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Stresses in an Elastic Circular Cylinder with a Prolate Spheroidal Cavity Under Tension*

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This paper contains an exact solution for the stresses arising in an elastic circular cylinder with a prolate spheroidal cavity under tension. The solution can be reduced to a combination of the solutions, one being regular in the outside of a cavity and the other regular in an infinite circular cylinder, and is deduced with the aid of Dougall's stress functions. Two sets of simple harmonic stress functions are given by expressions referred both to the cylindrical and the prolate spheroidal coordinates. The boundary conditions on the surfaces of the cylinder and of the cavity are satisfied with the aid of the relations between the cylindrical and prolate spheroidal harmonics. Numerical results are given for some different equatorial semiaxes and shape ratios, and the stress distributions in the neighbourhood of the cavity are shown graphically.

1. Introduction

The problem of stress concentrations in an elastic medium in the presence of variously shaped inclusions and cavities is very important in many areas of engineering and geophysics. Considerable amount of research has been done on the problems of determining the stress concentration produced in an elastic body by a spherical inclusion or cavity.⁽¹⁾⁽²⁾ However, the shape of the existing flaw is more complicated than the spherical shape, so it would seem to be of interest to have knowledge of the effect of arbitrarily shaped inclusions or cavities on the stress concentration. In the present paper we consider an elastic body which possesses a cavity in the shape of a spheroid. The problem presented by the stress concentration around a spheroidal cavity in a circular cylindrical shaft, was considered first in 1937 by Neuber⁽³⁾ for loading conditions corresponding to uniform axial tension, pure bending, Saint-Venant bending, and Saint-Venant torsion of a solid shaft. Neuber's results do not take into account the interference between the spheroidal cavity and the cylindrical boundary of the body, so they represent solutions for a cavity in a medium which otherwise occupies the entire space and is under the appropriate stress distributions at infinity. Some of Neuber's results related to the spheroidal cavity are later duplicated in a different manner by Miyamoto⁽⁴⁾, Ling-Yang⁽⁵⁾ and others.⁽⁶⁾

The asymmetric solution corresponding to a uniaxial state of tension at infinity, perpendicular to the polar axis of the spheroidal cavity, was given in 1947 by Sadowsky-Sternberg⁽⁷⁾, by using Boussinesq's stress function. Subsequently in 1951, Edwards⁽⁸⁾ presented a complete solution for the stress concentration due to an elastic spheroidal inclusion, induced by arbitrary uniform tractions at infinity.

Later on, Nisitani⁽⁹⁾ investigated the problem of an infinite body having an infinite row of spheroidal cavities under tension. The above listed papers are all concerned with the stress concentration around a spheroidal cavity or inclusion in a medium which occupies the entire space, but we have few analytic investigations related to the interference between a spheroidal cavity and an external boundary.

For example, the case of a spheroidal shell was treated by Muki⁽¹⁰⁾, and Nisida-Kim⁽¹¹⁾ presented photoelasticity results of the problem for a semi-infinite body having a spheroidal cavity.

Fig.1 Coordinate system

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In the present study, the authors deal with a problem of an infinite circular cylinder having a prolate spheroidal cavity under axial tension. The solution is obtained by use of Dougall's stress function approach. The boundary conditions both on the surface of the cylinder and of the cavity are satisfied with the aid of the relations between the cylindrical and the prolate spheroidal harmonics.

Numerical results are given for some different equatorial semi-axes and shape ratios and the stress distributions around a cavity are shown graphically.

2. Method of solution

Consider an infinite elastic circular cylinder having a symmetrically located prolate spheroidal cavity as shown in Figure 1. Denote as usual the cylindrical and prolate spheroidal coordinates of a point by (r, θ, z) and (α, β, γ) respectively. They are connected with each other by

$$\left. \begin{aligned} r &= c \sinh \alpha \sin \beta = c q \bar{p} \\ \theta &= \gamma \\ z &= c \cosh \alpha \cos \beta = c q p \end{aligned} \right\} \dots\dots\dots (1)$$

in which c is a positive constant and $q = \cosh \alpha$, $\bar{q} = \sinh \alpha$, $p = \cos \beta$, $\bar{p} = \sin \beta$ with $-1 \leq p \leq 1$ and $1 \leq q < \infty$. The parametric surfaces $\alpha = \text{const.}$, $\beta = \text{const.}$ and $r = \text{const.}$ form a triple orthogonal

conformal family of prolate spheroids, hyperboloids of two sheets, and meridional half planes, respectively. Take the axis of the cylinder as z -axis, and the origin O at the center of the cavity. For convenience, r , z , q and $\bar{q}$ will be regarded as dimensionless quantities referring to a typical length of radius of the cylinder.

The surface of the cylinder is thus represented by $r=1$ and the surface of the cavity by $\alpha=\alpha_0$. The major and minor semi-axes of the cavity are taken as a and b respectively.

For the mathematical formulation of the problem, Dougall's harmonic stress functions φ_0 and φ_4 are introduced, whose combinations form a general solution of the displacement equations of equilibrium in axisymmetric torsionless strain for solids of revolution. The components of the displacement vector in terms of them are expressed as follows:

$$\left. \begin{aligned} 2Gu_r &= \frac{\partial \varphi_0}{\partial r} + r \frac{\partial \varphi_4}{\partial z}, \quad \nu \varphi_0 = 0 \\ 2Gu_z &= \frac{\partial \varphi_0}{\partial z} - r \frac{\partial \varphi_4}{\partial r} - 4(1-\nu)\varphi_4 \end{aligned} \right\} \dots\dots\dots (2)$$

$$\text{where } \nabla^2 \varphi_0 = \nabla^2 \varphi_4 = 0, \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Here $(u_r, \nu \varphi_0, w_z)$ denote the components of displacement referred to the cylindrical coordinates (r, θ, z) , while G and ν denote the shear modulus and Poisson's ratio, respectively.

Using Eq.(1), the basic solutions, Equation (2), in prolate spheroidal coordinates now appear as follows:

$$\left. \begin{aligned} 2Gu_\alpha &= hq \frac{\partial \varphi_0}{\partial q} + chq \left\{ \bar{p} \frac{\partial \varphi_4}{\partial p} - 4(1-\nu)p\varphi_4 \right\}, \quad 2Gv_\beta = -h\bar{p} \frac{\partial \varphi_0}{\partial p} + ch\bar{p} \left\{ q \frac{\partial \varphi_4}{\partial q} + 4(1-\nu)q\varphi_4 \right\} \\ w_\gamma &= 0 \end{aligned} \right\} \dots\dots\dots (3)$$

$$\text{where } \nabla^2 \varphi_0 = \nabla^2 \varphi_4 = 0, \quad \nabla^2 \equiv h^2 q^2 \frac{\partial^2}{\partial q^2} + 2h^2 q \frac{\partial}{\partial q} + h^2 \bar{p}^2 \frac{\partial^2}{\partial p^2} - 2h^2 \bar{p} \frac{\partial}{\partial p}, \quad h^2 \equiv \frac{1}{c^2(q^2 - p^2)}$$

and u_α , v_β and w_γ are the components of the displacement in the prolate spheroidal coordinates (α, β, γ) .

The corresponding components of stress ($\sigma_\alpha \dots, \tau_{\alpha\beta} \dots$) are given as follows:

$$\left. \begin{aligned} \sigma_\alpha &= h^2 \left\{ q^2 \frac{\partial^2 \varphi_0}{\partial q^2} + c^2 h^2 \bar{p}^2 \left(q \frac{\partial \varphi_0}{\partial q} - p \frac{\partial \varphi_0}{\partial p} \right) \right\} + ch^2 \left\{ q^2 \bar{p}^2 \frac{\partial^2 \varphi_4}{\partial q \partial p} \right. \\ &\quad \left. + q^2 p (c^2 h^2 \bar{p}^2 - 2(2-\nu)) \frac{\partial \varphi_4}{\partial q} - q \bar{p}^2 (c^2 h^2 q^2 - (1-2\nu)) \frac{\partial \varphi_4}{\partial p} \right\} \\ \sigma_\beta &= h^2 \left\{ \bar{p}^2 \frac{\partial^2 \varphi_0}{\partial p^2} + c^2 h^2 q^2 \left(q \frac{\partial \varphi_0}{\partial q} - p \frac{\partial \varphi_0}{\partial p} \right) \right\} - ch^2 \left\{ q^2 \bar{p}^2 \frac{\partial^2 \varphi_4}{\partial q \partial p} \right. \\ &\quad \left. + q^2 p (c^2 h^2 \bar{p}^2 - (1-2\nu)) \frac{\partial \varphi_4}{\partial q} - q \bar{p}^2 (c^2 h^2 q^2 - 2(2-\nu)) \frac{\partial \varphi_4}{\partial p} \right\} \\ \sigma_r &= h^2 \left(q \frac{\partial \varphi_0}{\partial q} - p \frac{\partial \varphi_0}{\partial p} \right) + (1-2\nu) ch^2 \left(q^2 p \frac{\partial \varphi_4}{\partial q} + q \bar{p}^2 \frac{\partial \varphi_4}{\partial p} \right) \\ \tau_{\alpha\beta} &= h^2 q \bar{p} \left\{ - \frac{\partial^2 \varphi_0}{\partial q \partial p} + c^2 h^2 \left(q \frac{\partial \varphi_0}{\partial p} - p \frac{\partial \varphi_0}{\partial q} \right) \right\} + ch^2 q \bar{p} \left\{ \frac{1}{2} \left(q^2 \frac{\partial^2 \varphi_4}{\partial q^2} - \bar{p}^2 \frac{\partial^2 \varphi_4}{\partial p^2} \right) \right. \\ &\quad \left. + q (c^2 h^2 \bar{p}^2 + 2(1-\nu)) \frac{\partial \varphi_4}{\partial q} + p (c^2 h^2 q^2 + 2(1-\nu)) \frac{\partial \varphi_4}{\partial p} \right\} \end{aligned} \right\} \dots\dots\dots (4)$$

We first take the stress functions φ_0 and φ_4 in cylindrical coordinates as follows:

$$\left[\begin{aligned} \varphi_0 &= -\frac{1-2\nu}{4(1+\nu)} p_0 (2z^2 - r^2) \\ \varphi_4 &= -\frac{1}{2(1+\nu)} p_0 z \end{aligned} \right] \dots\dots\dots (5)$$

The components of displacement and stress generated by the stress function [I] are given as follows:

$$\left[\begin{aligned} u_r &= -\frac{\nu}{E} p_0 r, \quad v_\theta = 0, \quad w_z = \frac{1}{E} p_0 z \\ \sigma_r &= p_0, \quad \sigma_\theta = \sigma_\phi = \tau_{r\theta} = \tau_{r\phi} = \tau_{\theta\phi} = 0 \end{aligned} \right] \dots\dots\dots (6)$$

where E is Young's modulus.

This solution gives a stress state where a circular cylinder, in the absence of a cavity is subjected to a uniform tension of p_0 per unit area at both ends of the cylinder. Moreover, this gives no traction on the surface of the cylinder but yields the following stresses on the spheroidal surface $\alpha = \alpha_0 = \text{const.}$:

$$\left. \begin{aligned} \left(\frac{\sigma_{\alpha}}{p_0} \right)_{\alpha=\alpha_0} &= c^2 h_0^2 q_0^2 p^2, & \left(\frac{\sigma_{\beta}}{p_0} \right)_{\alpha=\alpha_0} &= c^2 h_0^2 q_0^2 \bar{p}^2 \\ \left(\frac{\tau_{\alpha\beta}}{p_0} \right)_{\alpha=\alpha_0} &= -c^2 h_0^2 q_0 q_0 p \bar{p} \end{aligned} \right\} \quad (7)$$

where $q_0 = (q)_{\alpha=\alpha_0}$, $\bar{q}_0 = (\bar{q})_{\alpha=\alpha_0}$, $h_0 = (h)_{\alpha=\alpha_0}$

The method of satisfying the boundary conditions when the cavity is present is to construct two sets of auxiliary stress functions by using suitable harmonic func-

tions such that they give no traction on the surface of the cylinder and at the same time give no stress at z -infinity or the ends of the cylinder.

These stress functions are combined linearly and added to the basic stress function [I]. These auxiliary stress functions must satisfy the following boundary conditions:

- (i) On the surface of the cylinder $r=1$

$$(\sigma_r)_{r=1} = (\tau_{rz})_{r=1} = 0 \quad (8)$$
- (ii) On the surface of the cavity $\alpha = \alpha_0$.

$$\left. \begin{aligned} \left(\frac{\sigma_{\alpha}}{p_0} \right)_{\alpha=\alpha_0} &= -c^2 h_0^2 q_0^2 p^2 \\ \left(\frac{\tau_{\alpha\beta}}{p_0} \right)_{\alpha=\alpha_0} &= c^2 h_0^2 q_0 q_0 p \bar{p} \end{aligned} \right\} \quad (9)$$

Now, in order to construct the required stress functions, we assume that the elastic body is the common region of two simply-connected domains, i.e. the domain of a circular cylinder and that of an infinite body excluding a cavity and, make use of two groups of the stress functions, which have no singularities in the circular cylinder and have singularities at the focuses of the spheroidal cavity. These are expressed by prolate spheroidal harmonics and cylindrical ones:

$$[\text{II}] \quad \varphi_0 = p_0 \sum_{m=0}^{\infty} A_m Q_{2m}(q) P_{2m}(p), \quad \varphi_1 = p_0 \sum_{m=0}^{\infty} B_m Q_{2m+1}(q) P_{2m+1}(p) \quad (10)$$

$$[\text{III}] \quad \varphi_0 = p_0 \int_0^{\infty} \psi_1(\lambda) I_0(\lambda r) \cos \lambda z d\lambda, \quad \varphi_1 = p_0 \int_0^{\infty} \lambda \psi_2(\lambda) I_0(\lambda r) \sin \lambda z d\lambda \quad (11)$$

where A_m , B_m and $\psi_1(\lambda)$, $\psi_2(\lambda)$ are coefficients of superposition and arbitrary functions of λ , which are specified by the boundary conditions. $P_n(p)$ and $Q_n(q)$ are the ordinary Legendre functions of the first and second kinds, respectively, defined as follows:

$$P_n(p) = \frac{1}{2^n} \frac{d^n}{dp^n} (p^2 - 1)^n$$

$$Q_n(q) = \frac{1}{2} P_n(q) \log \frac{q-1}{q+1} - W_n(q)$$

where $W_n(q)$ is the polynomial part of the expansion.

$I_n(\lambda r)$ represents a modified Bessel function of order n .

In a particular case, where the cavity is much smaller than the cylinder, that is to say, when the cavity is embedded in an infinite medium, the coefficients in Eq.(10) can be determined as follows:

$$\left. \begin{aligned} A_0 &= \frac{c^2}{2D} [(-3q_0^2 + (1+4\nu))q_0 Q_0 + (3q_0^2 - 4\nu)], & A_1 &= \frac{c^2}{D} [(1-2\nu)q_0 Q_0 - (3-2\nu)] \\ B_0 &= \frac{3c^2}{2D} \left(q_0 Q_0 - \frac{q_0^2 - 2}{q_0^2} \right), & A_m &= 0 \quad (m \geq 2), & B_m &= 0 \quad (m \geq 1) \end{aligned} \right\} \quad (12)$$

where $D = 3c \left\{ (1+\nu)q_0 Q_0^2 + \frac{(1-2\nu)q_0^2 + 1}{q_0^2} Q_0 - \frac{(2-\nu)q_0^2 - q_0}{q_0^4} \right\}$, $Q_0 \equiv Q_0(q_0) = \frac{1}{2} \log \frac{q_0+1}{q_0-1}$

The result is coincident with the one obtained by Edwards (8).

The auxiliary stress functions [II] and [III] are given by expressions referred to the prolate spheroidal and cylindrical coordinates respectively. Therefore, these functions must be transformed into one kind of the coordinates so as to satisfy the boundary conditions both on the surface of the cylinder and on the surface of the cavity.

In view of the relations (12)*

$$\left. \begin{aligned} \int_0^{\infty} j_{2m}(\lambda c) K_0(\lambda r) \cos \lambda z d\lambda &= (-1)^m \frac{\pi}{2c} Q_{2m}(q) P_{2m}(p) \\ \int_0^{\infty} j_{2m+1}(\lambda c) K_0(\lambda r) \sin \lambda z d\lambda &= (-1)^m \frac{\pi}{2c} Q_{2m+1}(q) P_{2m+1}(p) \end{aligned} \right\} \quad (13)$$

the stress function [II] is rewritten in the form:

$$[\text{IV}] \quad \varphi_0 = p_0 \sum_{m=0}^{\infty} a_m \int_0^{\infty} j_{2m}(\lambda c) K_0(\lambda r) \cos \lambda z d\lambda, \quad \varphi_1 = p_0 \sum_{m=0}^{\infty} b_m \int_0^{\infty} j_{2m+1}(\lambda c) K_0(\lambda r) \sin \lambda z d\lambda \quad (14)$$

where $j_n(\lambda c)$, $K_n(\lambda r)$ are the spherical Bessel functions of the first kind of order n and the modified Bessel functions of the second kind of order n respectively, and

$$a_m = (-1)^m \frac{2c}{\pi} A_m, \quad b_m = (-1)^m \frac{2c}{\pi} B_m$$

* For convenience of computation, we replace $J_{n+1/2}(\lambda c)$ in the expressions in Ref.[12] by $j_n(\lambda c)$.

By using Eq.(2) and appropriate stress-displacement relations for the stress functions [III] and [IV], the following expressions are obtained for the conditions of zero tractions on the surface of the cylinder:

$$\left(\frac{\sigma_r}{\rho_0}\right)_{r=1} = \int_0^\infty \lambda^2 \left[\left\{ I_0(\lambda) - \frac{I_1(\lambda)}{\lambda} \right\} \phi_1(\lambda) + \{ \lambda I_1(\lambda) + (1-2\nu) I_0(\lambda) \} \phi_2(\lambda) + \left\{ \frac{K_1(\lambda)}{\lambda} + K_0(\lambda) \right\} \sum_{m=0}^\infty a_m j_{2m}(\lambda c) + \left\{ (1-2\nu) \frac{K_0(\lambda)}{\lambda} - K_1(\lambda) \right\} \sum_{m=0}^\infty b_m j_{2m+1}(\lambda c) \right] \cos \lambda x d\lambda = 0 \quad (15)$$

$$-\left(\frac{\tau_{rz}}{\rho_0}\right)_{r=1} = \int_0^\infty \lambda^2 \left[I_1(\lambda) \phi_1(\lambda) + \{ \lambda I_0(\lambda) + 2(1-\nu) I_1(\lambda) \} \phi_2(\lambda) - K_1(\lambda) \sum_{m=0}^\infty a_m j_{2m}(\lambda c) + \left\{ K_0(\lambda) - 2(1-\nu) \frac{K_1(\lambda)}{\lambda} \right\} \sum_{m=0}^\infty b_m j_{2m+1}(\lambda c) \right] \sin \lambda x d\lambda = 0 \quad (16)$$

By the application of Fourier transform to Eqs.(15) and (16), respectively, the arbitrary functions $\phi_1(\lambda)$ and $\phi_2(\lambda)$ are found as follows:

$$\phi_1(\lambda) = \sum_{m=0}^\infty \{ f_1(\lambda) a_m j_{2m}(\lambda c) + f_2(\lambda) b_m j_{2m+1}(\lambda c) \}, \quad \phi_2(\lambda) = \sum_{m=0}^\infty \{ f_3(\lambda) a_m j_{2m}(\lambda c) + f_4(\lambda) b_m j_{2m+1}(\lambda c) \} \quad (17)$$

where

$$\left. \begin{aligned} F_0(\lambda) f_1(\lambda) &= -[\lambda^2 + 2(1-\nu)] I_1(\lambda) K_1(\lambda) + \lambda^2 I_0(\lambda) K_0(\lambda) + 2(1-\nu) \\ F_0(\lambda) f_2(\lambda) &= [\lambda^2 - 2(1-\nu)(1-2\nu)]/\lambda, \quad F_0(\lambda) f_3(\lambda) = 1 \\ F_0(\lambda) f_4(\lambda) &= -[\lambda^2 + 2(1-\nu)] I_1(\lambda) K_1(\lambda) + \lambda^2 I_0(\lambda) K_0(\lambda) - 2(1-\nu)/\lambda \\ F_0(\lambda) &= \lambda^2 [I_0(\lambda)]^2 - [\lambda^2 + 2(1-\nu)] [I_1(\lambda)]^2 \end{aligned} \right\} \quad (18)$$

$$F_0(\lambda) = \lambda^2 [I_0(\lambda)]^2 - [\lambda^2 + 2(1-\nu)] [I_1(\lambda)]^2 \quad (19)$$

These are the conditions of zero tractions derived from the auxiliary stress functions on the surface of the cylinder.

Next, to adjust the boundary conditions on the surface of the cavity, it is necessary to expand the stress function [III] in the prolate spheroidal coordinates. The following relations are useful: (12)*

$$\left. \begin{aligned} I_0(\lambda r) \cos \lambda z &= \sum_{m=0}^\infty (-1)^m (4m+1) j_{2m}(\lambda c) P_{2m}(q) P_{2m}(p) \\ I_0(\lambda r) \sin \lambda z &= \sum_{m=0}^\infty (-1)^m (4m+3) j_{2m+1}(\lambda c) P_{2m+1}(q) P_{2m+1}(p) \end{aligned} \right\} \quad (20)$$

The resulting expansion is expressed in the following form:

$$[V] \quad \varphi_0 = p_0 \sum_{n=0}^\infty \alpha_n P_{2n}(q) P_{2n}(p), \quad \varphi_4 = p_0 \sum_{n=0}^\infty \beta_n P_{2n+1}(q) P_{2n+1}(p) \quad (21)$$

where

$$\left. \begin{aligned} \alpha_n &= \frac{2c}{\pi} (4n+1) \sum_{m=0}^\infty (-1)^{m+n} \left\{ A_m \int_0^\infty f_1(\lambda) j_{2n}(\lambda c) j_{2m}(\lambda c) d\lambda \right. \\ &\quad \left. + B_m \int_0^\infty f_2(\lambda) j_{2n}(\lambda c) j_{2m+1}(\lambda c) d\lambda \right\} \\ \beta_n &= \frac{2c}{\pi} (4n+3) \sum_{m=0}^\infty (-1)^{m+n} \left\{ A_m \int_0^\infty \lambda f_3(\lambda) j_{2n+1}(\lambda c) j_{2m}(\lambda c) d\lambda \right. \\ &\quad \left. + B_m \int_0^\infty \lambda f_4(\lambda) j_{2n+1}(\lambda c) j_{2m+1}(\lambda c) d\lambda \right\} \end{aligned} \right\} \quad (22)$$

Further, the boundary condition (9) can be written in the form.

$$\left. \begin{aligned} \left(\frac{\sigma_a}{\rho_0 c^2 h^4}\right)_{a=a_0} &= \frac{8}{35} c^2 \bar{q}_0^2 P_4(p) - \left(\frac{2}{3} q_0^2 - \frac{4}{7}\right) c^2 \bar{q}_0^2 P_2(p) - \left(\frac{1}{3} q_0^2 - \frac{1}{5}\right) c^2 \bar{q}_0^2 P_0(p) \\ \left(\frac{\tau_{a\theta}}{\rho_0 c^2 h^4 \bar{q} \bar{p}}\right)_{a=a_0} &= -\frac{2}{35} c^2 q_0 P_4'(p) + \left(\frac{1}{3} q_0^2 - \frac{1}{7}\right) c^2 q_0 P_2'(p) \end{aligned} \right\} \quad (23)$$

Deriving the stresses from the stress functions [II] and [V] and making use of the recurrence relations for the Legendre functions, one can satisfy the boundary condition (23) on the surface of the cavity as follows:

$$\left. \begin{aligned} \left(\frac{\sigma_a}{\rho_0 c^2 h^4}\right)_{a=a_0} &= \sum_{n=0}^\infty \{ s_{A1} A_{n-1} + s_{A2} A_n + s_{A3} A_{n+1} + s_{B1} B_{n-2} + s_{B2} B_{n-1} + s_{B3} B_n + s_{B4} B_{n+1} \\ &\quad + s_{A1} \alpha_{n-1} + s_{A2} \alpha_n + s_{A3} \alpha_{n+1} + s_{B1} \beta_{n-2} + s_{B2} \beta_{n-1} + s_{B3} \beta_n + s_{B4} \beta_{n+1} \} P_{2n}(p) \\ &= \frac{8}{35} c^2 \bar{q}_0^2 P_4(p) - \left(\frac{2}{3} q_0^2 - \frac{4}{7}\right) c^2 \bar{q}_0^2 P_2(p) - \left(\frac{1}{3} q_0^2 - \frac{1}{5}\right) c^2 \bar{q}_0^2 P_0(p) \end{aligned} \right\} \quad (24)$$

$$\left. \begin{aligned} \left(\frac{\tau_{a\theta}}{\rho_0 c^2 h^4 \bar{q} \bar{p}}\right)_{a=a_0} &= \sum_{n=1}^\infty \{ t_{A1} A_{n-1} + t_{A2} A_n + t_{A3} A_{n+1} + t_{B1} B_{n-2} + t_{B2} B_{n-1} + t_{B3} B_n + t_{B4} B_{n+1} \\ &\quad + t_{A1} \alpha_{n-1} + t_{A2} \alpha_n + t_{A3} \alpha_{n+1} + t_{B1} \beta_{n-2} + t_{B2} \beta_{n-1} + t_{B3} \beta_n + t_{B4} \beta_{n+1} \} P_{2n}'(p) \\ &= -\frac{2}{35} c^2 q_0 P_4'(p) + \left(\frac{1}{3} q_0^2 - \frac{1}{7}\right) c^2 q_0 P_2'(p) \end{aligned} \right\} \quad (25)$$

where

$$\left. \begin{aligned} s_{A1} &= \frac{2n(2n-1)}{(4n-1)(4n-3)} \{ (2n-2) Q_{2n-2}(q_0) - q_0 Q_{2n-2}'(q_0) - \bar{q}_0^2 Q_{2n-2}''(q_0) \} \\ s_{A2} &= -\frac{1}{(4n+3)(4n-1)} [2n(2n+1) Q_{2n}(q_0) - 2(4n^2+2n-1) q_0 Q_{2n}'(q_0) \\ &\quad - \{ (4n+3)(4n-1) \bar{q}_0^4 + 2(4n^2+2n-1) \bar{q}_0^2 \} Q_{2n}''(q_0)] \end{aligned} \right\}$$

$$\begin{aligned}
s_{A3} &= -\frac{(2n+2)(2n+1)}{(4n+5)(4n+3)} \{ (2n+3)Q_{2n+2}(q_0) + q_0 Q_{2n+2}'(q_0) + \bar{q}_0^2 Q_{2n+2}''(q_0) \} \\
s_{B1} &= \frac{2n(2n-1)(2n-2)c}{(4n-1)(4n-3)(4n-5)} \{ (2n-3)(1-2\nu)q_0 Q_{2n-3}(q_0) + 2(n-\nu)\bar{q}_0^2 Q_{2n-3}'(q_0) \} \\
s_{B2} &= -\frac{2nc}{(4n+3)(4n-1)(4n-3)} [2(-2n-1)(4n+3)(4n-3)\nu q_0 \bar{q}_0^2 \\
&\quad + (2n-1)(6n^2+n-3)(1-2\nu)q_0 Q_{2n-1}(q_0) + \{ (2n-2\nu+3)(4n+3)(4n-3)\bar{q}_0^4 \\
&\quad + (24n^3+4[1-2\nu]n^2-14n-3[1-2\nu])\bar{q}_0^2 \} Q_{2n-1}'(q_0)] \\
s_{B3} &= -\frac{(2n+1)c}{(4n+5)(4n+3)(4n-1)} [2(-2n+2)(4n+5)(4n-1)\nu q_0 \bar{q}_0^2 \\
&\quad + (2n+2)(6n^2+5n-2)(1-2\nu)q_0 Q_{2n+1}(q_0) + \{ (4n+5)(4n-1)(2n+2\nu-2)\bar{q}_0^4 \\
&\quad + 2(12n^3+4[4+\nu]n^2+4\nu n-[1+2\nu])\bar{q}_0^2 \} Q_{2n+1}'(q_0)] \\
s_{B4} &= -\frac{(2n+3)(2n+2)(2n+1)c}{(4n+7)(4n+5)(4n+3)} \{ (2n+4)(1-2\nu)q_0 Q_{2n+3}(q_0) + (2n+2\nu+1)\bar{q}_0^2 Q_{2n+3}'(q_0) \} \\
t_{A1} &= \frac{(2n-1)(2n-3)}{(4n-1)(4n-3)} Q_{2n-2}'(q_0) \\
t_{A2} &= \frac{1}{(4n+3)(4n-1)} [(4n+3)(4n-1)q_0 Q_{2n}(q_0) - \{ (4n+3)(4n-1)q_0^2 - 4(2n-1)(n+1) \} Q_{2n}'(q_0)] \\
t_{A3} &= -\frac{(2n+4)(2n+2)}{(4n+5)(4n+3)} Q_{2n-2}'(q_0) \\
t_{B1} &= -\frac{(2n-1)(2n-2)c}{(4n-1)(4n-3)(4n-5)} \{ (2n-3)(n-2\nu)Q_{2n-3}(q_0) \\
&\quad + (3-2\nu)q_0 Q_{2n-3}'(q_0) + \frac{1}{2}\bar{q}_0^2 Q_{2n-3}''(q_0) \} \\
t_{B2} &= -\frac{c}{(4n+3)(4n-1)(4n-3)} \{ \{ (4n+3)(4n-3)(2n-1)(n-2\nu+2)\bar{q}_0^2 + 2n(2n-1)(6n^2 \\
&\quad + [1-4\nu]n-3) \} Q_{2n-1}(q_0) + 2\{ (4n+3)(4n-3)(1-\nu)q_0 \bar{q}_0^2 + (6n^2-n-3)(3-2\nu)q_0 \} Q_{2n-1}'(q_0) \\
&\quad + \{ \frac{1}{2}(4n+3)(4n-3)\bar{q}_0^4 + (6n^2-n-3)\bar{q}_0^2 \} Q_{2n-1}''(q_0) \} \\
t_{B3} &= -\frac{c}{(4n+5)(4n+3)(4n-1)} \{ \{ (n+1)(4n+5)(4n-1)(2n+4\nu-3)\bar{q}_0^2 + (2n+2)(2n+1)(6n^2 \\
&\quad + [5+4\nu]n-2[1-\nu]) \} Q_{2n+1}(q_0) + 2\{ (4n+5)(4n-1)(1-\nu)q_0 \bar{q}_0^2 \\
&\quad + (6n^2+7n-1)(3-2\nu)q_0 \} Q_{2n+1}'(q_0) + \{ \frac{1}{2}(4n+5)(4n-1)\bar{q}_0^4 + (6n^2+7n-1)\bar{q}_0^2 \} Q_{2n+1}''(q_0) \} \\
t_{B4} &= \frac{(2n+3)(2n+2)c}{(4n+7)(4n+5)(4n+3)} \{ (n+2)(2n+4\nu+1)Q_{2n+3}(q_0) \\
&\quad + (3-2\nu)q_0 Q_{2n+3}'(q_0) + \frac{1}{2}\bar{q}_0^2 Q_{2n+3}''(q_0) \}
\end{aligned}
\tag{26}$$

$$\begin{aligned}
t_{B1} &= -\frac{(2n-1)(2n-2)c}{(4n-1)(4n-3)(4n-5)} \{ (2n-3)(n-2\nu)Q_{2n-3}(q_0) \\
&\quad + (3-2\nu)q_0 Q_{2n-3}'(q_0) + \frac{1}{2}\bar{q}_0^2 Q_{2n-3}''(q_0) \} \\
t_{B2} &= -\frac{c}{(4n+3)(4n-1)(4n-3)} \{ \{ (4n+3)(4n-3)(2n-1)(n-2\nu+2)\bar{q}_0^2 + 2n(2n-1)(6n^2 \\
&\quad + [1-4\nu]n-3) \} Q_{2n-1}(q_0) + 2\{ (4n+3)(4n-3)(1-\nu)q_0 \bar{q}_0^2 + (6n^2-n-3)(3-2\nu)q_0 \} Q_{2n-1}'(q_0) \\
&\quad + \{ \frac{1}{2}(4n+3)(4n-3)\bar{q}_0^4 + (6n^2-n-3)\bar{q}_0^2 \} Q_{2n-1}''(q_0) \} \\
t_{B3} &= -\frac{c}{(4n+5)(4n+3)(4n-1)} \{ \{ (n+1)(4n+5)(4n-1)(2n+4\nu-3)\bar{q}_0^2 + (2n+2)(2n+1)(6n^2 \\
&\quad + [5+4\nu]n-2[1-\nu]) \} Q_{2n+1}(q_0) + 2\{ (4n+5)(4n-1)(1-\nu)q_0 \bar{q}_0^2 \\
&\quad + (6n^2+7n-1)(3-2\nu)q_0 \} Q_{2n+1}'(q_0) + \{ \frac{1}{2}(4n+5)(4n-1)\bar{q}_0^4 + (6n^2+7n-1)\bar{q}_0^2 \} Q_{2n+1}''(q_0) \} \\
t_{B4} &= \frac{(2n+3)(2n+2)c}{(4n+7)(4n+5)(4n+3)} \{ (n+2)(2n+4\nu+1)Q_{2n+3}(q_0) \\
&\quad + (3-2\nu)q_0 Q_{2n+3}'(q_0) + \frac{1}{2}\bar{q}_0^2 Q_{2n+3}''(q_0) \}
\end{aligned}
\tag{27}$$

in which $Q_n'(q)$ denotes a derivative of $Q_n(q)$ with regard to q .

The coefficients s_{Ai} , s_{Bi} , t_{Ai} , t_{Bi} are obtained from the coefficients s_{Ai} , s_{Bi} , t_{Ai} , t_{Bi} by replacing $Q_n(q_0)$ with $P_n(q_0)$ respectively.

Comparing the coefficients of the functions $P_{2n}(p)$ and $P_{2n}'(p)$ ($n=0,1,2,\dots$) in Eqs. (24) and (25), we obtain the infinite systems of algebraic equations. Solving the linear equations for the unknown constants A_n , B_n , the solution of this problem is obtained.

All displacements and stresses are derived by combining linearly the stress functions [I]~[III].

3. Numerical results

Numerical calculations are carried out for several values of the equatorial semiaxis a and of the shape ratio $s = a/b = \bar{q}_0/q_0$. Here Poisson's ratio ν is taken as 0.3.

The preceding infinite integrals in Eq. (22) are computed using Simpson's rule of numerical integration. In solving the linear equations for the coefficients A_n and B_n , 10 terms are required to obtain the values of stresses accurate to three significant figures for the largest value of a . It is now possible to compute the stresses at any point in the elastic cylinder.

The stresses σ_r and σ_θ at any point are given by

$$\begin{aligned}
\frac{\sigma_\theta}{p_0} &= c^2 h^2 q^2 \bar{p}^2 + c^2 h^4 \sum_{n=0}^{\infty} [\{ A_n Q_{2n}(q) + \alpha_n P_{2n}(q) \} \{ -\bar{q}^2 p P_{2n}'(p) + (\bar{q}^2 \bar{p}^2 + \bar{p}^4) P_{2n}''(p) \} \\
&\quad + \{ A_n Q_{2n}'(q) + \alpha_n P_{2n}'(q) \} q \bar{q}^2 P_{2n}(p)] - c^2 h^4 \sum_{n=0}^{\infty} [\{ B_n Q_{2n+1}(q) + \beta_n P_{2n+1}(q) \} \\
&\quad \times \{ (3-2\nu)q \bar{q}^2 \bar{p}^2 + 2(2-\nu)q \bar{p}^4 \} P_{2n+1}'(p) + \{ B_n Q_{2n+1}'(q) + \beta_n P_{2n+1}'(q) \} \{ (2\nu \bar{q}^2 p \bar{p}^2 \\
&\quad - [1-2\nu] \bar{q}^4 p \} P_{2n+1}(p) + \{ \bar{q}^4 \bar{p}^2 + \bar{q}^2 \bar{p}^4 \} P_{2n+1}''(p)] \tag{28}
\end{aligned}$$

$$\begin{aligned}
\frac{\sigma_r}{p_0} &= h^2 \sum_{n=0}^{\infty} [-\{ A_n Q_{2n}(q) + \alpha_n P_{2n}(q) \} p P_{2n}'(p) + \{ A_n Q_{2n}'(q) + \alpha_n P_{2n}'(q) \} q P_{2n}(p)] \\
&\quad + (1-2\nu) c h^2 \sum_{n=0}^{\infty} [\{ B_n Q_{2n+1}(q) + \beta_n P_{2n+1}(q) \} q \bar{p}^2 P_{2n+1}'(p) \\
&\quad + \{ B_n Q_{2n+1}'(q) + \beta_n P_{2n+1}'(q) \} \bar{q}^2 p P_{2n+1}(p)] \tag{29}
\end{aligned}$$

The stresses occurring on the surface of the cavity are obtained by setting $q = q_0$, $(=b/c)$, $\bar{q} = \bar{q}_0 (=a/c)$ in the above equations.

Figure 2 shows the variation of the stress σ_θ with φ for a shape ratio $s=0.5$ on the surface of the cavity. For comparison with the case of the spherical cavity, we introduce the angle φ of the ordinary spherical polar coordinates.

The angle φ is measured from the z-axis and has the relation with β on the surface of the cavity: $\tan \varphi = (a/b) \tan \beta$.

The stress σ_θ has a maximum compressive value at the pole $\varphi=0^\circ$ (Point B in Fig.1), is tensile for $\varphi > 10^\circ$ and has maximum tensile value at $\varphi=90^\circ$ (Point A). The value of σ_θ is little affected by the magnitude of a for $\varphi < 20^\circ$ but affected remarkably for $\varphi > 20^\circ$.

In Figure 3 the variation of σ_r with φ on the surface of the cavity is presented. The stress σ_r is compressive except for the case of $a=0.2$ for $\varphi > 65^\circ$. As in Fig.2, σ_r is strongly affected by the magnitude of a for $\varphi > 20^\circ$.

Figure 4 plots the variation of the stress $(\sigma_\theta)_{r=0} = (\sigma_\theta)_{z=0}$ along the r-axis for

$s=0.5$. It is seen from Fig.4 that the stress σ_θ has a maximum value on the surface of the cavity and diminishes as the distance r increases and attains the minimum tensile value on the cylindrical surface. The dashed line indicates the dependence of $(\sigma_\theta)_{\max}$ upon a . The value of $(\sigma_\theta)_{\max}$ increases rapidly with an increasing a . Figures 5 and 6 show the variations of σ_θ and σ_r on the cavity surface for $a=0.2$ with varying shape

Fig.4 Variation of σ_θ with r on the plane $z=0$ for $s=0.5$.

Fig.2 Variation of σ_θ on the spheroidal surface with φ for $s=a/b=0.5$.

Fig.5 Variation of σ_θ on the spheroidal surface with φ for $a=0.2$.

Fig.3 Variation of σ_r on the spheroidal surface with φ for $s=0.5$.

Fig.6 Variation of σ_r on the spheroidal surface with φ for $a=0.2$.

Fig.7 Variation of σ_{max} with the shape ratio s ($r=a$, $z=0$).

ratios $s=0.143$ ($1/7$), 0.5 and 1 . Here, for the limiting value $s=1$, the spheroid becomes a sphere, whose solution is given by Kawashima et al.⁽¹³⁾

From the above results, the maximum of all normal stresses is found to be σ_{θ} at the equator point A. Figure 7 shows the variation of the maximum stress $\sigma_{max} = (\sigma_{\theta})_{\theta=0, \varphi=90^\circ}$ with the shape ratio s .

The magnitude of σ_{max} is largest in the case of the spherical cavity ($s=1$), and decreases as s decreases. In the limiting case of $s=0$, the value of σ_{max} coincides with the corresponding stress σ_i for an infinite circular hollow cylinder with inner radius a and outer radius 1 .

The value is obtained by equating the resultant force on a cross section to the resultant of the uniform axial stress at infinity, and is equal to $p_0/(1-a^2)$.

4. Conclusions

In this paper, the axisymmetric problem of an infinite elastic circular cylinder having a prolate spheroidal cavity under uniform axial tension was analyzed on the basis of Dougall's stress functions. The computations were carried out successfully and the numerical results were presented for the stresses around the cavity. To sum up the results:

(1) The maximum tensile stress occurs at the surface of the cavity across the minimum section. This stress increases as the size of the cavity becomes larger. In contrast, the maximum compressive stress occurs at the poles on the surface of the cavity and is little affected by the size of the cavity.

(2) The curves of the stresses σ_{θ} and σ_r on the surface of the cavity become

constant except for the small range of φ , as the shape ratio $s=a/b$ becomes zero.

(3) The maximum tensile stress occurring on the surface of the cavity is largest in the case of the spherical cavity ($s=1$) and decreases as s diminishes.

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