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Response Analysis of a Vibrational System of Multi Degrees of Freedom Subjected to an Arbitrary Force*

By Kunihiro ISHIHARA** and Masaya FUNAKAWA***

The response of a vibrational system of multi degrees of freedom is analyzed, applying Cayley-Hamilton's theorem.

The results calculated by the method proposed in the present report are compared with the experimental ones to prove the validity of this method, and the following conclusions are obtained.

- (1) The present method leads to more strict solutions than the usual ones.
- (2) It is rather difficult by the usual methods to choose the appropriate time increment in order to get stable solutions, but by the present method, stable solutions can be obtained easily, provided the time increment represents the exciting forces exactly.

1. Introduction

The problem of obtaining the response of a vibrational system subjected to a known external force occupies an important position in a vibrational analysis as well as in the eigen value problem. Especially, the response analysis for sinusoidal forces is reduced to a problem of solving the multi dimensional non simultaneous equations. Therefore, there is no difficulty in this analysis.

For the response to an arbitrary external force, modal analysis and direct integration methods (Runge-Kutta-Gill(1) method, Newmark β method(2) etc.) have been often used. In these numerical calculations, if we solve the equation step by step in time regions, we can not necessarily obtain a solution of the vibrational equation of a system of multi degrees of freedom, because a divergent phenomenon is induced by the vibration of higher orders whose periods are shorter than the time increment of calculation. In such a case, we must adopt a very small time increment. But this increases the calculation time and is a futile attempt because such a higher vibration is of no significance actually.

On the contrary, the modal analysis is useful for such a case. The reason is that the system is resolved into independent equations of individual modes and the solution is obtained by adding together the calculation results of each mode.

The advantage of this method lies in that needless higher modes can be neglected. But in this case, we can not use it without the condition that the damping matrix can be made diagonal; i.e., each mode is not combined.

However, the damping matrix is not

always diagonal. In such a case, it is necessary to apply a complex modal analysis derived from the complex eigen values and eigen modes after analyzing the eigen value problem containing the damping matrix.

Ito(3) analyzed the earthquake random response of structures by this method. In his analysis, the equation governing the response is given by the first order differential equations described by the complex vector. In order to solve these equations, there are some methods such as the predictor corrector method, the mid point acceleration method, Newmark β method and Runge-Kutta-Gill method. But the instability is inherent in these methods as aforementioned, and it is necessary to be careful in choosing the time increment. In this paper, the authors attempt to solve a vibrational system described by the complex vector semi-analytically.

2. Method of analysis

The equation of motion for a vibrational system of multi-degrees of freedom is generally described by

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \quad \dots (1)$$

where M , C and K are the mass, the damping and the stiffness matrixes ($N \times N$) respectively, and x and F are the displacement and the external force vectors with N dimensions. The equation (1) is transformed as follows by replacing $Z = \{x, \dot{x}\}^T$

$$\dot{Z} = Z + A\alpha \quad (2)$$

where

$$[A] = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \quad \alpha = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}$$

The analytical solution of Eq. (2) with the initial vector Z_0 is given by

$$Z = e^{At}Z_0 + e^{At} \int_0^t e^{-A\tau} \alpha(\tau) d\tau \quad \dots (3)$$

We must obtain the matrix e^{At} before calculating Eq. (3). Its usual method is to calculate the following equation.

$$e^{At} = \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i \quad \dots (4)$$

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** Staff, The Technical Laboratory, Kawasaki Heavy Industries, Akashi,

*** Professor, The Faculty of Engineering, The Science University of Okayama, Okayama.

But the calculation of this equation takes so much time that it is not recommendable. Then we will here obtain the matrix $e^{\mathbf{A}}$ by applying "Cayley-Hamilton's theorem" (4) introduced next.

"Theorem"

An arbitrary square matrix $\mathbf{B}$ satisfies the characteristic equation of itself, i.e., the characteristic equation of $\mathbf{B}$ is

$$\lambda^n + b_{n-1}\lambda^{n-1} + b_{n-2}\lambda^{n-2} + \dots + b_1\lambda + b_0 = 0 \quad (5)$$

then

$$\mathbf{B}^n + b_{n-1}\mathbf{B}^{n-1} + b_{n-2}\mathbf{B}^{n-2} + \dots + b_1\mathbf{B} + b_0\mathbf{I} = 0 \quad (6)$$

is obtained. The matrix $e^{\mathbf{B}}$ being expanded by the Power Series as shown in Eq. (4) and $\mathbf{B}^n$ being described by a Power Series smaller than $(n-1)$ powers, $e^{\mathbf{B}}$ is also described by a power series smaller than $(n-1)$ powers and written by

$$e^{\mathbf{A}t} = c_0\mathbf{I} + c_1(\mathbf{A}t) + c_2(\mathbf{A}t)^2 + \dots + c_{n-1}(\mathbf{A}t)^{n-1} \quad (7)$$

where $\mathbf{B} = \mathbf{A}t$

$\mathbf{A}$ in Eq. (7) is the $(2N \times 2N)$ matrix, so n in Eq. (7) is $2N$.

Meanwhile Eq. (7) must be satisfied by the complex eigen values λ_i ($i=1 \sim 2N$) of $\mathbf{A}t$. Then we obtain

$$e^{\beta_i} = c_0 + c_1\tilde{\beta}_i + c_2\tilde{\beta}_i^2 + \dots + c_{2N-1}\tilde{\beta}_i^{2N-1} \quad (i=1 \sim 2N) \quad (8)$$

where $\lambda_i t = \tilde{\beta}_i$, $\sim$ represents complex number.

These equations are expressed in the matrix form,

$$\begin{pmatrix} e^{\beta_1} \\ e^{\beta_2} \\ \vdots \\ e^{\beta_{2N}} \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\beta}_1 & \tilde{\beta}_1^2 & \dots & \tilde{\beta}_1^{2N-1} \\ 1 & \tilde{\beta}_2 & \tilde{\beta}_2^2 & \dots & \tilde{\beta}_2^{2N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \tilde{\beta}_{2N} & \tilde{\beta}_{2N}^2 & \dots & \tilde{\beta}_{2N}^{2N-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2N-1} \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} e^{\beta_1} \\ e^{\beta_j} \\ e^{\beta_k} \\ \vdots \\ e^{\beta_{2N}} \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\beta}_1 & \tilde{\beta}_1^2 & \tilde{\beta}_1^3 & \dots & \tilde{\beta}_1^{2N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \tilde{\beta}_j & \tilde{\beta}_j^2 & \tilde{\beta}_j^3 & \dots & \tilde{\beta}_j^{2N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2\tilde{\beta}_k & 3\tilde{\beta}_k^2 & \dots & (2N-1)\tilde{\beta}_k^{2N-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \tilde{\beta}_{2N} & \tilde{\beta}_{2N}^2 & \tilde{\beta}_{2N}^3 & \dots & \tilde{\beta}_{2N}^{2N-1} \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_{j-1} \\ c_k \\ \vdots \\ c_{2N-1} \end{pmatrix} \quad (11)$$

The matrix $[\tilde{\beta}]$ in Eq. (11) is no longer singular, so $[\tilde{\beta}]^{-1}$ is available and we can obtain the coefficient vector $\{\mathbf{C}\}$ from Eq. (9).

It is considered that $\{\mathbf{C}\}$ becomes a complex number because of $[\tilde{\beta}]$ being a complex number. But it is proved in the following manner that $\{\mathbf{C}\}$ becomes a real number because of the conjugate nature of these complex eigen values. Eq. (8) is rewritten as ;

$$e^{\beta_i} = \sum_{k=0}^{2N-1} c_k \beta_i^k = \sum_{k=0}^{2N-1} c_k P_{k,i} \quad (i=1 \sim N) \quad (12)$$

where $P_{k,i} = \beta_i^k = P_{k,i}' + jP_{k,i}''$

still more $\bar{P}_{k,i} = P_{k,i}' - jP_{k,i}''$

If c_k is represented by $c_k' + jc_k''$, from Eq. (12) we obtain

$$\begin{aligned} e^{\beta_i} &= \sum_{k=0}^{2N-1} (c_k' + jc_k'')(P_{k,i}' + jP_{k,i}'') = \sum_{k=0}^{2N-1} (c_k' P_{k,i}' - c_k'' P_{k,i}'') \quad (13) \\ &+ j \sum_{k=0}^{2N-1} (c_k'' P_{k,i}' + c_k' P_{k,i}'') = e^{\sigma_i} \cos \theta_i + j e^{\sigma_i} \sin \theta_i \quad (i=1 \sim N) \end{aligned}$$

In the same manner

or simply

$$\{\mathbf{e}\} = [\tilde{\beta}]\{\mathbf{C}\} \quad (9.a)$$

The matrix $[\tilde{\beta}]$ is called "Van del monde matrix" and the value of determinant of $[\tilde{\beta}]$ is given by

$$\text{Det} [\tilde{\beta}] = (-1)^{(2N(2N-1))/2} \prod_{i < k} (\tilde{\beta}_i - \tilde{\beta}_k) \dots \quad (10)$$

As known from Eq. (10), $[\tilde{\beta}]^{-1}$ is available for a vibrational system with different eigen values from each other and we can obtain the coefficient vector $\{\mathbf{C}\}$ from Eq. (9).

In the problem of blade vibration, the natural frequencies concerning bending and torsion are sometimes identical. In this case, the independent degree of freedom reduces, and $[\tilde{\beta}]$ becomes singular. The treatment of such a case is explained next. Assuming that one pair double roots ($\tilde{\beta}_j = \tilde{\beta}_k$) is in existence, the j th line and the k th line in Eq. (9) become the same equations and a solution of Eq. (9) can not be obtained.

Then we use the j th line equation differentiated by $\tilde{\beta}_j$;

$$e^{\beta_j} = c_1 + 2c_1\tilde{\beta}_j + 3c_2\tilde{\beta}_j^2 + \dots + (2N-1)c_{2N-1}\tilde{\beta}_j^{2N-2}$$

instead of the k th line equation in order to complement the needless equation. (for example the k th line equation). Then we obtain

$$e^{\beta_i} = \sum_{k=0}^{2N-1} (c_k' P_{k,i}' + c_k'' P_{k,i}'') + j \sum_{k=0}^{2N-1} (c_k'' P_{k,i}' - c_k' P_{k,i}'') = e^{\sigma_i} \cos \theta_i - j e^{\sigma_i} \sin \theta_i \quad (i=N+1 \sim 2N) \quad (14)$$

where $\beta_i = \sigma_i + j\theta_i$

Equating real and imaginary parts of both side equations in Eq. (13) and Eq. (14) and making a simple operation, we obtain the next four equations.

$$\left. \begin{aligned} 2 \sum_{k=0}^{2N-1} P_{k,i}'' c_k'' &= 0 \quad (i=1 \sim N) \\ 2 \sum_{k=0}^{2N-1} P_{k,i}' c_k'' &= 0 \quad (i=N+1 \sim 2N) \end{aligned} \right\} \quad (15)$$

$$\left. \begin{aligned} e^{\sigma_i} \cos \theta_i &= \sum_{k=0}^{2N-1} P_{k,i}' c_k' \quad (i=1 \sim N) \\ e^{\sigma_i} \sin \theta_i &= \sum_{k=0}^{2N-1} P_{k,i}'' c_k' \quad (i=N+1 \sim 2N) \end{aligned} \right\} \quad (16)$$

In Eq. (15), $P_{k,i}'$ and $P_{k,i}''$ ($k=0 \sim 2N-1$) are linearly independent, so c_k'' must be equal to zero for all k . As c_k'' is the imaginary part of the coefficient c_k , it is proved that c_k is real. On the other hand, the real part c_k' is obtained by solving the simultaneous equations constructed by Eq. (16). Then we can obtain the function of matrix $e^{\mathbf{A}t}$ which is theoretically strict by substituting these coefficients into Eq. (7).

Above proof is carried out similarly also in the case that Eq. (11) has double roots. But only when $i = j$, Eq. (15) and Eq. (16) become the next equations.

$$\left. \begin{aligned} 2 \sum_{k=0}^{2N-2} (k+1) P_{k,j}'' c_{k+1}'' &= 0 \\ 2 \sum_{k=0}^{2N-2} (k+1) P_{k,j}' c_{k+1}'' &= 0 \end{aligned} \right\} \quad (15)^*$$

$$\left. \begin{aligned} \sum_{k=0}^{2N-2} (k+1) P_{k,j}' c_{k+1}' &= e^{\sigma_j} \cos \theta_j \\ \sum_{k=0}^{2N-2} (k+1) P_{k,j}'' c_{k+1}' &= e^{\sigma_j} \sin \theta_j \end{aligned} \right\} \quad (16)^*$$

Table 1 Conversion of $e^{\mathbf{B}}$ and comparison between usual method and present method.

$$\mathbf{B} = e^{\mathbf{B}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}$$

Conversion of a_{11}

Order	$\Delta t = 0.001$	$\Delta t = 0.005$	$\Delta t = 0.010$	$\Delta t = 0.015$	$\Delta t = 0.020$
3	0.876 274	-2.093 15	-11.372 6	-26.838 3	-48.490
7	0.878 805	-0.827 28	-6.905 2	-137.421 0	-987.348
11	conv.	-0.793 45	-0.161 2	-46.518 0	-1 225.222
15		-0.793 33	0.253 2	-3.335 4	329.370
19		conv.	0.259 9	1.080 9	111.271
23			0.259 4	0.380 9	-2.099
27			conv.	0.382 3	-0.889
31				conv.	-0.864
35					-0.864
					conv.
all 3 order	0.878 758	-0.793 63	0.260 32	0.380 66	-0.864 39

Values in lowest column is the present result.

$$T_1 = 0.127 58 \text{ s}, \quad T_2 = 0.012 63 \text{ s}$$

Fig. 1 Vibrational system with 2-degrees of freedom

These double roots are discriminated when complex eigen values are obtained. We obtain the coefficient vector $\{\mathbf{C}\}$ by using Eq. (9) for no double roots and Eq. (11) for double roots.

3. Comparison between the present results and another reference on the matrix $\exp(\mathbf{A}t)$

When $\exp(\mathbf{A}t)$ is expanded in a power series like Eq. (4), there is a certain relationship between the number of terms of a power series which makes each element of the matrix convergent and the time increment Δt . According to ref. (2), the number of terms of the element of matrix converging increases with Δt becoming rough, as shown in Table 1, in the vibrational system represented in Fig. 1. While solving the same system by using the present method, we can obtain the matrix $\exp(\mathbf{A}t)$ strictly, by adopting the number of terms which is one smaller than the order of $\mathbf{A}t$, regardless of the value Δt . The low column of Table 1 gives the values calculated by the present method.

4. Numerical calculations

The matrix $\exp(\mathbf{A}t)$ necessary in calculating Eq. (3) has been obtained up to the previous chapter. But in the case that the external force $\mathbf{a}(t)$ is given arbitrarily we are obliged to carry out the calculation numerically just as by another method. However, the feature of present method is that we can obtain the matrix $\exp(\mathbf{A}t)$ by a theoretically strict means. Therefore, it is desirable to give the external force exactly. Then we give the external force in each time interval as the linear variation. If we use such a force, we can obtain the analytical solutions in each time interval and the response in all time intervals except approximation of the force under the condition that the terminal value of the $(i-1)$ th interval is equal to the initial value of the i th interval. Next we will show the above explanation concretely. Now the external force vector $\mathbf{a}(t)$ in $\Delta t(i-1) \leq t \leq \Delta t i$ is given as follows

$$\mathbf{a}(t) = \mathbf{a}_i t + \mathbf{a}_{i-1} \quad (17)$$

where $\mathbf{a}_i = (\mathbf{a}_i - \mathbf{a}_{i-1}) / \Delta t$

Substituting Eq. (17) into Eq. (3) and replacing $\mathbf{Z}_0$ in Eq. (3) with $\mathbf{Z}_{i-1}$, t with Δt , and $\mathbf{Z}(t)$ with $\mathbf{Z}_i$ the next equation is introduced.

$$\mathbf{Z}_i = e^{\mathbf{A}\Delta t} (\mathbf{Z}_{i-1} + \mathbf{A}^{-1} \mathbf{a}_{i-1} + \mathbf{A}^{-2} \mathbf{a}_i) - \mathbf{A}^{-1} \mathbf{a}_i - \mathbf{A}^{-2} \mathbf{a}_i \quad (18)$$

If we want to obtain the acceleration, from Eq. (2)

$$\dot{\mathbf{Z}}_{i-1} = \mathbf{A}\mathbf{Z}_{i-1} + \mathbf{a}_{i-1} \quad (19)$$

Substituting $\mathbf{Z}_{i-1} = \mathbf{A}^{-1}(\dot{\mathbf{Z}}_{i-1} - \mathbf{a}_{i-1})$ into Eq. (18), we obtain

$$\dot{\mathbf{Z}}_i = e^{\mathbf{A}\Delta t} \dot{\mathbf{Z}}_{i-1} + \mathbf{A}^{-1}(e^{\mathbf{A}\Delta t} - \mathbf{I})\mathbf{a}_i \quad (20)$$

where $\mathbf{Z}_i = \langle \dot{\mathbf{x}}_i, \ddot{\mathbf{x}}_i \rangle$

Equation (20) is the same equation which gives the displacement and the velocity in the case of constant force in each interval. The calculation procedure mentioned above is shown in Fig. 3. Next we will refer to some calculation examples.

Fig. 2 Linear approximation of an external force

Fig. 3 Procedure of calculation

Fig. 4 shows the response of a 2-degrees of freedom system (Eq. 21) to an external force given discretely.

The parameter Δt is the time interval and k is defined as $\Delta t/T_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sin 0.5t \\ 0 \end{bmatrix} \quad (21)$$

where

$$\omega_1 = 0.61774, \quad T_1 = 10.17 \text{ s}$$

$$\omega_2 = 1.61273, \quad T_2 = 3.89 \text{ s}$$

The calculation was carried out for $k = 0.05, 0.1, 0.2, 0.3, 0.5, 1.0$.

Fig. 4, indicates that the solution is obtained stably for all values of k . Meanwhile in order to obtain the solution stably by using the Newmark β method, a time interval smaller than Δt in Eq. (22) must be given.

$$\frac{\Delta t}{T} = \frac{1}{\pi} \frac{1}{\sqrt{1-4\beta}} \quad \left(K = 0.551 \quad \beta = \frac{1}{6} \right) \quad (22)$$

On the contrary, in the present method, we can obtain a solution without divergent phenomenon even for $k = 1.0$.

In this way, we must always pay attention to the choice of time interval Δt in order to obtain the solution stably when the other numerical calculation methods are adopted. But the present method has

Fig. 4 Response for various values of $\Delta t/T_2$

the feature that the solution can be obtained stably regardless of Δt because $\exp(\Delta t)$ can be obtained strictly. If a very large Δt is adopted, however, we cannot obtain the wave form smoothly as shown in Fig. 4, though the value itself at any time can be obtained exactly. Δt must be chosen carefully for this reason, even if we adopt the present method. The choice of Δt in this case is not for the sake of the stability of solution, but for the sake of smooth wave form. Then it is easy to choose the time interval Δt .

Next we will try to compare the results obtained by the present method with the theoretical ones. For the sake of simplicity, neglecting the damping Eq. (21), we obtain the solution of Eq. (21) as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2-\omega^2 & -1 \\ -1 & 1-\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dots\dots\dots (23)$$

Here substituting $\omega = 0.5$ into Eq. (23), we obtain $x_1 = 2.4$, $x_2 = 3.2$. On the other hand, carrying out the calculation by the present method, we can obtain $x_1 = 2.3723$, $x_2 = 3.16303$. Both results agree well with each other, though the numerical values are slightly smaller than the theoretical values on account of damping.

Fig. 5 Response of 5-degrees of freedom system subjected to a sinusoidal external force

Fig. 6 Response to a periodic rectangular wave force

Next, we let us consider a system with many degrees of freedom, for example: 5 degrees of freedom system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} + \begin{bmatrix} 0.2 & -0.1 & 0 & 0 & 0 \\ -0.1 & 0.2 & -0.1 & 0 & 0 \\ 0 & -0.1 & 0.2 & -0.1 & 0 \\ 0 & 0 & -0.1 & 0.2 & -0.1 \\ 0 & 0 & 0 & -0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \sin \omega t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \dots\dots\dots (24)$$

The complex eigen values of this system are given as follows:

$$\begin{aligned} \bar{\lambda}_1 &= -0.004051 \pm j 0.284601 \\ \bar{\lambda}_2 &= -0.034514 \pm j 0.830113 \\ \bar{\lambda}_3 &= -0.085769 \pm j 1.306910 \\ \bar{\lambda}_4 &= -0.141542 \pm j 1.676543 \\ \bar{\lambda}_5 &= -0.184125 \pm j 1.910132 \end{aligned}$$

Now carrying out the calculation by substituting $\omega = 1$ into Eq. (24), Fig. 5 is obtained. The steady state values are

$$\begin{aligned} x_1 &= 0.299693(0.307262) \\ x_2 &= 0.999363(1.016957) \\ x_3 &= 0.993343(0.987889) \\ x_4 &= 0.075418(0.099755) \\ x_5 &= 0.962955(0.982678) \end{aligned}$$

respectively. The values in bracket () are obtained by solving Eq. (24) strictly (i.e., the solution of complex simultaneous equation). Both values are identical, so the present calculation methods is considerably accurate with a system more than 2 degrees too. We will next show the calculation examples for some external forces. Fig. 6 is the result for periodic rectangular forces and Fig. 7 for step input. Fig. 8 and Fig. 9 are the transient responses to the periodic rectangular forces with the first and the second resonance frequencies. From Fig. 7, we can understand that a 2-degrees of freedom system with dash pot at mass point 2 has large damping effect.

The present calculation method is not appropriate for solving a large system. The reason is that it is hard to obtain the coefficient vector $\{C\}$. Then we will consider the applicability of this method.

5. Applicability of the present method

In the present method, as previously mentioned, we must obtain the coefficient vector $\{C\}$. If elements of the coefficient matrix of $\{C\}$ are too large, it is impossible to calculate the matrix e^{At} . Its element is given by the product of the power of the eigen value and time increment; therefore if it is large, the overflow phenomenon arises for only a few degrees. A certain relation holds between the maximum eigen value and the number of freedoms N . In order to examine this matter, we rewrite Eq. (9) as follows by using the minimum eigen value β_1

(a) System with dashpot at mass point 1.

(b) System with dashpot at mass point 2.

Fig. 7 Response to a step external force

Fig. 8 Transient response to a periodic rectangular force with the first resonance frequency

Fig. 9 Transient response to a periodic rectangular force with the second resonance frequency

$$\begin{bmatrix} e^{\beta_1 t} \\ e^{\beta_2 t} \\ \vdots \\ e^{\beta_{2N} t} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \frac{\beta_2}{\beta_1} & \left(\frac{\beta_2}{\beta_1}\right)^2 & \cdots & \left(\frac{\beta_2}{\beta_1}\right)^{2N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\beta_{2N}}{\beta_1} & \left(\frac{\beta_{2N}}{\beta_1}\right)^2 & \cdots & \left(\frac{\beta_{2N}}{\beta_1}\right)^{2N-1} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{2N-1} \end{bmatrix} \quad (25)$$

where

$$d_0 = c_0, \quad d_1 = \beta_1 c_1, \quad d_2 = \beta_1^2 c_2, \quad \dots, \quad d_{2N-1} = \beta_1^{2N-1} c_{2N-1}$$

The maximum element of this coefficient matrix is $\epsilon = |\beta_{2N}/\beta_1|^{2N-1}$, so care should be taken so that this value does not overflow in computers. Fig. 10 shows the limits of this method for each value of ϵ . According to this result, the smaller ϵ is, the larger N is. The present method is generally suitable for analyzing a system of small number of degrees of freedom precisely. In the mechanical vibration, we often encounter the case of analyzing a system of some degrees of freedom. Therefore, the present method is effective in such a case.

6. Comparison between calculation results and experimental results

In order to confirm the validity of this method, we try to make some simple experiments and compare these results with calculated ones. The test rig is shown in Fig. 11. We use a flat plate with two concentrated masses as the vibrational system, and apply a rectangular wave force to the mass point 1.

Fig. 10 Applicability of the present method (the relation between the ratio $|\beta_{2N}/\beta_1|$ and the number of freedoms N)

Fig. 11 Experimental equipment

(a) Calculation result ($\omega=14.73$) (b) Experimental result ($\omega=14.73$)

Fig. 12 Comparison between calculation results and experimental ones

(a) Experimental result ($\omega=7.96$) (b) Calculation result ($\omega=7.96$)

Fig. 13 Comparison between calculation results and experimental ones

The rectangular wave force is given by the air ejected from nozzle, which acts on the mass point 1 through a slit cut on the rotating circular plate. We use the anemometer and displacement meter (capacitance type) for detecting the external force and mass point displacement respectively. These are amplified and then recorded by oscillograph.

The equation of motion for this model is given as follows by the influence coefficient method.

$$\{ \ddot{\mathbf{X}} \} = -[\mathbf{a}][\mathbf{M}]\{\dot{\mathbf{X}}\} + [\mathbf{a}]\{\mathbf{F}\} \quad (26)$$

where $[\mathbf{a}]$ is the influence coefficient matrix and given as follows:

$$[\mathbf{a}] = \frac{l^3}{48EI} \begin{bmatrix} 2 & 5 \\ 5 & 16 \end{bmatrix} \quad (27)$$

Substituting the numerical values into mass, bending stiffness, respectively, yields Eq. (28)

$$\begin{aligned} & \begin{bmatrix} 2.88 \times 10^{-4} & 0 \\ 0 & 1.44 \times 10^{-4} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \\ & + \begin{bmatrix} 1.0923 & -0.3413 \\ -0.3413 & 0.1365 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ & = \begin{Bmatrix} 0.03 \sin(0.5t) \\ 0 \end{Bmatrix} \end{aligned} \quad (28)$$

where the value 0.03 given in the external force term is a value obtained from

$$\{\mathbf{F}\} = [\mathbf{a}]^{-1}\{\mathbf{X}\}_s : \{\mathbf{X}\}_s \text{ is the static}$$

deflection for constant nozzle jet.

The natural frequencies of this model are as follows:

$$\omega_1 = 13.12(12.9), \quad \omega_2 = 67.59(67.55)$$

where the values in bracket () are experimental ones.

The natural frequencies, which are the characteristics of the system, are identical with the experimental values, so Eq. (28) is thought to describe the system precisely. Then we can confirm the validity of the present method by comparing the response calculation of Eq. (28) with the

experimental one. The response waves for rectangular forces with $\omega = 14.73$ and 7.96 are shown in Fig. 12 and Fig. 13. From these results, calculation results and experimental results are found identical. Thus the present method is recognized to be an excellent one.

7. Concluding remarks

The response of a vibrational system of multi-degrees of freedom subjected to an arbitrary wave force is analyzed, applying Cayley-Hamilton's theorem. The calculated results obtained from the method proposed in the present report are compared with the experimental ones to prove the validity of this method, and the following conclusions are obtained.

- (1) The present method leads to more strict solutions than the usual ones.
- (2) It is rather difficult by the usual methods to choose an appropriate time increment in order to get stable solutions, but by the present method, stable solutions can be obtained easily, provided the time increment represents the exciting force exactly.
- (3) The validity of this method is proved by comparing the calculating results with the experimental ones.

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