomes zero identically, then the equation for G_x is impossible to derive. Therefore, expanding Eq.(53) about the point of $AG_x=0$ and neglecting the higher terms than the second order, we have

$$2AG_s\{1+(BG_s)^2\}^{(\beta-1)/2}\{2+\beta+(BG_s)^2\} + C=0 \quad (60)$$

Eliminating A in Eq.(60) by use of the relation $C=AD$ between the coefficients A, C and D , we have

$$2G_s\{1+(BG_s)^2\}^{(\beta-1)/2}\{2+\beta+(BG_s)^2\} + D=0 \quad (61)$$

Defining the ratio of the two wall shear stresses by $(\nu\mu/h^2\partial p^*/\partial z)/|\tau_o|=n$, the relationship between n and G_z is

$$(n\alpha_p)^2 R_h^{2(\beta+2)}\{1+(BG_s)^2\}^{\beta+1} + 16(n^2-4)\left(\frac{\partial p^*}{\partial z}\right)^2=0 \quad (62)$$

Using Eqs.(59), (61) and (62), we can obtain the turbulent coefficients based on the assumption (ii).

4.2 Figures of turbulent coefficients

Let us discuss the computed results obtained from the above equations for the turbulent coefficients.

Fig.9 shows the relations between the turbulent coefficients and the Reynolds number in infinitely long width bearings. The values of G_x and G_z decrease with an increase of $|\nabla p^*|$, and the tendency is most pronounced in superlaminar flows. Also, by investigating the values G_x and G_z according to $|AG_x|=2|\nabla p^*G_x/R_h|$, these values coincide with each other for $|AG_x|>1$ in which the Reynolds number is relatively small in comparison with the value of $2\nabla p^*G_x$, and the discrepancy between these values comes out at the point of $|AG_x|=1$ and a peculiar peak appears in the curves of G_z . For $|AG_x|<1$, both G_x and G_z approach asymptotically the curves of $\nabla p^*=0$. These results may be understood from the fact that the effect of turbulence on the increase of lubricant viscosity is mainly due to pressure flows for $|AG_x|>1$ and due to shear flows for $|AG_x|<1$ respectively.

Fig.10 shows the computed results in infinitely short width bearings. The value G_x coincides with G_z for $|BG_z|>1$ in which the lubricant flow is mainly dominated by

Fig.9 Turbulent coefficients in the case of $\partial p^*/\partial x \gg \partial p^*/\partial z$

Fig.10 Turbulent coefficients in the case of $\partial p^*/\partial x \ll \partial p^*/\partial z$

Fig.11 Turbulent coefficients in the case of $\partial p^*/\partial x \gg \partial p^*/\partial z$ or $\partial p^*/\partial x < \partial p^*/\partial z$

pressure flows, and these values approach gradually the curves of $Vp^*=0$ for a range of $|BG_z| < 1$ in which the lubricant flow is mainly dominated by shear flows. However, there is no peculiar peak as shown in Fig. 9.

A comparison between the results computed by this theory and those based on the other representative theories is shown in Fig.11. The characteristics of G_x and G_z mentioned above can also be obtained in each theory. However, Hirs' results are different from the other results for small values of $|Vp^*|$ and the Reynolds number because the approximation of experimental data in the superlaminar flow regime ($10^3 < Rh < 10^4$) is not available.

5. Conclusions

A turbulent lubrication equation is developed in a wide range of the Reynolds numbers by introducing a concept of the equivalent pressure flow based on the friction law of fluid, and the effects of pressure gradients on the turbulent coefficients are discussed. Conclusions are as follows;

(1) It is convenient to divide the flow regime into three regimes according to the Reynolds number in order to evaluate the friction factor by the friction law of fluid.

(2) By introducing the concept of equivalent pressure flow, the turbulent lubrication equation can be derived without discussing the nonlinearity in superposition of pressure flows on shear flows in Boussinesq's assumption.

(3) When pressure flows are dominant, it is necessary to take account of the effects of pressure gradients in order to derive the turbulent coefficients.

(4) The computed results by the theory in this paper are in good agreement with those of the nonlinear theories presented already for various pressure gradients.

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