The Torsion of a Hollow Cylinder with an External Crack*

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This paper deals with the axially symmetric torsion of a hollow cylinder with an external crack. The mixed boundary value problem is reduced to dual series equations. These equations are shown to be equivalent to an infinite system of simultaneous equations. Numerical results are presented for stresses, displacement and stress intensity factors.

1. Introduction

In recent years, the problem of determining the distribution of stresses in the vicinity of an external crack in a solid cylinder under torsion has been considered by a number of authors (1)-(5). The results of the stress intensity factor obtained by Benthem and Koiter (4) and Yamamoto and Sumi (5) agree best with those obtained by Okuya et al. (1). On the other hand, the approximate expression for stress intensity factor obtained by using the stress concentration factor of a body of revolution containing an external notch is liable to have a relatively large error. For a hollow cylinder with an external crack, Harris (6) gives an approximate expression of the stress intensity factor using the equation given by Neuber (7).

In this paper we consider the axially symmetric torsion of a hollow cylinder with an external crack. It is assumed that the curved surfaces are free from stress. The boundary value problem is shown to be equivalent to one of solving a pair of dual series equations. These dual series equations are reduced to an infinite system of simultaneous equations.

2. Derivation of equations and reduction to dual series equations.

The axially symmetric problem of an infinite hollow cylinder of radii \( r \) and \( r_0 \) with an external crack subjected to a torque \( T \) is considered (Fig.1). A cylindrical coordinate system \((r, \theta, z)\) is used in such a way that the axis of the hollow cylinder coincides with the \( z \)-axis. The crack is assumed to lie in a plane perpendicular to the hollow cylinder axis and occupy the region \( z = 0, r < r < r_0 \). The state of stress can be obtained from an analysis of the semi-infinite hollow cylinder \((z > 0)\) which is partially bonded to a rigid surface in an annular region \( r \leq r \leq r_0 \).

Since this problem is one of an axially symmetric torsion, there will be no dependence upon \( \theta \). The nonvanishing displacement and stresses are given by

\[
\begin{align*}
\tau_r &= -\frac{\partial \lambda_1}{\partial r}, \quad \tau_\theta = -G \frac{\partial \lambda_1}{\partial \theta}, \quad \tau_z = G \left( \frac{\partial^2 \lambda_1}{\partial z^2} + \frac{2 \partial \lambda_1}{r \partial r} \right) \\
\end{align*}
\]

where \( G \) is the shear modulus and \( \lambda_1(r, z) \) is Boussinesq's stress function satisfying

\[
\frac{\partial^2 \lambda_1}{\partial z^2} + \frac{2 \partial \lambda_1}{r \partial r} = 0
\]

The boundary conditions of the problem are

\[
\begin{align*}
(\tau_\theta)_{r=r_0} &= (\tau_z)_{r=r_0} = 0, \quad (0 \leq z < \infty) \\
(\tau_\theta)_{r=r} &= 0, \quad (r \leq r \leq r_0) \\
(\tau_z)_{r=r} &= 0, \quad (r_0 < r \leq r_0).
\end{align*}
\]

It is convenient to represent in the following form:

\[
\lambda_1 = A(r^2 - z^2) + B(2z^2 - r^2) + \sum_{n=1}^{\infty} A_n C_n (r_0/r) \exp(-n \pi z)
\]

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Fig.1 Geometry of an infinite hollow cylinder with an external crack.
where

\[ C(\xi_r) = (Y_d(\xi_0)I(\xi_r) - J_1(\xi_0)Y_d(\xi_r)) / Y_d(\xi_0) \]  

\[
\sum_{\xi_r} \xi_1^2 \xi_0^2 C(\xi_r) \exp(-\xi_r \xi_2) \]  

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The applied torque \( T \) is given by

\[
T = \int_{r_0}^{r_1} \frac{2\pi r^2 \rho_m \sigma_r}{(r_0 - r)} \, dr = 3\pi GB(r_1 - r_0) \]  

then, we obtain

\[ B = T \sigma \gamma C(r_1 - r_0) \]  

It follows immediately from Eq.(10) that the solution (6) will satisfy the boundary condition (3) if \( \xi_0 \) are the positive zeros of the equation

\[ C(\xi_0) = 0 \]  

The remaining two boundary conditions (4) and (5) may be written in the form:

\[ -2\pi r_i \sum_{\xi_r} \xi_0 \xi_1 A_0 C(\xi_r) = 0, \quad (r_i \leq r \leq r_f) \]  

\[ 2\pi \sigma \gamma (r_1 - r_0) - G \sum_{\xi_r} \xi_0 \xi_1 A_0 C(\xi_r) = 0, \quad (r_f < r \leq r_0) \]  

To solve the dual series equations (13) and (14), the technique (8) is used. We make the following transformation,

\[ b = \frac{r_1 + r_2}{2}, \quad w = \frac{r_1 - r_2}{2} \]  

\[ r^2 = b^2 + w^2 - 2bw \cos \phi, \quad (0 \leq \phi \leq \pi) \]  

Using Eq.(15), the variable \( r \) in \( r_i \leq r \leq r_f \) can be exchanged for a new one \( \phi \) in \( 0 \leq \phi \leq \pi \), when \( r = r_1 \) corresponds to \( \phi = 0 \) and \( r = r_2 \) holds for \( \phi = \pi \).

On the other hand, we can assume that \( \xi_0 \) has a singularity of the form \((r_1 - r)^{-1/2}\) and takes continuous and finite values in \( r \leq r_1 \). Then, the shear stress \( \tau_{(r_1),0} \) can be expressed by Fourier series with respect to \( \phi \) as follows:

\[ \tau_{(r_1),0} = \sum_{m=1}^{\infty} \beta_m \frac{2}{r_1} \cos \left( n \frac{\pi}{2} \right) H(r_1 - r), \quad (r_1 \leq r \leq r_f) \]  

where \( H(x) \) denotes Heaviside's unit function and \( \beta_m (m = 0, 1, 2, \ldots) \) are unknown coefficients. Eq.(16) may be rewritten

\[ \tau_{(r_1),0} = \sum_{m=1}^{\infty} \beta_m \frac{2}{r_1} \cos \left( n \frac{\pi}{2} \right) H(r_1 - r), \quad (r_1 \leq r \leq r_f) \]  

where

\[ \beta_m = (a_m - \delta_m) \nu V(bw)(2m + 1), \quad (n = 0, 1, 2, \ldots) \]  

and \( \delta_m \) denotes the Kronecker delta.

Eq.(17) is equivalent to the equation

\[ \frac{2T_j}{\pi r_1^2 (r_1 - r)} - G \sum_{\xi_r} \xi_0 \xi_1 A_0 C(\xi_r) = \sum_{m=1}^{\infty} \beta_m \frac{2}{r_1} \cos \left( n \frac{\pi}{2} \right) H(r_1 - r), \quad (r_1 \leq r \leq r_f) \]  

After a few manipulations, we get

\[ T = 2\pi \sum_{\xi_r} \beta_r \left( -r_1^2 + \frac{8bw}{(2n + 1)2n - 1} \right) \]  

\[ \sum_{m=1}^{\infty} \beta_m \left[ 2r C(\xi_0) + \xi_0 \xi_1 \sum_{m=1}^{\infty} (2 - \delta_m) J_1(\xi_0) C_m(\xi_r) F(n,k) \right] \]  

\[ a_m = \sum_{m=1}^{\infty} \beta_m \left[ 2r C(\xi_0) - r_1 C(\xi_0) \right] \]  

where

\[ F(n,k) = \frac{2n + 1}{2n - 1} \]  

If the coefficients \( a_m \) are given by Eq.(20), the boundary condition of \( \tau_{(r_1),0} \) is satisfied for arbitrary value of \( \beta_m (m = 0, 1, 2, \ldots) \). Therefore, to determine the coefficients \( \beta_m \), we use the condition of \( \tau_{(r_1),0} \).

Using the relationship

\[ C(\xi_r) = 2r \sum_{m=0}^{\infty} (p + 1) \frac{1}{2 \sin \phi} C_m (\xi_0) A_m (\xi_r) \]  

and substituting the above equation into Eq.(13), we find that

\[ A = \frac{3}{2} \frac{A_m}{\xi_0^2} \sum_{m=0}^{\infty} (p + 1) \frac{1}{2 \sin \phi} J_m (\xi_0) C_m (\xi_r) = 0, \quad (r_f < r \leq r_0) \]  

Since the above equation must hold for arbitrary value of \( \phi \), we get the following equation
\[ \sum_{\nu=0} A_\nu J_{\nu}(\xi w) C_{\nu} = Ab \delta_{\nu \nu}, \quad (p = 0, 1, 2, \ldots) \]  

(21)

If we now substitute the expression (20) for the coefficients \( A_\nu \) into Eq. (21) and interchange the order of summations, we obtain an infinite system of simultaneous equations

\[ \sum_{\nu=0} D(n, p) \beta_n = AG\beta_{np}, \quad (p = 0, 1, 2, \ldots) \]  

(22)

where

\[ D(n, p) = \sum_{\nu=0} J_{\nu}(\xi w) C_{\nu} \left( \frac{C_{\nu}(\zeta_{np})}{C_{\nu}(\zeta_{np})} \right) \left( \frac{1}{\delta} \int \frac{2 \pi \sin \phi}{J_{\nu}(\xi w) C_{\nu}(\xi w)} \int_0^\infty \cos \left( n + \frac{1}{2} \right) \phi \sin \phi J_{\nu}\left( 2\pi \sin \frac{\phi}{2} \right) d\phi \right) \]  

(23)

Once \( \beta_n \) is obtained, the coefficients \( A_\nu, A_\eta \) and the stresses and displacement at any point may be calculated by using Eqs. (8)-(10), (19) and (20).

Stress intensity factor \( K_{\infty} \) is defined as

\[ K_{\infty} = \lim_{r \to 0} \sqrt{2\pi (r - r\delta)/(2\pi r)} \]  

(24)

Using Eq. (17), Eq. (24) can be rewritten as

\[ K_{\infty} = \sqrt{\frac{2\pi}{r_{e} - r_{e}/2}} \sum_{\nu=0} (-1)^{\nu} (2\nu + 1) \beta_n \]  

(25)

4. The special case \( \nu = 0 \)

When the inner radius tends to zero, Eq. (17) and (15) become the following forms

\[ b = w = \frac{r_{e}^2}{r_{e}}, \quad \lim \frac{C_{\nu}(\zeta_{np})}{J_{\nu}(\xi w) C_{\nu}(\xi w)} = J_{\nu}(\xi w) \]  

(26)

where \( \xi_n \) are positive roots of the equation

\[ J_{\nu}(\xi_n) = 0 \]  

(27)

Putting \( \gamma_n = (-1)^{\nu} (2\nu + 1) \beta_n \) and \( \zeta_n = \xi_n/2 \), we find that Eq. (22) becomes

\[ \sum_{\nu=0} (-1)^{\nu} \frac{1}{2\nu} \left( \frac{2\pi}{\xi_n} \sin \frac{\phi}{2} \right) \cos \left( n + \frac{1}{2} \phi \right) J_{\nu}(\xi w) \sin \frac{\phi}{2} d\phi = AG\beta_{np}, \quad (p = 0, 1, 2, \ldots) \]  

(28)

Eq. (28) can also be written as

\[ \sum_{\nu=0} E(n, p) \beta_n = AG\beta_{np}, \quad (p = 0, 1, 2, \ldots) \]  

(29)

where

\[ E(n, p) = (-1)^{\nu} \sum_{\nu=0} \frac{1}{2\nu} \left( \frac{2\pi}{\xi_n} \sin \frac{\phi}{2} \right) \cos \left( n + \frac{1}{2} \phi \right) J_{\nu}(\xi w) \sin \frac{\phi}{2} d\phi \]  

(30)

and \( \lambda_n, \mu_n \) are the spherical Bessel functions of the first and second kind of order \( \nu \) respectively. In deriving Eq. (29), the following relations are used

\[ \int \sin \phi J_{\nu}(2\pi \sin \frac{\phi}{2}) d\phi = 2 \sin \frac{\phi}{2} J_{\nu}(2\pi \sin \frac{\phi}{2}) \]  

\[ \int \sin \left( n + \frac{1}{2} \phi \right) J_{\nu}(2\pi \sin \frac{\phi}{2}) d\phi = \frac{\pi}{2} \left( \mu_{n+1}(\zeta_n) \lambda_{n+1}(\zeta_n) - \mu_n(\zeta_n) \lambda_n(\zeta_n) \right) \]  

The stress intensity factor in this case is expressed in the following form

\[ K_{\infty} = \sqrt{\pi} \sum_{\nu=0} \gamma_n \]  

(31)

Eqs. (29) and (31) are in good agreement with those obtained by Okuya et al.\(^{(1)}\).

5. Numerical results and discussion

The large positive zeros of Eq. (12) may be calculated by the asymptotic expansion\(^{(3)}\)

\[ \xi_n = \rho \zeta_n \left( 1 + \frac{\eta}{\xi_n} \right) \left( 1 + \frac{1}{\xi_n^2} \right) \cdots \]  

(32)

where

\[ \rho = \frac{r_{e}}{r_{e}^2}, \quad \gamma = 2\pi (r_{e} - r_{e}/2), \quad \delta = \frac{15}{128\rho (r_{e} - r_{e}/2)}, \quad \eta = \frac{45(1 - \rho^2)}{128\rho (r_{e} - r_{e}/2)} \quad \xi_n = \frac{1024\rho (r_{e} - r_{e}/2)}{128\rho (r_{e} - r_{e}/2)} \]  

and the small zeros have to be calculated by Newton's method. Table 1 gives the zeros for \( \nu_{s}, \nu_{e} = 0.1, 0.5 \) and 0.9. It is seen that about thirty zeros for \( \nu_{s}, 0.1 \) and the first two zeros for \( \nu_{s}, 0.5 \) have to be calculated using Newton's method.

In the numerical examples the main question is the behavior in convergence of the coefficients and the series giving the stress intensity factor. Since an infinite system of simultaneous equations is to be truncated to a finite number of terms, let Eq. (22) be approximated by

\[ \sum_{\nu=0} D(n, p) \beta_n = AG\beta_{np}, \quad (p = 0, 1, 2, \ldots; N \geq 1) \]  

(33)

The behavior in convergence of the stress intensity factors is shown in Tables 2 and 3. Table 2 shows the convergence behavior of the non-dimensional stress intensity factor \( K_{\infty}\sqrt{\zeta_{np}} \) with \( r_{e} - r_{e} = (r_{e} - r_{e}) \) for \( \nu_{s}, 0.1, 0.5 \) and \( 0.9 \), where \( r_{e} = 2\pi (r_{e}^2 - r_{e}^2) \) is the net-section stress. Table 3 gives the convergence behavior of \( K_{\infty}\sqrt{\zeta_{np}} \) with \( \nu_{s}, 0.5 \) for \( \nu_{s}, 0.5 \), where \( r_{e} = 2\pi (r_{e}^2 - r_{e}^2) \).
Table 1. The roots of $C_0(\delta,r_0) = 0$.
(* By Newton’s method)

<table>
<thead>
<tr>
<th>$\delta/r_0$</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.423 38</td>
<td>0.332 26</td>
<td>0.293 35</td>
</tr>
<tr>
<td>2</td>
<td>0.347 41</td>
<td>0.263 55</td>
<td>0.216 24</td>
</tr>
<tr>
<td>3</td>
<td>0.278 54</td>
<td>0.206 97</td>
<td>0.170 82</td>
</tr>
<tr>
<td>4</td>
<td>0.219 92</td>
<td>0.162 87</td>
<td>0.130 51</td>
</tr>
<tr>
<td>5</td>
<td>0.164 93</td>
<td>0.120 23</td>
<td>0.093 28</td>
</tr>
</tbody>
</table>

Table 2. $K_{H0}(\tau_0,\gamma(\varphi_0))$ for various $N$.
($\tau_0 = 0.5$ and $0.9$)

<table>
<thead>
<tr>
<th>$N$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
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</thead>
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<tr>
<td>1</td>
<td>0.400</td>
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<td>5</td>
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</tr>
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Table 3. $K_{H0}(\tau_0,\gamma(\varphi_0))$ for various $N$.
($\gamma = 0.5$)

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The tables show a rapid convergence of the results even for relatively small values of $N$.

Distributions of $\nu$, $\tau_0$, and $\gamma_0$ are shown in Figs. 2 to 4 when $\tau_0 = 1.0$, $\tau_0 = 0.5$ and $\tau_0 = 0.7$. It is seen that $\tau_0$ and $\gamma_0$ rapidly approach the results of an uncracked hollow cylinder as $r_0$ increases. This implies that the crack would have only a negligible effect on the distribution of the shear stress on $z/r_0 = 1.0$.

For $r_0 = 0.5$ and $0.9$ the variation of $K_{H0}(\tau_0,\gamma(\varphi_0))$ with $(r_0 - \tau_0)/\ell$ is given in Fig. 5. The values shown by the dashed lines are the approximate expression

$$K_{H0} = \sqrt{1 - (\tau_0/\ell)}$$

Fig. 2 The distribution of $\nu$.

Fig. 3 The distribution of $\tau_0$.

Fig. 4 The distribution of $\gamma_0$.

Fig. 5 The variation of $K_{H0}(\tau_0,\gamma(\varphi_0))$ with $(r_0 - \tau_0)/\ell$.
Fig. 6 shows the variation $K_{II}/\tau_0 V_0$ with $(r_0 - r)/t$ for $r_0/t = 0.5$ and 0.9. The results obtained in this paper decrease monotonously with an increasing ratio $r_0/t$. As $(r_0 - r)/t = 1.8$, $K_{II}/\tau_0 V_0$ approaches an exact solution of the torsion problem of a body of revolution containing an external shallow notch. The approximate results obtained by Harris have the minimum values for $r_0/t = 0.8-0.9$.

Fig. 7 shows the variation $K_{II}/\tau_0 V_0$ with $r_0/t$ for $c/t = 0.5$. It will be observed that the results have maximum value 2.984 for $r_0/t = 0$, that is, for the solid cylinder with an external crack\(^{(1)}\) and the value decreases monotonously as $r_0/t$ increases. The results for $r_0/t = 1$ are in good agreement with the exact solution for an edge crack in a plane subjected to a constant shear stress $\tau_0$ at infinity as follows:

$$\frac{K_{II}}{\tau_0 V_0} = \sqrt{\frac{2t}{\pi} \tan \left( \frac{\pi c}{2t} \right)} = 1.128, \ (c/t = 0.5) \quad (35)$$

The results obtained by Harris are liable to have a relatively large error, especially for the ratio of inner to outer radii between 0.1 and 0.6.

References

(3) Kudriashev, B.K. and Parton, V.Z., PMM, 37, 316 (1973).
(9) McMahon, J., Annals Math, 9, 23 (1895).