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Application of the Infinitesimal Operators

of Translation and Rotation*

(No.1 The Lineage of Dynamic Equations)

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Various mechanical quantities and conservation equations of dynamics can be arranged in good order by means of appropriate infinitesimal operators. In dynamics of point mass systems, the infinitesimal operators of translation and rotation can yield various dynamical quantities from the energy form, and the corresponding conservation equations from the energy conservation equation which form a family tree. Dynamics of a rigid body is also considered from the view-point of infinitesimal operators. The technique is useful in studying dynamics of continua with microstructures.

1. Introduction

Basic equations in the dynamics of fluid mixtures can be derived from the energy conservation equation by applying the infinitesimal operator of translation. Translation is the most principal operation in the classical dynamics. The infinitesimal operator of translation yields momentum from kinetic energy and mass from momentum. This chain of dynamic quantities is very characteristic. Since the equation of motion of a point mass is invariant with respect to this infinitesimal operator, all the principles in the classical dynamics are invariant with respect to translation. This consideration suggests that a chain of conservation equations can be yielded from the energy conservation equation by the infinitesimal translational operation.

In carrying out this scheme, there is a difficulty to be overcome, that is, the commutation rule between the infinitesimal operator and the differentiation with respect to time. To examine the commutation rule, it is useful to begin with the dynamics of point mass systems before treating continuum mechanics. This is the aim of the present paper.

It is inferred that the rotational transformation of a coordinate system will yield important dynamical laws as in the case of translational transformation. In fact, the infinitesimal operator of rotation produces an angular momentum and then the moment of inertia tensor from the kinetic energy. Similarly, the operator can yield a chain of conservation equations from the energy conservation equation. There is, however, a situation different from the case of translation, that is, an accelerating state occurs by the rotational transformation. To overcome the difficulty, the technique of handling Coriolis's force is devised.

Dynamics of rigid bodies is recapitulated by means of the infinitesimal operators. The method of infinitesimal operators will offer an appropriate technique for studying dynamics of continua with microstructures.

2. Recapitulation of dynamics of point mass systems

The point masses are discriminated by the symbol $a (= 1, 2, \dots)$. Denote the mass and the position of a -particle by m_a and $\mathbf{x}_i(a)$, respectively. A coordinate system to be used is rectilinear and orthonormal. The index $i (= 1, 2, 3)$ represents the i -th component in the coordinate system. The force exerted on the a -particle from the outside will be denoted by $f_i(a)$, and the internal force produced by the b -particle by $f_i(ab)$. The equation of motion of the a -particle is written as

$$m_a \ddot{\mathbf{x}}_i(a) = f_i(a) + \sum_b f_i(ab) \quad (1)$$

It is noted that

$$f_i(ab) = -f_i(ba), \quad f_i(ab) \parallel (\mathbf{x}_i(a) - \mathbf{x}_i(b)) \quad (2)$$

The dot on the top of $\mathbf{x}_i(a)$ means the differentiation with respect to time t . By summing up Eqs (1) over all the particles, the equation of total momentum is obtained, which will be written in the following form:

$$\dot{\mathbf{X}}_{i(P)} \equiv \dot{\mathbf{P}}_i - \dot{\mathbf{F}}_i - \dot{\mathbf{W}}_{i(P)} = 0, \quad (3)$$

where

$$\mathbf{P}_i \equiv \sum m_a \dot{\mathbf{x}}_i(a), \quad \mathbf{F}_i \equiv \sum f_i(a), \quad (4)$$

$$\mathbf{W}_{i(P)} \equiv \sum w_{i(a)} \dot{\mathbf{x}}_i(a), \quad w_{i(a)} \equiv \dot{m}_a. \quad (5)$$

Here, $\mathbf{P}_i$ is the total momentum, $\mathbf{F}_i$ the resultant external force. To extend the application to the continuum theory, it is useful to consider open systems which admit going in and out of masses. We have, therefore, introduced $w_{i(a)}$ meaning the variation of particle mass, which causes the creation of momentum $w_{i(P)}$. In particular if $w_{i(b)} = \pm m_b \delta(t - t')$, it is meant that the b -particle plunges into the system at time t' for (+), or jumps out when (-). Considering the definition of $w_{i(a)}$, the following conservation equation of total mass is obtained

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$$\dot{X}_{(M)} \equiv \dot{M} - w_{(M)} = 0, \quad (6)$$

where

$$M \equiv \sum m_a, \quad w_{(M)} \equiv \sum w_{(a)}, \quad (7)$$

M meaning the total mass of the system and $w_{(M)}$ the rate of mass supply from the outside.

Summing up of Eqs. (1) after being multiplied by $\dot{x}_{i(a)}$ gives the conservation equation of total energy, which will be written in the following form:

$$\dot{X}_{(E)} \equiv \dot{K} - W - w_{(K)} = 0, \quad (8)$$

where

$$K \equiv \frac{1}{2} \sum m_a \dot{x}_{(a)}^2, \quad w_{(K)} \equiv \frac{1}{2} \sum w_{(a)} \dot{x}_{(a)}^2, \\ W \equiv W_{ex} + W_{int}, \quad W_{ex} \equiv \sum \dot{x}_{i(a)} f_{i(a)}, \\ W_{int} \equiv \sum (x_{i(a)} - x_{i(b)}) f_{i(ab)}, \quad (9)$$

K meaning the total kinetic energy of the system, $w_{(K)}$ the rate of supply of the kinetic energy from the outside, W the rate of work, which consists of W_{ex} done by external forces and W_{int} done by internal ones.

Next, let us take the moment of Eqs. (1) around the origin of the coordinate system. The vector product $\mathbf{x} \times \mathbf{y}$ is written in the index notation as follows:

$$(\mathbf{x} \times \mathbf{y})_i = \epsilon_{ijk} x_j y_k,$$

where ϵ_{ijk} is the unit alternating tensor, that is, it takes the value +1 when (i, j, k) is an even permutation of $(1, 2, 3)$ and -1 when it is an odd permutation. Summing up the vector product of Eqs. (1) with $x_{i(a)}$ for all a gives the conservation equation of total angular momentum, which will be written as

$$\dot{X}_{(H)} \equiv \dot{H}_i - N_i - w_{i(H)} = 0, \quad (10)$$

where

$$H_i \equiv \epsilon_{ijk} \sum m_a x_{j(a)} v_{k(a)}, \quad (11) \\ N_i \equiv \epsilon_{ijk} \sum x_{j(a)} f_{k(a)}, \\ w_{i(H)} \equiv \epsilon_{ijk} \sum w_{(a)} x_{j(a)} v_{k(a)},$$

H_i meaning the total angular momentum, N_i the moment of the external forces, and $w_{i(H)}$ the rate of supply of angular momentum from the outside. The internal forces have no contribution to N_i by virtue of (2). In the above $v_{k(a)} \equiv \dot{x}_{k(a)}$.

Let us introduce the total second moment of masses M_{ij} :

$$M_{ij} \equiv \sum m_a (x_{(a)}^2 \delta_{ij} - x_{i(a)} x_{j(a)}). \quad (12)$$

By differentiating the above expression with respect to time and applying the definitions (5), the following equation is obtained:

$$\dot{X}_{ij(M)} \equiv \dot{M}_{ij} - Z_{ij(M)} - w_{ij(M)} = 0, \quad (13)$$

where

$$Z_{ij(M)} \equiv \sum m_a (2 x_{(a)} v_{(a)} \delta_{ij} - x_{i(a)} v_{j(a)} - v_{i(a)} x_{j(a)}), \\ w_{ij(M)} \equiv \sum w_a (x_{(a)}^2 \delta_{ij} - x_{i(a)} x_{j(a)}), \quad (14)$$

$Z_{ij(M)}$ meaning the rate of internal production of the second moment of masses and $w_{ij(M)}$ the rate of supply from the outside.

3. The infinitesimal operator of translation G_i

Let us translate the coordinate system with a uniform and constant velocity $-V_i$. The quantities observed in the moving coordinate system are distinguished by putting prime ($'$). The position and the velocity of the a -particle in both coordinate system are in the following relations:

$$x_{i(a)}' = x_{i(a)} + V_i t, \quad (15) \\ v_{i(a)}' = v_{i(a)} + V_i.$$

No changes arise in the derivatives of $x_{i(a)}$ higher than the second order as well as the forces $f_{i(a)}$ and $f_{i(ab)}$ in the case of this coordinate transformation. Hence, the equation of motion (1) maintains the same form for translation.

Denote the derivative of $x_{i(a)}$ of the p -th order by $x_{i(a)}^{[p]}$:

$$(d^p/dt^p) x_{i(a)} \equiv x_{i(a)}^{[p]}, \quad p=0, 1, \dots,$$

in particular

$$x_{i(a)}^{[0]} = x_{i(a)}, \quad x_{i(a)}^{[1]} = v_{i(a)}.$$

Let $\mathcal{F}$ be an arbitrary function of $x_{i(a)}^{[p]}$, t :

$$\mathcal{F} = \mathcal{F}(x_{i(a)}^{[p]}, t).$$

The value of $\mathcal{F}$ observed in the moving coordinate system is denoted by $\mathcal{F}' = \mathcal{F}(x_{i(a)}'^{[p]}, t)$. The infinitesimal translation operator G_i is defined as follows:

$$G_i \mathcal{F} \equiv \lim_{(t \rightarrow 0, V_i \rightarrow 0)} (\mathcal{F}' - \mathcal{F}) / V_i. \quad (16)$$

By considering (15), the following formula follows immediately:

$$G_i \mathcal{F} = \sum_a \partial \mathcal{F} / \partial v_{i(a)}. \quad (17)$$

The arguments other than $v_{i(a)}$ have no effects in translation.

By virtue of the above formula, we can derive easily the following relations:

$$G_i K = P_i, \quad G_i P_j = M \delta_{ij}, \\ G_i M = 0, \quad (18)$$

where K is the kinetic energy (9), P_i the total momentum (4), and M the total mass (6). This shows the principal mechanical quantities can be arranged in a chain forming a family tree by the infinitesimal translational operator. Some other relations

are listed as follows:

$$G_i W = G_i W_{ex} = F_i, \quad G_i W_{in} = 0, \\ G_i F_j = 0, \quad (19)$$

$$G_i w_{(k)} = w_{(k)}(P), \quad G_i w_{j(P)} = w_{(M)} \delta_{ij}, \\ G_i w_{(M)} = 0. \quad (20)$$

The commutation rule between the operator G_i and the temporal differentiation d/dt is formulated as follows:

$$G_i \frac{d\mathcal{F}}{dt} = \frac{d}{dt} G_i \mathcal{F} + \sum_a \frac{\partial \mathcal{F}}{\partial x_{i(a)}}. \quad (21)$$

The proof is straightforward. Namely,

$$\frac{d\mathcal{F}}{dt} = \frac{\partial \mathcal{F}}{\partial t} + \sum_{b,p} \frac{\partial \mathcal{F}}{\partial x_{j(b)}^{[p]}} x_{j(b)}^{[p+1]} \quad (21)'$$

Hence

$$G_i \frac{d\mathcal{F}}{dt} = \sum_a \frac{\partial^2 \mathcal{F}}{\partial v_{i(a)} \partial t} + \sum_{a,b,p} \frac{\partial^2 \mathcal{F}}{\partial v_{i(a)} \partial x_{j(b)}^{[p]}} x_{j(b)}^{[p+1]} + \sum_a \frac{\partial^2 \mathcal{F}}{\partial x_{i(a)}}.$$

On the other hand, differentiation of Eq. (17) with respect to time gives the first two terms in the right hand side of the above equation. Then the formula (21) follows.

Now successive application of G_i on the energy conservation form $X_{(E)}$ gives the following relations:

$$G_i X_{(E)} = X_{(P)}, \quad G_i X_{j(P)} = X_{(M)} \delta_{ij}, \\ G_i X_{(M)} = 0, \quad (22)$$

where $X_{(P)}$ and $X_{(M)}$ are the conservation forms of total momentum and total mass defined by (3) and (6), respectively. The above shows that the principal conservation forms can be arranged in a lineage by the infinitesimal operator G_i , which is depicted in Fig. 1.

Fig. 1 Chains of dynamic quantities and conservation equations.

4. The infinitesimal operator of rotation R_i

In this section, we will rotate the coordinate system around the origin with a constant angular velocity $-\Omega$. The quantities in the rotating coordinate system are distinguished by prime ($'$) as in the preceding section. The basis vectors e_i' of the rotating system are related to those of the fixed system e_i in the following way:

$$e_i' = e_i - \Omega \times e_i \Delta t \\ = e_j (\delta_{ji} - \epsilon_{jki} \Omega_k \Delta t),$$

providing that Δt is infinitesimal. Inversely, we have

$$e_j = e_j' (\delta_{ji} + \epsilon_{jki} \Omega_k \Delta t). \quad (23)$$

Fig. 2 A rotation of the coordinate system

The position of the a -particle x_a is expressed in the two coordinate system respectively as follows

$$x_a(t+\Delta t) = \sum x_{i(a)}(t+\Delta t) e_i \\ = \sum x_{i(a)}'(t+\Delta t) e_i'.$$

From (23), it follows that

$$x_{i(a)}'(t+\Delta t) = x_{i(a)}(t+\Delta t) + \epsilon_{ijk} \Omega_k x_{j(a)}(t+\Delta t) \Delta t. \quad (24)$$

Differentiation with respect to Δt gives

$$x_{i(a)}'^{[p]} = x_{i(a)}^{[p]} + p \epsilon_{hji} \Omega_h x_{j(a)}^{[p-1]} \quad (p \geq 1) \quad (25)$$

For $p=0$, Eq. (25) gives

$$x_{i(a)}' = x_{i(a)}. \quad (26)$$

Let $\mathcal{F}$ be again an arbitrary function of t and $x_{i(a)}^{[p]}$. The infinitesimal rotational operator R_h is defined as follows:

$$R_h \mathcal{F} \equiv \lim (\mathcal{F}' - \mathcal{F}) / \Omega_h \Delta t, \quad (\Delta t \rightarrow 0, \Omega_h \rightarrow 0) \quad (27)$$

From (25) it follows that

$$R_h x_{i(a)}^{[p]} = p \epsilon_{hji} x_{j(a)}^{[p-1]}, \quad (28)$$

in particular,

$$R_h v_{i(a)} = \epsilon_{hji} x_{j(a)}, \quad (29.a)$$

$$R_h \dot{v}_{i(a)} = 2 \epsilon_{hji} \dot{v}_{j(a)}, \quad (29.b)$$

$$R_h x_{i(a)} = 0, \quad (29.c)$$

where the last one comes from (26). By applying the above relations, the following general formula is obtained:

$$R_h \mathcal{F} = \epsilon_{hjk} \sum_{p \geq 1} p \frac{\partial \mathcal{F}}{\partial x_{k(a)}^{[p]}} x_{j(a)}^{[p-1]}. \quad (30)$$

The operation of R_h on the kinetic energy (9) produces a line of dynamic quantities:

$$R_h K = H_h, \quad R_h H_i = M_{hi}, \\ R_h M_{ij} = 0, \quad (31)$$

where H_h is the total angular momentum defined by (11), and M_{hi} the second moment of masses defined by (12). In the above, use has been made of the identity

$$\epsilon_{ijm} \epsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$$

Let us carry out cross operations, which yield

$$G_i H_j = -R_i P_j = \epsilon_{ijk} M_k \quad (32)$$

where M_i is the first moment of masses defined by

$$M_i \equiv \sum m_a x_{i(a)}. \quad (33)$$

We will discuss somewhat about Coriolis' force. The equation of motion (1) in the rotating coordinate system becomes by using Eq. (29.b)

$$m_a \ddot{x}_{i(a)}' = f_{i(a)}' + \sum_b f_{i(ab)}, \quad (34.a)$$

where

$$f_{i(a)}' = f_{i(a)} + 2\epsilon_{hji} m_a \dot{x}_{h(a)} \dot{x}_{j(a)}. \quad (34.b)$$

In the above, the equation of motion has been arranged such as to give the same form as that in the coordinate system at rest. For that purpose, the virtual external force $f_{i(a)}'$ has to be (34.b), the last term therein being Coriolis' force. From (34) an important formula follows:

$$\begin{aligned} R_h f_{i(a)} &= 2\epsilon_{hji} m_a \dot{x}_{j(a)}, \\ R_h f_{i(ab)} &= 0. \end{aligned} \quad (35)$$

So far as the above rules are observed, all the dynamic relations are invariant with respect to the rotational transformation.

5. Commutation rule between R_i and d/dt . The hierarchy of the dynamic equations

As in Sec. 3, let $\mathcal{F}$ be an arbitrary function of t and $x_{i(a)}^{[p]}$. The commutation rule between the operator R_i and the differentiation d/dt is formulated as follows:

$$\begin{aligned} R_h \frac{d\mathcal{F}}{dt} &= \frac{d}{dt} R_h \mathcal{F} \\ &+ \epsilon_{hij} \sum_{p \geq 0} x_{i(a)}^{[p]} \frac{\partial \mathcal{F}}{\partial x_{j(a)}^{[p+1]}}. \end{aligned} \quad (36)$$

The proof is straightforward. Operate R_h on Eq. (21), then we have

$$\begin{aligned} R_h \frac{d\mathcal{F}}{dt} &= \epsilon_{hij} \sum p x_{i(a)}^{[p-1]} \frac{\partial^2 \mathcal{F}}{\partial x_{i(a)}^{[p-1]} \partial t} \\ &+ \epsilon_{hij} \sum p x_{i(a)}^{[p-1]} \frac{\partial^2 \mathcal{F}}{\partial x_{i(a)}^{[p-1]} \partial x_{j(b)}^{[p]}} x_{j(b)}^{[p+1]} \\ &+ \epsilon_{hij} \sum (p+1) \frac{\partial \mathcal{F}}{\partial x_{i(a)}^{[p+1]}} x_{i(a)}^{[p]}. \end{aligned}$$

On the other hand, the differentiation of Eq. (30) gives

$$\begin{aligned} \frac{d}{dt} R_h \mathcal{F} &= \epsilon_{hij} \sum p x_{i(a)}^{[p-1]} \frac{\partial^2 \mathcal{F}}{\partial t \partial x_{i(a)}^{[p-1]}} \\ &+ \frac{\partial^2 \mathcal{F}}{\partial x_{j(b)}^{[p]} \partial x_{i(a)}^{[p+1]}} x_{j(b)}^{[p+1]} + x_{i(a)}^{[p]} \frac{\partial \mathcal{F}}{\partial x_{i(a)}^{[p+1]}}. \end{aligned}$$

Eq. (36) is obtained by taking the difference between the above two equations into consideration.

Let us operate R_i on the energy conservation form (8). First, by utilizing (9), (30) and (36), we get

$$R_i \dot{K} = \dot{H}_i, \quad R_i W_{\epsilon\alpha} = \dot{N}_i.$$

$$R_i W_{\alpha\alpha} = 0, \quad R_i w_{(k)} = w_{i(k)}.$$

With the aid of the above, the following important relation can be obtained

$$R_i X_{(E)} = X_{i(M)}, \quad (37)$$

where $X_{i(M)}$ is defined by Eq. (10).

Next, by utilizing Eqs. (11), (36) and (21), we have

$$\begin{aligned} R_i \dot{H}_j &= \dot{M}_{ij} \\ &+ \epsilon_{ikn} \epsilon_{jln} \sum m_a (x_{i(a)} \dot{x}_{n(a)} - x_{n(a)} \dot{x}_{i(a)}), \end{aligned}$$

and from Eqs. (11), (30) and (35), we have

$$\begin{aligned} R_i \dot{N}_j &= 2\epsilon_{ikn} \epsilon_{jln} \sum m_a x_{i(a)} \dot{x}_{n(a)}, \\ R_i w_{j(M)} &= w_{ij(M)}, \end{aligned}$$

where $w_{ij(M)}$ is defined by Eq. (14). Now operate R_h on $X_{i(M)}$. From the Above relations we get easily

$$R_i X_{j(M)} = X_{ij(M)}. \quad (38)$$

The right hand side is an expression defined by (13). Operation of R_h annihilates $X_{ij(M)}$:

$$R_h X_{ij(M)} \equiv 0. \quad (39)$$

This comes from the following relations derived from Eqs. (12), (14), (30) and (35):

$$\begin{aligned} R_h w_{ij(M)} &= 0, \\ R_h \dot{M}_{ij} &= R_h X_{ij} = \epsilon_{hkp} \sum m_a x_{k(a)} \\ &\times (2x_{p(a)} \delta_{ij} - x_{j(a)} \delta_{ip} - x_{i(a)} \delta_{jp}). \end{aligned}$$

According to Eqs. (37) and (38), the conservation equations of angular momentum and the second moment of masses can be automatically derived from the energy conservation equation $X_{(E)} = 0$ by operating R_i :

$$\begin{aligned} R_i : X_{(E)} = 0 &\rightarrow X_{i(M)} = 0 \\ &\rightarrow X_{ij(M)} = 0. \end{aligned} \quad (40)$$

This is a branch of the family tree or the hierarchy of the conservation equations.

Let us consider cross operation. The following relations can be easily proved

$$G_i X_{j(M)} = -R_i X_{j(P)} = \epsilon_{ijk} X_{k(M)}, \quad (41)$$

where $X_{i(M)}$ is a conservation form of the first moment of masses defined by

$$\begin{aligned} X_{i(M)} &\equiv \dot{M}_i - P_i - Z_{i(M)}, \\ Z_{i(M)} &\equiv \sum w_{(a)} x_{i(a)}. \end{aligned} \quad (42)$$

Finally, the following relations are also easily shown:

$$\begin{aligned} G_i X_{j(M)} &= 0, & R_i X_{j(M)} &\equiv 0, \\ R_i X_{(M)} &= 0, & G_h X_{ij(M)} &\equiv 0. \end{aligned}$$

The results obtained so far are arranged in Fig. 3 and Fig. 4, which show in good shape

the structures existing among principal dynamic quantities and conservation forms.

Fig. 3 The lineage of dynamic quantities

Fig. 4 The lineage of conservation equations

6. Dynamics of rigid bodies

In the dynamics of rigid bodies there appears a new variable of motion, that is the angular velocity and also we have a new kinematical quantity, i.e., the moment of inertia tensor adhered to the body. The dynamics of a rigid body with a fixed point will be formulated by the method of applying the infinitesimal rotational operator. This treatment affords a useful idea to construct a general dynamical theory of polar fluids.

Denote the angular velocity by ω or ω_i . The velocity of a point of the rigid body located at x is given by

$$v = \omega \times x$$

$$\text{or } v_i = \epsilon_{ijk} \omega_j x_k \quad (44)$$

where the origin of the position vector is the fixed point of a rigid body. The angular momentum and the kinetic energy are obtained by substituting the above relation into the expressions (11) and (9):

$$H_i = M_{ij} \omega_j, \quad (45.a)$$

$$K = \frac{1}{2} M_{ij} \omega_i \omega_j, \quad (45.b)$$

$$M_{ij} = \int (x^2 \delta_{ij} - x_i x_j) dm, \quad (45.c)$$

where M_{ij} is the moment of inertia tensor of a rigid body. The energy conservation equation is in the same form as Eq. (8), where

$$W = N_i \omega_i, \quad N_i = \epsilon_{ijk} \int x_j f_k dm, \quad (46)$$

$$w_{(K)} = \frac{1}{2} w_{ij(M)} \omega_i \omega_j.$$

Let us examine how the angular velocity changes in rotational transformation of the coordinate system. For that purpose, apply the infinitesimal rotational operator R_h to Eq. (44). From Eq. (29.a) we have

$$\epsilon_{hji} x_j = \epsilon_{ijk} (R_h \omega_j) x_k,$$

which must hold for any x_k . Hence

$$R_h \omega_j = \delta_{hj}. \quad (47)$$

We shall consider in what manner the temporal derivatives of spacelike quantities behave in the rotating coordinate system. For example, take a tensor of the second order ϕ_{ij} , which is also expressed as $\phi = \phi_{ij} e_i e_j$. By comparing the components in the fixed and the rotating coordinate systems as in Sec. 4, we have

$$\phi'_{ij}(t+\Delta t) = \phi_{ij}(t+\Delta t) + \Delta t (\epsilon_{hki} \phi_{kj} + \epsilon_{hjk} \phi_{ik}) \delta_{h\Delta}.$$

Differentiate with respect to Δt and apply the definition (27), then

$$R_h \dot{\phi}_{ij} = (R_h \phi_{ij})' + \epsilon_{hki} \phi_{kj} + \epsilon_{hjk} \phi_{ik}. \quad (48)$$

The extension to quantities of an arbitrary order is easily inferred. In particular

$$R_h \dot{\phi} = (R_h \phi)' \quad \text{for scalars}$$

$$R_h \dot{\phi}_i = (R_h \phi_i)' + \epsilon_{hji} \phi_j \quad \text{for vector}$$

The commutation between the operator R_h and the differentiation d/dt can be carried out by combining the relations (48) and (49). A unified technique can be given by the Lie differentiation, but it is not described here. It is noted that the components of spacelike quantities are invariant for themselves in applying R_h :

$$R_h \phi_{ij} = 0. \quad (50)$$

Operate R_h on the kinetic energy of (45.b): From Eq. (47), we have

$$\begin{aligned} R_h K &= M_{ij} \omega_j = H_i, \\ R_h H_j &= M_{ij}, \quad R_h M_{ij} = 0. \end{aligned} \quad (51)$$

Now, let us calculate the moment of Coriolis' force. When the relation (44) is considered, the formula (35) gives

$$\begin{aligned} R_h f_i &= 2 \epsilon_{hji} \epsilon_{jkl} \omega_k x_l \\ &= 2 (x_l \omega_i - \omega_l x_i). \end{aligned}$$

Hence from Eq. (46)

$$R_h N_i = 2 \epsilon_{ijk} \omega_k \int x_j x_h dm. \quad (52)$$

From the identity $(\epsilon_{ijk} \epsilon_{hkl}) \epsilon_{lmn} = \epsilon_{ijk} (\epsilon_{hkl} \epsilon_{lmn})$, we get the relation

$$\begin{aligned} \epsilon_{hmi} \delta_{fn} - \epsilon_{hni} \delta_{fm} \\ + \delta_{hf} \epsilon_{imn} - \delta_{if} \epsilon_{hmn} = 0. \end{aligned}$$

By using the above relation Eq. (52) becomes

$$R_h N_i = (\epsilon_{ikj} M_{km} - \epsilon_{jkm} M_{ki} + \epsilon_{imk} M_{jk}) \omega_m \quad (53)$$

Operate R_i on the energy conservation equation (8), then it follows from the formulas (49) and (52) that

$$R_i X(\epsilon) = X_{i(M)}, \quad (54)$$

where

$$X_{i(M)} \equiv \dot{H}_i - N_i - W_{i(M)},$$

$$W_{i(M)} \equiv W_{ij} \omega_j.$$

Successive application gives

$$R_i X_{j(M)} \equiv X_{ij(M)}, \quad (55)$$

where

$$X_{ij(M)} \equiv \dot{M}_{ij} + \epsilon_{ikj} H_k - \hat{Z}_{ij} - W_{ij},$$

$$\hat{Z}_{ij} \equiv R_i N_j.$$

By substituting Eq. (53), the above form becomes

$$X_{ij(M)} \equiv \dot{M}_{ij} + (\epsilon_{ikm} M_{km} - \epsilon_{imk} M_{jk}) \omega_m - W_{ij} \quad (55)'$$

This is the usual form for the change of moment of inertia tensor. An important point in the present treatment is that this form is derived from the energy equation by applying the infinitesimal operator and not through kinematical handling. This technique will become very useful to derive the mass equation in the continuum mechanics

with complicated structure. Operation R_i yields a series of conservation equations from the energy equation $X(\epsilon) = 0$:

$$R_i : \quad X(\epsilon) = 0 \rightarrow X_{i(M)} = 0 \\ \rightarrow X_{ij(M)} = 0.$$

7. Conclusions

The operations of translation and rotation are very useful to derive various dynamic quantities from the energy expression and also various conservation equations from the energy equation. Before applying the technique to continuum mechanics, the dynamics of point mass systems and that of a rigid body are examined by this technique and a lineage of conservation equation is obtained. It is noted that introduction of Coriolis' force is necessary to maintain the invariance of form of the basic equation in the case of rotational operation. The treatment of Coriolis' force in applying the infinitesimal operators is described. According to the present work, the technique will be applicable to the operators of higher orders.

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