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Analysis of An Unsteady Aerodynamic Force on a Blade

Due to Ununiform Amplitude Gusts*

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Lift fluctuations of a cambered blade with angle of attack under periodic gusts with ununiform amplitude were studied by using vortex theory under the assumption that the gust amplitude on an airfoil varies linearly as described $W(b+ax)$. Consequently it was clarified that the developed theory is valid by comparing with other results and that the ununiformity of gust amplitude gives strong effects on the fluctuating lift for small reduced frequency but does not for large reduced frequency.

1. Introduction

In an axial turbomachine, it is well known that the unsteady lift acts on rotor blades which move through the wakes shed from the preceding stator blade row. Such a fluctuating lift induces a blade vibration and gives rise to fatigue failure, so it is necessary to evaluate such a force quantitatively. The study on the unsteady aerodynamic force has been made theoretically by Karman and Sears. [1]

Sears investigated the unsteady lift on a single airfoil in a sinusoidal traveling gust in which the velocity perturbation is normal to the undisturbed flow. Horlock[2] analyzed the fluctuating lift on a single airfoil with angle of attack due to a longitudinal gust and clarified that it also had considerable effects on the unsteady lift.

H. Nauman and H. Yeh[3] obtained the unsteady lift on a cambered airfoil with angle of attack moving through both longitudinal and transverse gusts; the knowledge gained by this analysis clearly indicates design trends to obtain minimum lift fluctuations.

In the above analysis, a single airfoil was considered as an object of investigation. On the contrary, there has been analysis of cascade recently. S. Murata and others[4] obtained the unsteady lift on a cascade blade by using the acceleration potential method combined with conformal mapping method.

Authors[5,6] applied these theoretical results to problems of blade vibration and obtained resonance stresses and compared them with experimental results. However in this calculation, an amplitude of fluctuating gust was required as input data. Therefore this evaluation is very important

to obtain the resonant stress exactly. In the previous analysis, it was assumed that the amplitude of fluctuating gust is uniform on the blade. But according to Kawashima's investigation[7], it was pointed out that the Sears function did not agree with experimental results for small reduced frequency. Moreover they showed that this resulted from the uniformity of fluctuating gust and obtained a modified Sears function which agreed with experimental results.

However in this calculation, dealing with a flat plate parallel to the flow, the effect of camber and angle of attack could not be discussed and the report did not clarify the reason why the effect of ununiformity of gust amplitude was strong for small reduced frequency and, on the contrary, nothing at all for large reduced frequency.

Then in this paper, the unsteady lift on the cambered airfoil with angle of attack in the disturbed flow which has an ununiform amplitude of fluctuating gust is analyzed by vortex theory and these effects are discussed.

2. Assumptions and Notations

Assumptions used here are as follows:

- (1) The flow is 2-dimensional, incompressible and inviscid, and the amplitude of fluctuating gust is small as compared with mean flow.
- (2) The gust amplitude on an airfoil varies linearly.

Fig. 1. Amplitude variation of the fluctuating wind (S. Kawashima & others)

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Notations

- a : gradient when variation of gust amplitude is assumed as $y=ax+b$
 b : y-cut point when variation of gust amplitude is assumed as $y = ax + b$
 ω : circular frequency of the gust
 U : main flow velocity
 k : reduced circular frequency ($=\omega/U$)
 v_p : amplitude of fluctuating velocity perpendicular to the chord of airfoil
 u_p : amplitude of fluctuating velocity parallel to the chord of airfoil
 v : induced velocity
 v_s : steady component of v
 v_t : unsteady component of v
 Γ_t : unsteady vorticity
 Γ_b : bound vorticity on an airfoil
 Γ_f : free vorticity on an airfoil and in the wake
 $C(k)$: Theodorsen function
 J_0, J_1 : Bessel functions of the first kind of zero and first order.
 F_a, F_f, S : influence factors of angle of attack and camber respectively on the unsteady lift coefficient and Sears function in the case of uniform gust amplitude.
 G_a, G_f, G_s : influence factors of angle of attack and camber respectively on unsteady lift coefficient and Sears function due to the deviation from uniform amplitude gust.

The assumption (2) used here is reasonable for small chord length from the experimental result shown in Fig.1. S.Kawashima made a sinusoidal gust in the wind tunnel by using a cascade blade and measured the gust amplitude at some points apart from the cascade with anemometer. Fig.1 shows experimental values of amplitude of gust W against distance from the cascade; the amplitude decreases with k being smaller than 0.45 and increases with k being larger than 0.45. This variation being considered to be approximately linear, for the sake of simplicity of analysis, it is assumed that this variation is linear.

3. Analysis

3.1 Expression of gust

When the gust amplitude on an airfoil varies linearly as shown in Fig.2 and the origin of the airfoil is taken as basis, the fluctuating velocity at an arbitrary point x on the airfoil is given as follows:

$$w(x, t) = \bar{w}_p (b + ax) e^{j\omega(t - x/U)} \quad (1)$$

where U is the main flow velocity, $\bar{w}_p$ the fluctuating gust amplitude at leading edge, ω the circular frequency of gust, a and b are parameters of a linear function of x , the coordinate of cross point of the y axis and envelope (straight line) of gust

gust fluctuation, a the gradient of the linear function. The following relationship holds between a and b :

$$b = a + 1 \quad (2)$$

Where $a > 0$ and $a < 0$ correspond to $k > 0.45$ and $k < 0.45$ in Fig.1 respectively. Describing components of gust which are perpendicular and parallel to the chord of airfoil as v_p and u_p respectively, u_p and v_p in turbo-machine are given as follows:

$$v_p = w_p \sin \phi, \quad u_p = w_p \cos \phi \quad (3)$$

Where w_p is the gust amplitude of the wake and ϕ is an angle between rotor and stator blades.

3.2 Boundary Conditions on an airfoil

Boundary conditions on a cambered airfoil with angle of attack positioned in a flow which has mean velocity U and small fluctuating velocities u_p and v_p , can be described as follows:

$$\frac{dY}{dx} = \frac{U\alpha + v_p + v}{U + u_p} \quad (4)$$

Where $Y(x)$ is a camber line, α the angle of attack, v the induced velocity. This induced velocity v is composed of steady part v_s and time-dependent part v_t , i.e.,

$$v(x, t) = v_s(x) + v_t(x, t) \quad (5)$$

Subtracting the mean flow relation

$$\frac{dy}{dx} = \frac{U\alpha + v_s}{U}$$

from Eq.(4) yields:

$$v_t(x, t) = \left(\frac{dY}{dx} - \bar{u}_p - \bar{v}_p \right) (b + ax) e^{-j k x} e^{j \omega t} \quad (6)$$

For a thin airfoil with parabolic mean camber line, the camber line is described as

$$y = f(-x^2 + 1)$$

where f is the ratio of maximum camber to the half-chord length. Hence

$$v_t = (2fx\bar{u}_p + \bar{v}_p)(b + ax)e^{-j k x} \quad (7)$$

where $v(x, t) = \bar{v}(x) e^{-j k x} e^{j \omega t}$. Hereafter, fluctuating quantities will be described as

$$\delta(x, t) = \bar{\delta}(x) e^{j \omega t}$$

3.3 The relationship between bound and free vortices and induced velocity

The velocities induced by distributed vortices are also considered as the sum of steady and unsteady components. Moreover its unsteady component consists of bound and free vortices, i.e.,

$$\gamma_t(x, t) = \gamma_b(x, t) + \gamma_f(x, t)$$

$$\text{domains} \quad \begin{cases} \gamma_b(x, t) & -1 < x < 1 \\ \gamma_f(x, t) & -1 < x < \infty \end{cases} \quad (8)$$

When the vorticity distribution on any segment of an airfoil changes with time, free vorticities γ_f will be shed and carried away with the mean velocity U . Consider, for instance, a segment dx of the airfoil: the free vortices enter at the rate of $U\gamma_f$ and leave at the rate of $U[\gamma_f - (\partial\gamma_f/\partial x)dx]$, hence the net vorticity flux during time dt is $U(\partial\gamma_f/\partial x)dxdt$.

This represents a net outward flux and, for inviscid flow, it must be equal to a decrease of total vorticity within $d\bar{x}$ during dt , i.e., $-(\partial\gamma/\partial t)dxdt$. Noting that $\partial\gamma/\partial t = \partial\gamma_t/\partial t$, we obtain

$$\frac{\partial\gamma_t}{\partial t} + U \frac{\partial\gamma_f}{\partial x} = 0 \quad (9)$$

$$\text{From Eq. (9)} \quad U\gamma_f(1, t) = - \int_{-1}^1 \frac{\partial\gamma_t(\xi, t)}{\partial t} d\xi \quad (10)$$

This is carried downstream with the velocity U , hence

$$\gamma_f(x, t) = \gamma_f\left(1, t - \frac{x-1}{U}\right) \quad (11)$$

According to Eqs. (10) and (11), free vorticity in the wake is described by

$$\tilde{\gamma}_f(x) = -jk e^{jk(1-x)} \int_{-1}^1 \tilde{\gamma}_t(\xi) d\xi \quad (x > 1) \quad (12)$$

Induced velocity being made by γ_t on the airfoil and γ_f in the wake, we get the following equation:

$$\bar{v}_t(x) = \frac{1}{2\pi} \int_{-1}^1 \frac{\tilde{\gamma}_t(\xi)}{\xi - x} d\xi + \frac{1}{2\pi} \int_1^\infty \frac{\tilde{\gamma}_f(\xi)}{\xi - x} d\xi \quad (13)$$

Substituting Eq. (12) into Eq. (13) we get

$$\frac{1}{2\pi} \int_{-1}^1 \frac{\tilde{\gamma}_t(\xi)}{\xi - x} d\xi = \frac{1}{2\pi} jk e^{jk} \int_{-1}^1 \tilde{\gamma}_t(\xi) d\xi \int_1^\infty \frac{e^{-jk\xi}}{\xi - x} d\xi + \bar{v}_t(x) \quad (14)$$

The solution of Eq. (14), considering Kutta's condition, is obtained by Söhnngen's formula.

$$\begin{aligned} \text{i.e.,} \quad \tilde{\gamma}_t(x) = & - \left[\frac{2}{\pi} \sqrt{\frac{1-x}{1+x}} \int_{-1}^1 \bar{v}_t(\xi) \sqrt{\frac{1+\xi}{1-\xi}} \frac{d\xi}{\xi - x} \right. \\ & \left. - \frac{4}{\pi^2 (jH_0^{(2)} + H_1^{(2)})} \sqrt{\frac{1-x}{1+x}} \int_{-1}^1 \bar{v}_t(\xi) \sqrt{\frac{1+\xi}{1-\xi}} d\xi \int_1^\infty e^{-jk\eta} \sqrt{\frac{\eta+1}{\eta-1}} \frac{d\eta}{x-\eta} \right] \end{aligned} \quad (15)$$

On the other hand, $\tilde{\gamma}_b(x)$ is obtained from Eq. (8) and Eq. (9) as follows.

$$\tilde{\gamma}_b(x) = \tilde{\gamma}_t(x) + jk \int_{-1}^x \tilde{\gamma}_t(\xi) d\xi \quad (-1 \leq x \leq 1) \quad (16)$$

Substituting Eq. (15) into Eq. (16)

$$\tilde{\gamma}_b(x) = - \frac{2}{\pi} \int_{-1}^1 \bar{v}_t(\xi) \left[jk \frac{1}{2} \ln \frac{1-\xi x + \sqrt{1-\xi^2} \sqrt{1-x^2}}{1-\xi x - \sqrt{1-\xi^2} \sqrt{1-x^2}} + \sqrt{\frac{1-x}{1+x}} \sqrt{\frac{1+\xi}{1-\xi}} \left\{ \frac{1}{\xi-x} - (1-C(k)) \right\} \right] d\xi \quad (17)$$

where $C(k)$ is Theodorsen function and defined as :

$$C(k) = H_1^{(2)}(k) / \{H_1^{(2)}(k) + jH_0^{(2)}(k)\} \quad (18)$$

and $H_0^{(2)}$, $H_1^{(2)}$ are Hankel functions of the second kind of zero and first order, respectively.

$\bar{v}_t(x)$ in Eq. (17) is given by Eq. (7), therefore substituting this into Eq. (17) and using the transformation $x = -\cos \theta$, $\xi = -\cos \vartheta$ we get

$$\begin{aligned} \tilde{\gamma}_b(\theta) = & - \frac{2}{\pi} \oint_0^\pi (2f \cos \vartheta \bar{u}_p - \bar{v}_p)(b - a \cos \vartheta) e^{jk \cos \vartheta} \left[jk \sin \vartheta \frac{1}{2} \ln \frac{1 - \cos(\theta + \vartheta)}{1 - \cos(\theta - \vartheta)} + \frac{\sin \theta}{\cos \theta - \cos \vartheta} \right. \\ & \left. + \frac{1}{2} [1 + \cos \vartheta + (1 - \cos \vartheta) \{2C(k) - 1\}] \cot \frac{\theta}{2} \right] d\vartheta \end{aligned} \quad (19)$$

Fig. 2. Airfoil subjected to ununiform amplitude gusts.

The integration of each term in the forgoing equation is laborious; Appendix 1 gives an outline of the steps. The final result for $\bar{r}_b$ is

$$\begin{aligned}\bar{r}_b(\theta) = & \left[2f\bar{u}_p \left\{ F(k) \cot \frac{\theta}{2} + \frac{4}{k} \sum_{n=1}^{\infty} j^{n-1} J_n(k) \sin n\theta \right\} + 2\bar{v}_p S(k) \cot \frac{\theta}{2} \right] \times b + \bar{v}_p \left[F(k) \cot \frac{\theta}{2} \right. \\ & + \frac{4}{k} \sum_{n=1}^{\infty} j^{n-1} J_n(k) \sin n\theta \left. \right] \times a + f\bar{u}_p \left[\{J_0(k) + jJ_1(k)\} + \{2C(k) - 1\} \{J_0(k) - jJ_1(k)\} \right] \cot \frac{\theta}{2} \times a \\ & + 2f\bar{u}_p \left[-\frac{4}{jk} \left\{ -\frac{2}{k} \sum_{n=1}^{\infty} j^{n-1} J_n(k) \sin n\theta + J_0(k) \sin \theta + 2 \cos \theta \sum_{n=0}^{\infty} j^n J_n(k) \sin n\theta \right\} \right. \\ & \left. + \frac{1}{2} \{A(k) + jB(k) + \{2C(k) - 1\} \{A(k) - jB(k)\} \cot \frac{\theta}{2} \} \right] \times a\end{aligned}\quad (20)$$

where $F(k) = \{2C(k) - 1\} \left[J_0(k) - \frac{J_1(k)}{k} - jJ_1(k) \right] - \left[J_0(k) - \frac{J_1(k)}{k} + jJ_1(k) \right]$

$S(k)$: Sears function which is given as follows.

$$S(k) = [J_0(k) - jJ_1(k)]C(k) + jJ_1(k)$$

and $A(k) = J_0(k) - 2J_1(k)/k$, $B(k) = (1 - 4/k^2)J_1(k) + 2J_0(k)/k$

3.4 Pressure distribution and lift on an airfoil

Pressure distribution arising on an airfoil from $r_b(\theta)$ is obtained by applying Euler's equation the flow on the upper and lower sides of an airfoil as follows.

$$\bar{\pi}(\theta) = P_{us} - P_{ls} = \rho[U\bar{r}_b(\theta) + \bar{u}_p(\theta)\gamma_s(\theta)] \quad (21)$$

Substituting $\bar{r}_b(\theta)$ of Eq.(20), $r_s(\theta)$ of the well known results from steady airfoil theory, i.e.,

$$\gamma_s(\theta) = 2U[\alpha \cot(\theta/2) + 2f \sin \theta]$$

and $\bar{u}_p(\theta) = \bar{u}_p(b - a \cos \theta)e^{jk \cos \theta}$ into the aforementioned equation (21) and integrating this equation with respect to x from $x=-1$ to $x=1$, we obtain the lift as follows.

$$\bar{L} = \int_{-1}^1 \bar{\pi}(x) dx = \oint_0^{2\pi} \bar{\pi}(\theta) \sin \theta d\theta = (L_f + L_a + L_s) \times b + (L_f^* + L_a^* + L_s^*) \times a \quad (22)$$

where

$$\begin{aligned}L_f &= 2\pi\rho U \bar{u}_p f F_f(k) & F_f(k) &\equiv F(k) + 4J_1(k)/k \\ L_a &= 2\pi\rho U \bar{u}_p \alpha F_a(k) & F_a(k) &\equiv J_0(k) + jJ_1(k), \quad F_{-a}(k) \equiv J_0(k) - jJ_1(k) \\ L_s &= 2\pi\rho U \bar{v}_p S(k) \\ L_f^* &= 2\pi\rho U \bar{u}_p f G_f(k) & G_f(k) &\equiv \frac{1}{2} [F_a(k) + \{2C(k) - 1\} F_{-a}(k) - j4 \{J_0(k) - 2J_1(k)/k\} / k \\ & & & + [A(k) + jB(k) + \{2C(k) - 1\} \{A(k) - jB(k)\}]] \\ L_a^* &= -2\pi\rho U \bar{u}_p \alpha G_a(k) & G_a(k) &\equiv J_0(k) + jJ_1(k) - J_1(k)/k \\ L_s^* &= 2\pi\rho U \bar{v}_p G_s(k) & G_s(k) &\equiv \frac{1}{2} [F(k) + 2J_1(k)/k]\end{aligned}$$

From these results, the unsteady lift in the case of varying gust amplitude on the airfoil is described by Eq.(22) and Eq.(3) as follows.

$$\frac{\bar{L}}{2\pi\rho U w_p \sin \phi} = [\cot \phi \{fF_f(k) + \alpha F_a(k)\} + S(k)]b + [\cot \phi \{fG_f(k) - \alpha G_a(k)\} + G_s(k)]a \quad (23)$$

$F_f(k)$, $F_a(k)$ and $S(k)$ in Eq.(23) have already been given in ref.(3), but $G_f(k)$, $G_a(k)$ and $G_s(k)$ are newly obtained in the present analysis.

4. Calculation results

In this chapter, we will compare the calculation results obtained here with other calculation results and experimental results and confirm that present calculation results are reasonable and discuss the effects of the ununiformity of gust amplitude on the unsteady aerodynamic force in the case of the blade with camber and angle of attack.

4.1 Comparison between present result and other calculation or experimental results.

Fig.3. Comparison of present method and other results on Sears function.

The modified Sears function represented in ref.(7) (dotted line in Fig. 3) and the experimental result (mark o. in Fig.3) are shown in Fig.3. In order to compare present calculation results with these ones, we need the values of parameters a and b which represent the ununiformity of gust amplitude for each reduced frequency from Fig. 1. Reduced frequencies in Fig. 1 were obtained by using the chord length (15 cm) of cascade blade which was used in making gusts. On the other hand, reduced frequencies in Fig. 3 were obtained by using the chord length (20 cm) of the blade which was used in measuring the aerodynamic force. Therefore these are not identical. Table 1 gives the modified values of a and b taking account of this fact.

On the other hand, putting $f=\alpha=0$ in Eq.(23), we can obtain the unsteady lift arising from a flat plate without angle of attack (the same as ref.(7)).

The functions $S(k)$ and $G(k)$ represented in Eq.(23) are shown in Fig. 4.

k	a	b
0.067	-0.167	0.833
0.307	-0.135	0.865
0.427	-0.073	0.927
0.600	0.000	1.000
0.987	0.107	1.107

Table 1. Parameters of the gust amplitude (obtained from Fig.1)

The lift for each reduced frequency, obtained by using these values, is indicated by mark • in Fig. 3. As shown in Fig. 3, the present results agree with Kawashima's results and they are proved to be reasonable.

Fig. 5 shows the unsteady lift coefficient for each frequency in the case of an ununiform fluctuating gust acting normal to a flat plate. The parameter a in this figure, as previously shown, represents the degree of ununiformity. Naturally the larger the absolute value of a is, the stronger the ununiformity is. Negative sign represents that the gust amplitude decreases on the airfoil, while positive sign represents that the gust amplitude increases on the airfoil.

As shown in this figure, the influence of the ununiformity of the gust amplitude is strong for small k but weak for large k . It is also natural that the influence becomes stronger with the absolute value of a becoming large. Moreover we can understand that in the case of $a < 0$ the absolute value tends to be small and the phase of the unsteady lift coefficient tends to advance over that in the case of $a = 0$ (the Sears function). When a becomes positive this tendency reverses though it is not shown here.

4.2 On the effect of camber

For a cambered airfoil, as known from Eq.(6), the induced velocity can be obtained by the fluctuating velocity u_p whose direction is chordwise.

This corresponds to the case of $\phi \neq 90^\circ$ in Eq.(23). Fig.6 shows the unsteady lift coefficient due to the effect of the camber.

The function $F_f(k)$, described by dotted line, and obtained by H.Naumann and H.Yeh,

Fig. 4. Unsteady lift coefficient due to fluctuating gust perpendicular to the chord of airfoil.

Fig. 5. Unsteady lift coefficient due to ununiform amplitude gusts.

is the unsteady lift coefficient due to the uniform amplitude gust. On the other hand, the function $G_f(k)$, described by solid line shows the effect due to the variation from the uniform amplitude gust (ununiformity). This is obtained by the present analysis.

Fig. 7 and Fig. 8 show the unsteady lift coefficient in the case of $\phi = 45^\circ$, $f = 0.1$ and $\phi = 135^\circ$, $f = 0.1$ respectively. The curve plotted by marks o is the unsteady lift coefficient due to the uniform amplitude gust. From these, we can understand that, in both cases, the absolute value of the unsteady lift coefficient tends to be small and the phase of that tends to advance for small reduced frequency with a decreasing value of a . Moreover the unsteady lift coefficient becomes large for small reduced frequencies and becomes small for large reduced frequencies with ϕ being smaller than 90° .

Fig. 6. Unsteady lift coefficient due to the effect of camber.

Fig. 7. Unsteady lift coefficient due to ununiform amplitude gusts.

Fig. 8. Unsteady lift coefficient due to ununiform amplitude gusts.

4.3 On the effect of angle of attack

In the case of the airfoil with angle of attack, also the unsteady lift suffers the influence of the fluctuating velocity u_p . Therefore the degree of the influence varies with the value of ϕ as in the case of the cambered airfoil. Fig. 9 shows the unsteady lift coefficient due to the effect of angle of attack; $F_a(k)$ described by dotted line is the effect of the uniform amplitude gust, and $G_a(k)$ described by solid line gives the effect of the deviation from this uniform amplitude gust. By using these, we can obtain Fig. 10 and Fig. 11 which are calculated for $\phi=45^\circ$ and $\phi=135^\circ$ in the case of $\alpha=10^\circ$ respectively. In this case, as well, the absolute value of the unsteady lift tends to be small and the phase of that tends to advance with the value of a becoming small for small reduced

frequency. Moreover it is found that the absolute value of the unsteady lift coefficient in the case of $\phi < 90^\circ$ is larger than that in the case of $\phi > 90^\circ$ over all range of reduced frequencies.

4.4 On the reason why the ununiformity of gust amplitude does not give its effect over the range of large reduced frequency.

As is seen from above results, the ununiformity of gust amplitude gives a big effect for small reduced frequency, but no effect for large reduced frequency. This can be understood from the fact that these results are already pointed out in ref.(7). In this study we applied these results to the blade vibration; the result obtained by using the unsteady lift for the uniform amplitude gust with large reduced frequencies also agrees fairly well with the experimental result.

However the reason for this has not been given clearly. Then we will consider it in this section.

The unsteady lift due to the ununiform amplitude gust is given by Eq.(23). Rewriting this by using Eq.(2), we obtain

$$\frac{L}{2\pi\rho U w_p \sin\phi} = [\cot\phi \{fF_f(k) + \alpha F_a(k)\} + S(k)] + a[\cot\phi \{f(F_f(k) + G_f(k)) + \alpha(F_a(k) - G_a(k))\} + \{S(k) + G_s(k)\}] \quad (24)$$

Fig. 9. Unsteady lift coefficient due to the effect of angle of attack.

The first bracket [] in Eq.(24) denotes the unsteady lift coefficient due to the uniform gust and three braces { } in [] express the effect of the ununiformity of gust amplitude on the unsteady lift coefficient due to camber, angle of attack, and the Sears function. Now we can recognize from Figs.(4), (6), (9) that the absolute values of F_f , S are equal to those of G_f , G_s and the phases of F_f , S are reverse to those of G_f , G_s respectively, for $k > 3$. On the contrary, the absolute value and the phase of F_a are the same with those of G_a . This fact implies that the content in each braces { } in Eq.(24) is nearly equal to zero. Therefore the unsteady lift is independent of the value of a , i.e., the degree of ununiformity of gust amplitude. On the contrary, it is recognized that the value of a gives a big effect on the un-

steady lift for small reduced frequencies because the aspects of both figures are very different from each other.

5. Conclusions

The unsteady lift for a cambered airfoil with angle of attack in the disturbed flow which has the ununiform amplitude gust, was analyzed by vortex theory assuming that the gust amplitude varies linearly like $W(b+ax)$. Then we obtained the following results.

- (1) It is made clear that the present theory is reasonable by comparing with other calculation and experimental results.
- (2) The ununiformity of the gust amplitude has strong effect for small reduced frequencies including the case of having camber and angle of attack, but, little effect for large reduced frequencies ($k > 3$)

Fig. 10. Unsteady lift coefficient due to ununiform amplitude gusts.

Fig. 11. Unsteady lift coefficient due to ununiform amplitude gusts.

- (3) In the case that the gust amplitude on an airfoil decreases, the absolute value of the unsteady lift tends to be small and the phase of it tends to advance for small reduced frequency in comparison with the uniform gust, and vice versa in the case that the gust amplitude increases.
- (4) On the effect of camber, the unsteady lift for $\phi > 90^\circ$ is larger than that for $\phi < 90^\circ$ in the case of small reduced frequency, and vice versa in the case of large reduced frequency.
- (5) As to the effect of angle of attack, the unsteady lift for $\phi > 90^\circ$ is larger than that for $\phi < 90^\circ$ for all reduced frequencies.

Finally, we show the values of G_s , G_f and G_a for each reduced frequency in Table 2.

k	$R(G_s)$	$I(G_s)$	$R(G_f)$	$I(G_f)$	$R(G_a)$	$I(G_a)$
0.00	0.500 0	0.000 0	1.000 0	0.000 0	0.500 0	0.000 0
0.01	0.495 8	-0.027 8	0.991 6	-0.043 2	0.499 9	0.005 0
0.02	0.485 0	-0.051 2	0.970 4	-0.077 6	0.499 9	0.010 0
0.04	0.458 5	-0.081 3	0.917 9	-0.113 3	0.499 7	0.020 0
0.06	0.436 7	-0.098 3	0.874 7	-0.123 4	0.499 3	0.030 0
0.08	0.420 2	-0.111 1	0.841 8	-0.125 2	0.498 8	0.040 0
0.10	0.406 5	-0.122 5	0.814 5	-0.124 4	0.498 1	0.049 9
0.20	0.350 5	-0.167 4	0.700 6	-0.098 8	0.492 5	0.099 5
0.30	0.307 0	-0.192 3	0.606 0	-0.038 0	0.483 2	0.148 3
0.40	0.277 5	-0.206 4	0.533 1	0.040 7	0.470 3	0.196 0
0.50	0.262 2	-0.220 1	0.483 4	0.175 4	0.453 9	0.242 3
0.60	0.250 2	-0.230 1	0.434 9	0.196 9	0.434 2	0.286 7
0.70	0.243 6	-0.239 8	0.392 5	0.272 2	0.411 2	0.329 0
0.80	0.240 7	-0.249 4	0.353 2	0.342 5	0.385 2	0.368 8
0.90	0.240 4	-0.258 7	0.315 4	0.407 5	0.356 5	0.405 9
1.00	0.241 9	-0.267 7	0.277 9	0.466 9	0.325 1	0.440 1
1.20	0.248 7	-0.283 1	0.202 3	0.569 1	0.255 9	0.498 3
1.40	0.258 2	-0.294 0	0.124 7	0.648 9	0.179 7	0.541 9
1.60	0.268 7	-0.299 2	0.045 6	0.706 2	0.099 2	0.569 9
1.80	0.279 0	-0.297 8	-0.033 3	0.741 0	0.016 9	0.581 5
2.00	0.288 0	-0.289 6	-0.110 1	0.753 6	-0.064 5	0.576 7
2.50	0.298 8	-0.240 2	-0.278 5	0.694 2	-0.247 2	0.497 1
3.00	0.285 4	-0.156 8	-0.389 2	0.527 8	-0.373 1	0.339 1
3.50	0.243 3	-0.055 5	-0.422 1	0.295 5	-0.419 4	0.137 4
4.00	0.174 9	0.043 9	-0.373 4	0.047 2	-0.380 6	-0.066 0
4.50	0.088 6	0.122 7	-0.256 0	-0.168 1	-0.269 2	-0.231 1
5.00	-0.002 2	0.166 8	-0.097 1	-0.312 2	-0.112 1	-0.327 6
6.00	-0.139 1	0.134 9	0.206 1	-0.324 2	0.196 8	-0.276 7
7.00	-0.150 8	-0.002 5	0.299 9	-0.057 6	0.300 7	-0.004 7
8.00	-0.045 1	-0.119 5	0.135 4	0.216 2	0.142 3	0.234 6
9.00	0.082 9	-0.121 3	-0.123 5	0.264 2	-0.117 6	0.245 3
10.00	0.128 9	-0.018 9	-0.250 9	0.074 6	-0.250 3	0.043 5

Table 2. Real and imaginary values of functions $G_s(k)$, $G_f(k)$, $G_a(k)$

Appendix 1.

Replacing

$$H(\theta, \vartheta) = e^{jk \cos \vartheta} \left[jk \sin \theta \frac{1}{2} \ln \frac{1 - \cos(\theta + \vartheta)}{1 - \cos(\theta - \vartheta)} + \frac{\sin \theta}{\cos \theta - \cos \vartheta} + \frac{1}{2} [1 + \cos \vartheta + (1 - \cos \vartheta) \{2C(k) - 1\}] \cot \frac{\theta}{2} \right] \quad (\text{AP-1})$$

Equation (19) is rewritten as follows.

$$\begin{aligned} \bar{r}_b(\theta) = & -\frac{2}{\pi} \oint_0^\pi (2f \cos \vartheta \bar{u}_p - \bar{v}_p) H(\theta, \vartheta) d\vartheta \times b - \frac{2}{\pi} \oint_0^\pi \bar{v}_p \cos \vartheta H(\theta, \vartheta) d\vartheta \times a \\ & + \frac{2}{\pi} \oint_0^\pi 2f \bar{u}_p \cos^2 \vartheta H(\theta, \vartheta) d\vartheta \times a \end{aligned} \quad (\text{AP-2})$$

The integration of the first term of Eq.(AP-2) is given in ref.(3) and the integration of the second term is obtained by substituting $\bar{v}_p$ for $2f\bar{u}_p$ and zero for $\bar{v}_p$.
Integration of the first term I

$$I_1 = [2f\bar{u}_p \{F(k) \cot \frac{\theta}{2} + \frac{4}{k} \sum_1^\infty j^{n-1} J_n(k) \sin n\theta\} + 2\bar{v}_p S(k) \cot \frac{\theta}{2}] \quad (\text{AP-3})$$

Integration of the second term I₂

$$I_2 = \bar{v}_p \{F(k) \cot \frac{\theta}{2} + \frac{4}{k} \sum_1^\infty j^{n-1} J_n(k) \sin n\theta\} \quad (\text{AP-4})$$

Next in order to integrate the third term of Eq.(AP-2), transforming $\cos^2 \vartheta$ into $(1 + \cos 2\vartheta)/2$ and substituting this into the third term, we obtain

$$I_3 = \frac{2}{\pi} \oint_0^\pi f \bar{u}_p H(\theta, \vartheta) d\vartheta \times a + \frac{2}{\pi} \oint_0^\pi f \bar{u}_p \cos 2\vartheta H(\theta, \vartheta) d\vartheta \times a \quad (\text{AP-5})$$

Putting $\bar{u}_p = 0$, $\bar{v}_p = f\bar{u}_p$ in the first term of Eq.(AP-2), we obtain (the first term of I₃) as follows.

$$I_{31} = 2f\bar{u}_p S(k) \cot \frac{\theta}{2} = f\bar{u}_p [\{J_0(k) + jJ_1(k)\} + \{2C(k) - 1\} \{J_0(k) - jJ_1(k)\}] \cot \frac{\theta}{2} \quad (\text{AP-6})$$

In order to integrate the second term of I₃, I₃₂ is represented as the sum of three integrations by using Eq.(AP-1)

$$\beta_1 = \frac{2}{\pi} f \bar{u}_p \oint_0^\pi \cos 2\vartheta e^{jk \cos \vartheta} jk \sin \theta \frac{1}{2} \ln \frac{1 - \cos(\theta + \vartheta)}{1 - \cos(\theta - \vartheta)} d\vartheta \quad (\text{AP-7})$$

$$\beta_2 = \frac{2}{\pi} f \bar{u}_p \oint_0^\pi \cos 2\vartheta e^{jk \cos \vartheta} \frac{\sin \theta}{\cos \theta - \cos \vartheta} d\vartheta \quad (\text{AP-8})$$

$$\beta_3 = \frac{2}{\pi} f \bar{u}_p \oint_0^\pi \cos 2\vartheta e^{jk \cos \vartheta} \frac{1}{2} [1 + \cos \vartheta + (1 - \cos \vartheta) \{2C(k) - 1\}] \cot \frac{\theta}{2} d\vartheta \quad (\text{AP-9})$$

Using the relationship $\cos 2\vartheta \cos \vartheta = (\cos \vartheta + \cos 3\vartheta)/2$ $\oint_0^\pi e^{jk \cos \vartheta} \cos n\vartheta d\vartheta = \pi j^n J_n(k)$.

$$\beta_3 = f \bar{u}_p [\{A(k) + jB(k)\} + \{2C(k) - 1\} \{A(k) - jB(k)\}] \cot \frac{\theta}{2} \quad (\text{AP-10})$$

where

$$A(k) \equiv J_0(k) - 2J_1(k)/k, \quad B(k) \equiv (1 - 4/k^2)J_1(k) + 2J_0(k)/k$$

Integrating β_1 by parts, we obtain

$$\beta_1 = -\beta_2 - f \bar{u}_p \frac{8}{jk\pi} \oint_0^\pi \cos \vartheta e^{jk \cos \vartheta} jk \sin \theta \frac{1}{2} \ln \frac{1 - \cos(\theta + \vartheta)}{1 - \cos(\theta - \vartheta)} d\vartheta \quad (\text{AP-11})$$

Integrating by parts, the integration of Eq (AP-11) becomes as follows.

$$\beta^* = \frac{1}{jk} \sin \theta \oint_0^\pi e^{jk \cos \vartheta} \frac{1}{\cos \vartheta - \cos \theta} d\vartheta - \oint_0^\pi e^{jk \cos \vartheta} \cos \vartheta \frac{\sin \theta}{\cos \theta - \cos \vartheta} d\vartheta \quad (\text{AP-12})$$

Using $e^{jk \cos \vartheta} = J_0(k) + 2 \sum_1^\infty j^n J_n(k) \cos n\vartheta$, and Glauert's formula $\oint_0^\pi \frac{\cos n\vartheta}{\cos \vartheta - \cos \theta} d\vartheta = \pi \frac{\sin n\theta}{\sin \theta}$

$$\beta^* = \pi \left\{ -\frac{2}{k} \sum_0^\infty j^{n-1} J_n(k) \sin n\theta + J_0(k) \sin \theta - 2 \cos \theta \sum_0^\infty j^n J_n(k) \sin n\theta \right\} \quad (\text{AP-13})$$

Finally we obtain

$$\begin{aligned} I_{32} = \beta_1 + \beta_2 + \beta_3 = & 2f \bar{u}_p \left[-\frac{4}{k} \left\{ -\frac{2}{k} \sum_1^\infty j^{n-1} J_n(k) \sin n\theta + J_0(k) \sin \theta + 2 \cos \theta \sum_0^\infty j^n J_n(k) \sin n\theta \right\} \right. \\ & \left. + \frac{1}{2} [A(k) + jB(k) + \{2C(k) - 1\} \{A(k) - jB(k)\}] \cot \frac{\theta}{2} \right] \end{aligned}$$

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