Molecular Mean Free Path Effects in Gas Lubricated Slider Bearings

(An Application of the Finite Element Solution)

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Variational formulation is derived for the steady state compressible Reynolds equation under slip-flow conditions. Finite element techniques and a numerical procedure, using Newton-Raphson iteration method to solve the non-linear finite element equations, are described.

These techniques are applied to conventional slider bearings and the effects of the molecular mean free path (MMFP) for a wide range of the compressibility numbers ($A = 10^{-7}$) are shown. Then, MMFP effects for different bearing configurations are compared. As a result, it is found that non-dimensional load $W$ is useful for estimating MMFP effects.

The load or velocity versus spacing characteristics are calculated for a cylindrical slider bearing. These calculations reveal that when spacing decreases with velocity for a constant load, MMFP effects increase monotonously. However, when spacing decreases with an increase in load at a constant velocity, MMFP effects only increase slightly.

The new calculation procedure developed here has wide applications, converges rapidly and saves computer memory.

1. Introduction

Self-acting gas lubricated slider bearings have been applied successfully to read/write heads. These bearings fly over a high speed rotating medium, such as the disk or drum in magnetic storage equipment, and help maintain a steady, close spacing between the head and the medium. In order to achieve a high recording density and large memory capacity, it is necessary to minimize the spacing. Recently a $0.5 \sim 0.2 \mu m$ spacing has been achieved.

The fundamental characteristics of slider bearings were investigated by Gross based on the classical Reynolds equation which holds true for continuum flow. When the spacing, however, is reduced to several times the molecular mean free path (MMFP $\lambda_m = 0.064 \mu m$) thickness, slip-flow at the surface mainly caused by MMFP may be observed. Therefore, the classical Reynolds equation is not sufficient to describe such bearing characteristics. Burgdorf derived a modified Reynolds equation taking slip-flow conditions into account. In line with this, slip-flow effects have been investigated analytically for a simple film configuration and for spiral groove bearings by means of the finite difference method.

In this paper, a new application of the finite element method (FEM) to the lubrication problem taking slip-flow effects into account is described. The finite element approaches to conventional lubrication problems were developed by Reddy and Wada and applied to some lubrication problems.

In this work, the finite element procedure for a modified Reynolds equation is derived and a powerful calculation procedure is developed using Newton-Raphson iteration method and a diagonalized partition technique. Newton-Raphson method is applied to linearize a set of non-linear algebraic equations, and the partition technique saves computer memory and time. In addition, calculation results for three conventional slider bearings over a wide compressibility range ($A = 10^{-7}$) are reported. Pressure distribution and performance curves are calculated and experimental data are compared with the calculation results.

This new calculation method has general applications, converges rapidly and requires few main memory allocations.

2. Nomenclature

$x$ : bearing width
$H$ : normalized film thickness $h/h_0$
$h$ : local film thickness
$h_0$ : minimum film thickness (spacing)
$I$ : bearing length in direction of motion
$M$ : Knudsen number $\lambda_m/\lambda$
$n$ : outward normal unit vector
$P$ : local pressure
$P_s$ : ambient pressure
$S$ : lubricant region area or element area
$s$ : length along boundary
$U$ : surface velocity
$w$ : load carrying capacity of bearing
$W$ : normalized load carrying capacity of bearing $w/p_s$b
$A$ : compressibility number in direction of motion $b$ $p_s U/l/b h_s$
$A_t$ : compressibility number perpendicular to direction of motion
$\lambda$ : two-dimensional compressibility number
$\lambda_m$ : molecular mean free path (MMFP)
$\mu$ : viscosity

Suffixes
$m$ : element number
$n$ : iteration number

3. Numerical Solution Method

3.1 Variational Formulation

Isothermal and steady state conditions are
assumed to be in effect in lubricant region \( A \) in Figure 1. Then, the modified Reynolds equation derived by Burgdorfer can be written in a non-dimensional form as follows:

\[
\frac{\partial}{\partial x} \left[ \rho \left( 1 + \frac{6M}{\rho H} \right) \frac{\partial P}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \rho \left( 1 + \frac{6M}{\rho H} \right) \frac{\partial P}{\partial y} \right] = \frac{\partial}{\partial x} \left[ \rho \frac{\partial P}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \rho \frac{\partial P}{\partial y} \right]
\]

where the boundary conditions are:

(a) Pressure is prescribed for boundary \( B_1 \); and
(b) \( B_2 \) lies along a line of symmetry, as shown in the figure.

The boundary problem for solving the differential equation (1) with the prescribed boundary conditions is equivalent to a variational problem for finding stationary function \( \Phi(P) \). Here, \( \Phi \) minimizes the following functional \( \Phi(P) \) on \( A \):

\[
\Phi(P) = 0
\]

where:

\[
\Phi(P) = \int \left( \rho \left( 1 + \frac{6M}{\rho H} \right) \frac{\partial P}{\partial x} \right) \left( \rho \left( 1 + \frac{6M}{\rho H} \right) \frac{\partial P}{\partial y} \right) ds + 6M H \rho \left( \frac{\partial P}{\partial x} \right)^2 ds
\]

A proof for this variational formulation is described below. The following is obtained when applying Green's theorem to equation (3):

\[
\Phi(P) = -\int \left( \rho \left( 1 + \frac{6M}{\rho H} \right) \frac{\partial P}{\partial x} \right) \left( \rho \left( 1 + \frac{6M}{\rho H} \right) \frac{\partial P}{\partial y} \right) ds + 6M H \rho \left( \frac{\partial P}{\partial x} \right)^2 ds
\]

If \( P \) is the stationary function of functional \( \Phi(P) \), equation (4) implies that the first and second terms are both zero. Then, \( P \) satisfies equation (1) as a result of the arbitrariness of \( \partial P \) in the first term and satisfies the boundary conditions due to the second term. Conversely, if \( P \) satisfies equation (1) with boundary conditions (a) and (b), then equation (2) obviously holds as a result of equation (4). Equation (2) is a variational problem corresponding to the lubrication problem where slip-flow effects, shown in the third term of equation (3), are taken into account.

3.2 Discretization by FEM and Linearization by Newton-Raphson Method

In this study, FEM is applied using triangular elements. This approach simplifies analytical treatment for computational programming. Point \( Q(X, Y) \) in an element is defined uniquely by the area ratio between two (of three) sub-divided triangles. These sub-divided triangles are formed, as shown in Figure 2, using \( Q \) as the apex and one side of the original triangle as the side opposite the apex.

Let \( i, j, k \) be the apexes (nodal points) of the original triangles, and \( S \) the area of the original triangle, and \( S_i, S_j, S_k \) the areas of the sub-divided triangles. Then, area coordinates, \( f_i, f_j, f_k \) are defined as follows:

\[
f_i = S_i / S, \quad f_j = S_j / S, \quad f_k = S_k / S \quad \cdots(5)
\]

Using these coordinates, the following is obtained:

\[
X = \sum_{i=1}^{3} f_i X_i, \quad Y = \sum_{i=1}^{3} f_i Y_i \quad \cdots(6)
\]

These equations give:

\[
\begin{align*}
&f_i = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a_{1i} & a_{2i} \end{pmatrix} \begin{pmatrix} X_i & Y_i \end{pmatrix} \\
f_i &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a_{1i} & a_{2i} \\
& a_{3i} & a_{3i} \end{pmatrix} \begin{pmatrix} X_i & Y_i \end{pmatrix} \end{align*} \quad \cdots(7)
\]

where:

\[
a = \begin{pmatrix} a_{1i} & a_{2i} & a_{3i} \\
& a_{3i} & a_{3i} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\
& 1 & 1 \end{pmatrix}
\]

Then, pressure \( P_m \) and film thickness \( H_m \) in element \( m \) are assumed to be:

\[
P_m = \frac{1}{\alpha} \sum_{i=1}^{3} f_i P_i, \quad \cdots(9)
\]

\[
H_m = \frac{1}{\gamma} \sum_{i=1}^{3} f_i H_i, \quad \cdots(10)
\]

where \( P_i \) and \( H_i \) are respective values at the \( i \)th nodal point. Equation (9) can be extended to all nodal points by the following expression:

\[
P_m = f_m TP \quad \cdots(11)
\]

where:

![Fig.1 Lubricant region and coordinate system](image1)

![Fig.2 Cartesian coordinates and area coordinates for a triangular element](image2)
\[ P = (P_1, P_2, \ldots, P_n)^T, \]
\[ L: \text{total number of nodal points} \]
\[ f_{m} = (f_0, f_0, \ldots, f_n, \ldots, f_0)^T \]

Equation (11) is used to obtain:
\[ F = (i, j)^T \left( \frac{\partial P}{\partial X} \frac{\partial P}{\partial Y} \right) \]
\[ = (i, j)^T \left( \frac{\partial f_{m}}{\partial X} \frac{\partial f_{m}}{\partial Y} \right) P \]

While:
\[ A = (t, j)^T \left( A_1, A_2 \right)^T \]

Substituting equations (11), (14) and (15) into equation (2), the following matrix equation is obtained due to the arbitrariness of \( \partial P \):
\[ \sum_m \int_{\Omega_m} (B_m^T C_m B_m P - B_m^T U_{m} f_{m}) \partial f_{m} = 0 \]

where:
\[ B_m = \begin{pmatrix} 0 & a_{m0} & 0 & a_{m1} & \cdots & 0 & a_{m,n-1} & 0 & \cdots & 0 \\ 0 & 0 & a_{m0} & 0 & a_{m1} & \cdots & 0 & a_{m,n-1} & \cdots & 0 \end{pmatrix} \]
\[ C_m = \begin{pmatrix} H_m & 0 \\ 0 & H_m \end{pmatrix} \]
\[ U_{m} = \begin{pmatrix} \frac{H_m P}{H_m} \\ 0 \end{pmatrix} \]
\[ E_{m} = \begin{pmatrix} H_m P \\ 0 \end{pmatrix} \]

It should be noted that the second term in equation (16) is an additional term caused by consideration of MMPEP.

Equation (16), which is non-linear in terms of \( P \), can be reduced to a linear set of iteration equations by means of Newton-Raphson method. Put:
\[ K_m = B_m^T C_m B_m, \quad G_m = B_m^T U_{m} \]
\[ J_m = B_m^T E_m B_m \]

Then, Newton-Raphson iteration for equation (16) is:
\[ \phi(P^{(k+1)}) = \phi(P^{(k)}) + \left( P^{(k+1)} - P^{(k)} \right) \frac{\partial \phi(P^{(k)})}{\partial P^{(k)}} = 0 \]

where:
\[ \phi(P^{(k)}) = \sum_m \int_{\Omega_m} \left( (K_m P^{(k)}) \\ -G_m f_{m} \right)^2 + J_m P^{(k)} \sum_m (K_m P^{(k)})^2 \\ +K_m P^{(k)} f_{m} + G_m f_{m} + J_m dS_{m} \]

Next, \( D_m \) and \( M_m \) are used to replace the integrand, in equation (23) and (24) respectively, so that:
\[ \sum_m \int_{\Omega_m} M_m P^{(k)} dS_{m} = \sum_m \int_{\Omega_m} M_m P^{(k)} dS_{m} - \sum_m \int_{\Omega_m} D_m dS_{m} \]

When transforming the cartesian coordinates \((X, Y)\) into area coordinates \((f, f_0, f_1)\) in element \( m \), the areal integration is transformed as:
\[ \int \int_{\Omega_m} f(X, Y) dX dY = \int \int_{\Omega_m} F(f, f_0, f_1) \frac{\partial (X, Y)}{\partial f} dX dY \]
\[ = 2S_m \int \int_{\Omega_m} f_0 f_1 f dX dY \]
This integration is required only for terms which involve \( f_0, f_1, f_1 \) and \( H = \sum \frac{1}{2} f_i d_{li} \) and can be obtained simply by means of arithmetic calculus. This is because the areal integration of the polynomial for \( f_0, f_1, f_2 \) gives a reciprocal of an integer.

Next, \( M_m \) and \( D_m \) are defined as the integrations of \( M_n \) and \( D_n \), respectively, in equation (25). Then elements \( M_m \) of matrix \( M_m \), and elements \( D_m \) of vector \( D_m \), can be expressed as follows:
\[ M_m = M_n + M_n + M_n + M_n \]
\[ D_m = D_n + D_n + D_n \]

where:
\[ M_m = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1,n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{n,n} \end{pmatrix} \]
\[ D_m = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & \cdots & d_{1,n} \\ d_{21} & d_{22} & d_{23} & d_{24} & \cdots & d_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & d_{n3} & d_{n4} & \cdots & d_{n,n} \end{pmatrix} \]

Elements \( M_m \) and \( D_m \) are zero except when \( i = j \) or \( k \); and \( m = i, j \) or \( k \). \( F_m, F_m, F_m, F_m, F_m, F_m \) are shown in the appendix. Upon adding \( M_m \) and \( D_m \), respectively, for all elements, the following linear matrix equation is obtained:
\[ \tilde{M} P^{(k+1)} = \tilde{V} \]

where:
\[ \tilde{M} = \sum_m M_m, \quad \tilde{V} = \sum_m \left( F_0 f_1 f_1 + M_0 d_0 d_0 \right) \]

### 3.3 Calculating Procedures

In constructing the computer program, the lubricated field was partitioned into finite parts. This realized extreme savings in computer time and memory.

The flow diagram in Figure 3 shows the calculation procedures. In order to reduce input operations, automatic grids formation was employed. First, selected parameters which specify the scheme for dividing the field into elements were fed in. Then, elements and nodal points were labeled and coordinates, the film thickness at nodal points were determined. Next, nodal coordinates \( X, Y (i, j) \), film thickness \( H = H(i,j) \), area \( A \) \( A(i,j) \), \( F_m, F_m, F_m, F_m, F_m, F_m \), which are unique values for each element, were calculated for each element. These values were stored on magnetic tape after combined in each partition, then, read from the tape at the step when \( \tilde{M} \) and \( \tilde{D} \) or load carrying capacity and pressure center were calculated.
In this way, the working memory for matrix operations becomes independent of the total number of nodal points. Consequently, fine grids could be used despite the limitations of the computer memory. According to the calculation results for the slider bearings, the convergence rate hardly depends on the values of $A$. Convergence can be achieved after 1 or 2 iterations for $A = 1$, after 3 - 5 iterations for $A = 10^4$.

4. Calculation Results for Slider Bearings

4.1 Influence of MMFP on Load Carrying Capacity

The pressure distribution for typical low and high $A$'s in a tapered flat slider bearing are shown in Figures 4(a) and (b), respectively. One-half of the bearing surface was divided into regular rectangular...
grids, consisting of 330 nodes (30 nodes in the $X$ direction, 11 nodes in the $Y$) and 540 elements. The nodal points where pressure gradient was maximum were concentrated. For example, in Figure 4(a), grid spacing was finest at the trailing edge ($\Delta X = 0.002$), while $\Delta X = 0.1$ for intermediate grid spacing.

Load carrying capacity errors were estimated to be within 2% by extrapolating the results of finer grid models.

The influences of MMFP on load carrying capacity are shown in Figures 5~7. Here, the ordinate is the ratio of load $W$, taking MMFP into account, to load $W_{1,0}$ where MMFP $\lambda = 0$. These figures show that MMFP effects are dependent on bearing profile. Generally, however, the effects tend to decrease as $\lambda$ increases. It should be noted that the step slider bearing reveals an interesting phenomenon that the effects are not monotonous with $\lambda$. For low $\lambda$ values, the effects increase with an increasing $\lambda$.

Finitely wide bearings ($b/l = 1, 0.1$) and an infinitely wide bearing ($b/l = \infty$) are compared in Figure 8 for the same value of $W_{1,0}$, but where the value of $\lambda$ is not the same. Under these conditions, differences in MMFP effects caused by the differences in bearing width were not noticeable, despite different side flow effects. This is because MMFP effects compared at the same $W_{1,0}$ are not highly dependent on slider width.

These results hold for different bearing profiles. For example, the three bearings, shown in Figures 5~7, have nearly equal load ratios $W/W_{1,0}$.
i.e. within 24 ~ 26%, when compared at $W_{s=0} = 0.1$, $M=0.64$.

From these results, it can be concluded that MMFP effects depend almost uniquely on $W_{s=0}$. Hence non-dimensional load $W$ is a convenient parameter for evaluating MMFP effects.

### 4.2 Influence of MMFP on Static Characteristics of Slider Bearings

Static characteristics of slider bearings can be represented by the relation of film thickness $h_s$ versus velocity $U$ with a constant load or load $W$ at a constant velocity. In both cases, Knudsen number $M$ increases as compressibility number $A$ increases. This means that the growth of MMFP effects with an increasing $M$ and the decrease of MMFP effects with an increasing $A$ tend to cancel each other. The influence of MMFP on static characteristics, therefore, cannot be explained merely using $M$ and $A$.

The velocity versus spacing characteristics are shown in Figure 9. In this case, the growth of MMFP effects is larger than the decrease of MMFP effects as velocity decreases. The load versus spacing characteristics are shown in Figure 10. Here, both effects are nearly equal. Then, the load reduction rate caused by MMFP is almost constant for varying $h_s$.

### 5. Comparison of Calculation and Experimental Results

A schematic diagram of the experimental apparatus is shown in Figure 11. Here a precision glass disk, with a surface roughness of less than 0.003 $\mu$m CLA, was mounted on a precision ball bearing spindle. The surface runout of the glass disk was adjusted to less than 40 $\mu$m. A cylindrical slider bearing made of two rectangular shoes flies over the rotating glass disk. The dimensions of the bearing are shown in Figure 12.

Film thickness was measured with a conventional microscope observing the light fringes caused by optical interference between the glass and bearing surfaces. A mercury lamp was used as the light source.

Typical experimental and calculation results are compared in Figure 12. Surface correction coefficient $\sigma$ is the factor by which surface slip is multiplied and for air on glass, $\sigma=1.24$; for air on

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Fig.9 Velocity versus spacing characteristics

Fig.8 Comparison of MMFP effects between finitely and infinitely wide bearings

Fig.10 Load versus spacing characteristics
shellac, \( a = 1 \). Since \( a \) varies within a limited range for different materials, \( a \) is estimated to be within 1.0 ~ 1.24 for the experimental interfaces, i.e. air on glass (disk) and on ceramic (slider).

The experimental results are in good agreement with the slip-flow calculation results based on \( a = 1.0 \) or \( a = 1.24 \). This confirms that the solution method developed here is practically sufficient for most design purposes under slip-flow conditions.

6. Conclusions

This paper deals with the application of FEM to compressible lubrication problems under slip-flow conditions. The results can be summarized as follows:
1. Variational formulation is derived for Reynolds equation taking MMFP effects into account and a calculation procedure is developed by applying Newton-Raphson iteration method for linearization and diagonalized partition technique for solving the related equations. This procedure has general applicability, converges rapidly and saves main memory allocations.
2. Non-dimensional load \( W \) is a useful parameter for estimating MMFP effects. If \( W \) is equal for different configurations, MMFP effects are nearly equal.
3. If film thickness decreases with velocity for a constant load, MMFP effects increase monotonously. If film thickness decreases with an increasing load for a constant velocity, the effects increase slightly.
4. The numerical solution method reported here is satisfactory for designing practical bearings to operate under slip-flow conditions.

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Appendix

\[
PHS = \int \sum_{\infty}^{H(1)} f_{1} P(1) dS_{1} = \frac{S_{1}}{60} \int \sum_{\infty}^{H(1)} FPHS(1) P_{1} dS_{1} \quad \cdots (34)
\]

\[
PHL = \int \sum_{\infty}^{H(1)} f_{1} P(1) dS_{1} = \frac{S_{1}}{12} \int \sum_{\infty}^{H(1)} FHL(1) P_{1} dS_{1} \quad \cdots (35)
\]

\[
FPHS(i) = \int \sum_{\infty}^{H(1)} f_{1} P(1) dS_{1} = \frac{S_{1}}{60} \left[ H_{1} P_{1} + H_{1} P_{1} + H_{1} P_{1} + 3H_{1} P_{1} (H_{1} + H_{1}) + 2H_{1} P_{1} (H_{1} + H_{1}) \right] \quad \cdots (36)
\]

\[
FHL(i) = \int \sum_{\infty}^{H(1)} f_{1} P(1) dS_{1} = \frac{S_{1}}{12} \left[ H_{1} P_{1} + H_{1} P_{1} + H_{1} P_{1} + 2H_{1} P_{1} (H_{1} + H_{1}) \right] \quad \cdots (37)
\]

\[
H_{1}^{2} = \int \sum_{\infty}^{H(1)} f_{1} P(1) dS_{1} = \frac{S_{1}}{60} \left[ H_{1} P_{1} + H_{1} P_{1} + H_{1} P_{1} \right] + H_{1} P_{1} (H_{1} + H_{1}) + H_{1} P_{1} (H_{1} + H_{1}) \quad \cdots (38)
\]

Load carrying capacity

\[
W = \int (P - 1) dS_{1} = \frac{S_{1}}{3} \left( P + P + P - 3 \right) \quad \cdots (39)
\]

Pressure center

\[
x = \frac{1}{W} \int (P - 1) X dS_{1} = \frac{1}{W} \int X dS_{1} \left( \sum_{1}^{H_{1}} P, X_{1} \right) - \frac{1}{6} \sum_{i}^{H_{1}} X_{i}
\]

\[
+ \frac{P_{1}(X_{1} + X_{1}) + X_{1} + P_{1}(X_{1} + X_{1}) + P_{1}(X_{1} + X_{1})}{12} \quad \cdots (40)
\]

Fig.11 Experimental apparatus

Fig.12 Comparison of calculation and experimental results
References


(11) Ibid., p. 1234.