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A Numerical Method for the Solution of Nonlinear Equations*

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In case of treating problems including strong fluid dynamical or constitutive nonlinearity, governing equations can be hardly solved analytically, and so it is necessary to get numerical solutions.

A new method for the solution of nonlinear equations is proposed, and it is found relatively stable.

1. Introduction

Various kinds of macroscopic phenomena dealt with frequently in engineering problems are governed by some natural laws. For this reason, in order to grasp such phenomena qualitatively and quantitatively, a system of governing equations is to be solved. Of these problems, motion of pendulums, rotational vibration of a machine, classical dynamics of particles or rigid bodies, and also a model on an increase or a decrease of biological individuals are expressed with ordinary differential equations or integro-differential equations. Systems of these equations involving no partial differential operations have been examined analytically for a long time not only in the case of linear equations but also in the case of nonlinear equations⁽¹⁾.

On the other hand, problems on heat or fluid flows encountered in engineering and technology, on plastic deformation, and on elastic deformation are expressed with the aid of partial differential operators due to the character of media as a continuum. However, the existence and uniqueness of a solution for partial differential equations except in linear cases are hardly verified except for special cases and so many difficulties arise in analytical treatment^{(2), (3)}. In spite of these backgrounds, it is increasingly necessary to treat problems including strong nonlinearity due to constitutive equations of media or due to large deformation, and as a virtual solution a numerical solution is frequently sought.

For numerical analyses, finite-difference methods, numerical integration methods, methods of weighted residuals⁽⁴⁾ such as the Galerkin method, and variational methods including those with a finite-element method are utilized. Of these, finite-element methods and finite-difference methods produce new approximate governing equations by discretizing original equations with the values at a limited

number of spacial points and the original solutions are replaced by solutions for simultaneous equations with boundary conditions or initial conditions. For an original nonlinear equation, the replaced simultaneous equations are also nonlinear, which results in solving the simultaneous nonlinear equations as for a numerical solution. Especially in case of treating problems with strong nonlinearity such as hydrodynamical one or constitutive (thermal or kinematic) one, development of a new method has recently become necessary for solving simultaneous nonlinear equations very stably and smoothly.

In this report, a new relatively stable method for solving simultaneous nonlinear equations is proposed, and several points are compared with other existing methods.

Notations

h	: grid spacing
n	: index that indicates non-Newtonian viscosity
$\bar{n}$	: iteration times
R	: Reynolds number
R_e	: real part
U	: x -component of velocity vector
x	: Cartesian coordinate
x^n	: unknown(s)
x_0, x_0^n	: exact solution(s)
x^*, x^{*n}	: approximate solution(s)
y	: Cartesian coordinate
δ	: criterion parameter for convergence
ϵ	: parameter for iteration
θ	: rate of convergence for iteration
μ	: viscosity function
ϕ	: acceleration coefficient
ψ	: stream function

2. Theory

2.1 Equations applicable for numerical analyses

Simultaneous nonlinear equations considered here have in general the following form

$$A^i_n x^n = H^i(x^*) \quad (i=1, 2, \dots, N) \quad \dots\dots\dots (1)$$

which can be derived from nonlinear equations, e.g. momentum equations, an equation of conservation of mass, an energy equation (these equations are applied

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regardless of steady or unsteady state), constitutive equations, and an ordinary differential equation for two-point boundary value problems, by discretizing those with the aid of finite-difference methods, methods of weighted residuals, or finite-element methods. Here $[A^i_j]$ is a constant matrix (which may be a zero matrix), x^i ($i=1,2,\dots,N$) are N unknowns and $H^i(x^r)$ ($i=1,2,\dots,N$) mean functions. Moreover, in Eq.(1) Einstein's summation convention is used, and the notation x^r means that the argument spreads over the vector space of unknowns.

2.2 Existing methods and necessary characters for solving methods

In many cases simultaneous nonlinear equations of the type (1) are solved by iterative procedures if the number N is large. Such an iterative procedure is expressed in general as

$$x^{i(\bar{n}+1)} = x^{i(\bar{n})} - K^i_m(x^{r(\bar{n})}) \{H^i(x^{r(\bar{n})}) - A^i_m x^{i(\bar{n})}\} \dots \dots \dots (2)$$

corresponding to Eq.(1), where $\bar{n}$ means a time of an iterative step. According to the contents of K^i_m and/or whether replacement of unknowns is simultaneous or successive, there are several iterative procedures such as the Newton method, a secant method, the Jacobi method, and the Gauss-Seidel method⁽⁵⁾. To be effective as a solving procedure, the iterative procedure like Eq.(2) should have such a character of local convergence that sequences $x^{i(\bar{n})}$ converge to x^i_0 if sufficiently accurate approximate values x^{*i} for exact solutions x^i_0 are given. Moreover, in case of treating equations having strong nonlinearity, the following are desired as the characters of Eq.(2): (i) Even starting from relatively rough values yields stable convergence. (ii) For a fixed function K^i_j , the character of local convergence should not depend strongly on the parameters if any parameter is included in the function H^i . Especially when the function H^i is reduced to a linear one for a special value of the parameter, the iterative procedure should converge irrespective of the approximate values x^{*i} . (iii) Local convergence should not be lost by a minute variation of the component of K^i_j . (iv) The rate of convergence should be greater than a suitable level and it should be easy to find out whether the proceeding process is convergent or not. (v) If possible, some acceleration parameters should be supplied. (vi) For practical use, K^i_m should be a relatively simple function of A^i_m and H^i and should not be composed of the characteristics of them if possible.

2.3 Solving Procedure proposed in this report

For a small parameter ϵ (where $R_e(\epsilon) > 0$) and a square constant matrix F^i_j with N columns, Eq.(1) is modified as

$$(F^i_m + \epsilon A^i_m) x^m = F^i_m x^m + \epsilon H^i(x^r) \dots \dots \dots (3)$$

$$x^i = (F + \epsilon A)^{-1} \{F^i_m x^m + \epsilon H^i(x^r)\} \dots \dots \dots (4)$$

[$\det(F + \epsilon A) \neq 0$ is assumed.] A new proposed iterative procedure based on Eq.(4) is as follows:

$$\xi^{i(\bar{n})} \equiv (F + \epsilon A)^{-1} \{F^i_m x^{i(\bar{n})} + \epsilon H^i(x^{r(\bar{n})})\} \dots \dots \dots (5)$$

$$x^{i(\bar{n}+1)} \equiv \xi^{i(\bar{n})} + \phi(\epsilon^{-1} - 1) (\xi^{i(\bar{n})} - x^{i(\bar{n})}) \dots \dots (6)$$

where ϕ is a parameter for acceleration and $0 \leq \phi \leq 1$. At an early stage of iteration when strong stability is required, $\phi = 0$ during successive 2 to several times of iteration depending on the form and on initial approximate solutions. It is possible to accelerate the iteration after the early stage of iterations every time for a stable equation and every 2 to 5 times for a less stable equation. In order to satisfy local convergence of the iteration based on Eqs. (5) and (6) irrespective of the forms A and H , let F^i_j be

$$F^i_j = A^i_j - (\partial H^i / \partial x^j)_{x^*} \dots \dots \dots (7)$$

When a rather good approximate solution x^* is prescribed, any components of a matrix, which consists of elements 0 except a few elements with small values, may be added to the right-hand side of Eq.(7). With relatively accurate approximate values x^* , a residual $x^{r(\bar{n})} - x^r_0$ behaves with a good approximation as

$$x^{r(\bar{n})} - x^r_0 \propto (1 - \epsilon)^{\bar{n}} \dots \dots \dots (8)$$

if $|\epsilon|$ is small and $\phi = 0$. (See Appendix.) For this reason the following condition is adopted as a criterion for convergence of iteration:

$$\max_i \xi^{i(\bar{n})} - x^{i(\bar{n})} < \delta$$

2.4 Method for specifying initial approximate solutions

This iterative method does not necessarily require precise approximate solutions for a first step, but to reduce times of iteration and to improve the character of local convergence, better approximate solutions for initial values can be obtained with a so-called continuation method (ref.(5), p230). One of these methods is as follows: choose a new function $H^{*i}(x^r; \mu)$ possessing a new parameter μ so that $H^{*i}(x^r; 1) = H^i(x^r) \dots (9)$ and $H^{*i}(x^r; 0)$ is a linear function. Next, specify the sequence $\{\mu_{n^*}\}$ so that $0 = \mu_0 < \mu_1 < \dots < \mu_k = 1$. Then according to the order $n^* = 1, 2, \dots, k$, the solution of the

equation

$$A^i_n x^n = H^{*i}(x^r; \mu_n^*) \quad (10)$$

is found by the present iterative method, where the solution x^r of an equation

$$A^i_n x^n = H^{*i}(x^r; \mu_{n-1}^*)$$

is used as approximate solutions of Eq. (10). In the case of $n^*=0$ the equation of the type (10) becomes linear and its corresponding exact solutions can be found by a suitable method. In case that a parameter is included in the original equation, there is a case that an equivalent procedure is gained by varying the parameter itself.

3. Examples of numerical analyses and discussion

3.1 Applied examples

To examine the generality of the present method, first of all, this is applied to simultaneous quadratic equations with 10 elements. Next, the Navier-Stokes equation for incompressible two-dimensional steady flow and an equation of motion for power-law fluid flow through a square duct are supplied for application as examples derived from partial differential equations with boundary value problems.

3.2 In case of simultaneous quadratic equations with real coefficients and 10 elements

As a coefficient matrix A , a unit matrix, a zero matrix, and a band matrix are adopted. The absolute value of each element in A and that of each coefficient in the quadratic polynomials H^i are less than or equal to 10 and the absolute values of real roots sought are less than or equal to 3 and their combination is supplied at random. The approximate solutions for starting iteration are prepared by generating uniform random numbers with amplitude 0.1 to 0.25 around corresponding real roots. (The details of the coefficients are omitted here.) Table 1 shows the computational results on the percentage of convergence for iteration

etc. by the present method and by a simplified Newton method.

3.3 In case of the Navier-Stokes equations for incompressible two-dimensional steady flow.

So-called cavity flow shown in Fig. 1 is adopted as a flow field, which has been already made dimensionless. Eliminating pressure terms from the Navier-Stokes equations, we obtain, as an equation of motion,

$$\Delta^2 \psi = R \left(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \Delta \psi \quad (11)$$

where ψ is a stream function. Boundary conditions are

$$\left. \begin{aligned} \psi(x, 0) = \psi(x, H_0) = \psi_y(x, H_0) = 0 \quad (0 \leq x \leq 1) \\ \psi(0, y) = \psi(1, y) = \psi_x(0, y) = \psi_x(1, y) = 0 \quad (0 \leq y \leq H_0) \\ \psi_y(x, 0) = 1 \quad (0 < x < 1) \end{aligned} \right\} \quad (12)$$

In numerical analyses for Eqs. (11) and (12), a biharmonic operator, a harmonic operator, and partial differential operators of first order are replaced by 13-point approximation, 5-point approximation, and centered difference approximations all with truncating errors $O(h^2)$ respectively. At internal points of the domain the resulting equation for the analysis is

$$\begin{aligned} 20\psi_{i,j} - 8(\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1}) \\ + 2(\psi_{i+1,j+1} + \psi_{i-1,j+1} + \psi_{i-1,j-1} + \psi_{i+1,j-1}) \\ + \psi_{i+2,j} + \psi_{i-2,j} + \psi_{i,j+2} + \psi_{i,j-2} \\ = 0.25 R \{ (\psi_{i,j+1} - \psi_{i,j-1})(\psi_{i+2,j} + \psi_{i+1,j+1} \\ + \psi_{i+1,j-1} - 4\psi_{i+1,j} + 4\psi_{i-1,j} - \psi_{i-1,j+1} \\ - \psi_{i-1,j-1} - \psi_{i-2,j}) - (\psi_{i+1,j} - \psi_{i-1,j}) \\ \times (\psi_{i,j+2} + \psi_{i+1,j+1} + \psi_{i-1,j+1} - 4\psi_{i,j+1} \\ + 4\psi_{i,j-1} - \psi_{i-1,j-1} - \psi_{i-1,j-1} - \psi_{i,j-2}) \} \end{aligned} \quad (13)$$

where double suffixes indicate numbers for location of the grid point. The matrix A consists of the elements due to the biharmonic operator. Stream lines for the cases ($R=100$, $H_0=1$, $h=1/19$) and ($R=300$, $H_0=1$, $h=1/19$) are shown in Figs. 2 and 3 respectively. In case that R is large, the accuracy for approximation becomes worse unless h is chosen to be smaller⁽⁶⁾. Vorticity distribution at the moving wall

Table 1 Computational results for simultaneous quadratic equations with 10 elements

Method	ϵ	With or without acceleration	ϕ	Initial iteration times	Rate of acceleration	Total number of cases	Number of diverging cases	Percentage for divergence	Percentage for convergence
Present method	0.1~0.3	Without	0.8~1.0 1.0	9 5	1/5 1/1	210	10	4.8	95.2
	0.1~0.3	With				420	12	2.9	97.1
	0.1~0.3	With				70	12	17.1	82.9
Simplified Newton method						490	28	5.7	94.3

The rate of acceleration 1/5 indicates that acceleration is made every 5 times.

for the case ($R=20$, $H_0=1$) is shown in Fig. 4. The difference between the vorticity distribution by this report and that by other report⁽⁷⁾ ($h=1/128$) is not remarkable. Table 2 shows some conditions used and results for computation.

3.4 In case of the equations of motion for power-law fluid flow through a square duct

Figure 5 shows a coordinate system for this flow. Since there exists a similarity transformation rule for steady power-law fluid flow, without loss of generality equations of motion for the velocity $w(x,y)$ are expressed as

$$\bar{\mu} \Delta w + w_x \bar{\mu}_x + w_y \bar{\mu}_y = -1 \quad (14)$$

$$\bar{\mu} = (w_x^2 + w_y^2)^{(n-1)/2} \quad (15)$$

Boundary conditions are

$$w(1,y) = w(-1,y) = w(x,1) = w(x,-1) = 0 \quad (16)$$

At the internal points the following derived by finite-difference techniques using equal spacings are supplied for numerical computation.

$$\begin{aligned} & \bar{\mu}_{i,j}(w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4w_{i,j}) \\ & + 0.25(w_{i+1,j} - w_{i-1,j})(\bar{\mu}_{i+1,j} - \bar{\mu}_{i-1,j}) \\ & + 0.25(w_{i,j+1} - w_{i,j-1})(\bar{\mu}_{i,j+1} - \bar{\mu}_{i,j-1}) \\ & = -h^2 \dots \dots \dots (17) \\ & \ln \bar{\mu}_{i,j} - 0.5(n-1) \ln 0.25h^{-2} \{ (w_{i+1,j} \\ & - w_{i-1,j})^2 + (w_{i,j+1} - w_{i,j-1})^2 \} = 0 \dots \dots \dots (18) \end{aligned}$$

By the way various techniques for finite-difference methods are devised for each purpose and accuracy^{(8),(9)}. To strengthen nonlinearity, a function logarithm is introduced in Eq.(18). Taking the symmetry of the flow field into account, only solutions $w, \bar{\mu}$ in the first quadrant ($x>0, y>0$) are sought in computation. In this example, A is specified as a zero matrix. Moreover, to avoid the singularity of Eq.(15) at the origin, the whole square field is divided into odd number parts. Figure 6 shows an example of a flow pattern obtained. Table 3 shows some conditions used and results for computation. When $A = 0$ and $\varepsilon = 1$, the present procedure reduces to a simplified Newton method.

3.5 Discussion about the characters of the present solving technique

Fig.1 Field on cavity flow.

Fig.2 Cavity flow ($R=100$, $H_0=1$, $h=1/19$).

Fig.3 Cavity flow ($R=300$, $H_0=1$, $h=1/19$).

Fig.4 Vorticity distribution on a moving wall ($R=20$, $H_0=1$).

Fig.5 Coordinate system for flow through a square duct.

Fig.6 Flow pattern of lines possessing equal velocity ($n=0.75$).

Table 2 Computational results for the case of cavity flow

Number of series method	Solving	Program	R	Division	F	Initial-ization	Approx. solu-tions	ε	δ	Initial itera-tion times	Rate of accele-ration	φ	Total number of itera-tions	CPUT(s) for matrices	CPUT(s) for an itera-tion	Total CPUT(s)	Elapsed time (s)	
#01	Present method	I.O.	0	19×19	Standard	*	**		1.0D- 6				14		5.51	285	1080	
#02			50			Continu-ation method	R= 20	0.1	1.0D- 6	9	1/1	1.0	13				279	780
#03			100				R= 50		1.0D- 6				14	212			288	900
#04			100			Continu-ation method	R= 50	0.2	1.0D- 6	∞	0	—	41				438	1920
#05			300				R=100		1.0D- 7				19				318	1500
#06			300			*	**		1.0D- 7				diverging				—	—
#07		I.I.	0	15×15	Standard	*	**		1.0D- 6				14	47°	2.60°	84°	180	
#08		I.O.	100			Continu-ation method	R= 20		1.0D-10	9	1/1	1.0	21	37	2.31	85	780	
#09		I.I.	200				R=100	0.1	1.0D- 6				15	33	2.04	63	180	
#10		I.O.	300			Continu-ation method	R=100		1.0D-10				35	37	2.31	118	960	
#11		500	R=200					4.0D- 5				67	33	2.04	169	300		
#12		I.I.	0			*	*1		1.0D- 4	∞	0	—	57	5.7°	0.72°	43°	300	
#13	Simplified Newton Method	I.O.	100	11×11	Modified	Continu-ation method	R= 20	0.2	1.0D- 6	4	1/1	1.0	12	5.7°	0.72°	14°	60	
#14			100			Continu-ation method	**		1.0D- 7	9	1/1	1.0	25	11.4	1.02	37	660	
#15			500				R=100		1.0D- 6				31		21	180		
#16		I.I.	500	11×21	Modified	Continu-ation method	R=100	0.1	1.0D- 5	∞	0	—	118	5.7°	0.72°	90°	420	
#17		50	Continu-ation method			R= 20		1.0D- 6	9	1/1	1.0	14	36	2.03	64	120		
#18		50				*	**		1.0D- 7				17	91	3.52	151	720	
#19		Simplified Newton Method	100	11×11	Modified	Continu-ation method	R= 20		1.0D- 5				9			29°	120	
#20			500			R=100		1.0D- 6				29		71	120			
#21			500			R=100		1.0D- 5				diverging		—	—			

I.O. : In-core assembly, out-of-core solution
 I.I. : In-core assembly, in-core solution
 Division : Number of meshes in the whole field
 * : Approx. solutions are specified by input data.
 ** : Approx. solutions are specified as 0.
 *1 : Approx. solutions are specified by suitable data.

Initial iteration times : Iteration times when $\phi=0$ at an early stage of iteration
 ° : Without high speed algorithm (HSA), others with HSA.

Table 3 Computational results for the case of power-law fluid flow through a square duct

Number of series	Solving method	Program	n	Division	F	Initialization	Approx. solutions	ϵ	δ	Initial iteration times	Rate of acceleration	ϕ	Total number of iterations	CPUT(s) for matrices	CPUT(s) for an iteration	Total CPUT(s)	Elapsed time (s)	
#22	Present method	I. I.	1.0	15x15	Standard	*	**	0.1	2.0D-7	14	1/5	0.95	26	2.14	0.735	23.1	60	
#23			1.0		Continuation method	$n=0.9$	0.2	2.0D-5	24				29.5°			180		
#24			0.9		Standard	*	*1	0.1	2.0D-7				76			61.6	240	
#25			0.9				**	0.1	10	1/1			66			52.3	120	
#26			0.75					11					1/2			49	40.4	180
#27			0.75						1/3		79		63.6			420		
#28			0.75								100		79.0			420		
#29			0.75				*2	10	1/3	94	73.9		240					
#30			0.75				*2			diverging	—		60					
#31			0.75		$n=0.82$	14	1/5			71	56.6		120					
#32	0.65	Standard	$n=0.9$	191	143.9			240										
#33	0.65	Modified	$n=0.82$	157	119.1			240										
#34	Simplified Newton method	1.0	Standard	*	$n=0.9$	2.0D-5	2.0D-5	12	15.8°	120								
#35		0.9			**						22	18.3	120					
#36		0.75			$n=0.9$						47	38.4	120					
#37		0.75			*2	diverging	—	120										
#38		0.75			Continuation method	$n=0.82$	diverging	—	60									
#39		0.65			Standard	$n=0.9$	diverging	—	120									

* : Approx. solutions are specified by input data.

** : Approx. solutions are specified as $w=0.3\cos(\pi x/2)\cos(\pi y/2)$, $\bar{u}=1$.

*1 : Approx. solutions are specified by suitable data.

*2 : Approx. solutions are specified by the date of the type (**) + (uniform random number with amplitude 0.1).

3.5.1 Characters of dependence on approximate solutions for stability of iteration

In general, the Newton method and a simplified Newton method produce rapidly converging sequences for solutions of equations by iteration if starting approximate solutions have good accuracy; but if not, they often produce divergent sequences for heavily nonlinear equations because residuals propagate directly to the next iteration step. On the other hand, the method in this report gives rise to propagating residuals of order ϵ if there is none or a low rate of acceleration and the original residuals will be distributed into the domain of stability during 2 or 3 times of iteration. For this reason, iteration by the present method supplies a rather good character for stability (convergence) against randomness of starting approximate solutions as shown in Table 1. In case of cavity flow (Table 2), stable convergence (#14, #18) up to about 200 of R is obtained with the present method even if $x^* = 0$ is roughly specified as starting approximate solutions. The reason why the case #16 ($R = 300$) is a diverging case is whether Eq. (24) does not hold due to the dependence of F on x^* or residuals of higher than second power become large. Solutions for cases with a larger value R where nonlinearity is stronger can be obtained by the present technique, using the values x^* derived by the continuation method. Another solving method by Kawaguchi⁽¹⁰⁾ shows a diverging case at $R=128$ where $A=E$ (a unit matrix) and $F=0$ (a zero matrix). Suppose that small uniform random numbers are added to x^* . Then the present technique can lead to a solution with a suitable rate of acceleration (#29), whereas a simplified Newton method yields a diverging case (#37). Under these circumstances even the present method may lead to a diverging case, e.g. when accelerated every time (#30), because in such a case the variables have tendency to move by iteration toward a domain without local convergence. With the present method rough continuation yields converging cases (#31-#33), whereas with a simplified Newton method it yields a diverging case (#39).

3.5.2 Stability and parameters originally included in nonlinear equations

In contrast with other methods, e.g. the Gauss-Seidel method where nonlinear terms are all included on an iterative side, the present method, regardless of parameters such as R and n in these examples, yields convergence stably if a

Fig.7 Diagram for norm of errors (Cavity flow).

Fig.8 Diagram for norm of errors (Flow through a square duct).

Table 4 Values of θ obtained in various nonlinear problems

Solving procedure	Problem	Total number of grid points	θ (approx.)	Reference
Point-Jacobi method	Nonlinear heat conduction	961	0.97	(11)
Overrelaxation method			0.92	
ADI method			0.55	
Stone method			0.70	
Multi-level adaptive technique	($R=32$)	1024	0.934	(12)
	Cavity ($R=80$)		0.958	
	flow ($R=160$)		0.984	
	($R=160$)	256	0.983	

small parameter ϵ is so chosen that $R_e(\epsilon) > 0$ and if prescribed starting approximate solutions are relatively accurate, e.g. specified by the continuation method. In case that specified simultaneous equations are all real functions with real roots, it is more preferable for computation that $\epsilon > 0$. Moreover, in case that those equations are all linear, the iteration procedure is stable for convergence except in a special case $\det F=0$, if $\epsilon (R_e(\epsilon) > 0)$ is chosen sufficiently small.

3.5.3 Characters for local convergence

The characters of dependence on approximate solutions for local convergence are as indicated in §3.5.1. Since it is not desirable that the degree of convergence varies radically as a little change of boundary conditions etc., the degree of variation on the characters for convergence is examined by adding a few terms to F . These examined cases correspond to those "modified" in the column in Tables 2 and 3. From these it is found that the degrees of convergence and stability do not vary a great deal in case of the present proposed iterative procedure whereas the simplified Newton method shows a destabilizing tendency. According to this result, when constructing F from Eq.(7), there is a possibility of some minor terms being omitted.

3.5.4 Rate of convergence on iteration

Figures 7 and 8 show the behaviour of residuals against iteration time $\bar{n}$, where a norm of errors defined as

$$\text{a norm of errors} = \max_i |\xi_i^{(k)} - x_i^{(k)}|$$

is adopted as a criterion for a residual. θ in the figures represents an asymptotic value for ratios of neighbouring two norms. The maximum error is related to the norm of errors as

$$\text{the maximum error} \approx \{\theta/(1-\theta)\} \times (\text{the norm of errors})$$

if no acceleration is made. For the case of weak nonlinearity and relatively accurate initial approximate solutions, $\theta \approx 1 - \epsilon$ if $\epsilon > 0$ and if no acceleration is made. In a favourable case a value $0.1 \sim 0.2$ of θ can be obtained through acceleration as shown in Fig.7. In case of strong nonlinearity such as in the examples derived from power-law fluid flow, it is hard to get a small value of θ even by acceleration as shown in Fig.8. For reference approximate values of θ reported in other examples are shown in Table 4. So the values of θ in this report are considered rather good. Since much time is required for specifying the iteration matrices, in principle the once specified matrices should not be modified through a process of iteration. Nevertheless for equations with strong nonlinearity as in Fig.8, the rate of convergence is improved if the iteration matrix F is modified once in the midst of the whole process. In the case

shown in Fig.8, F is modified once after 25 times of iteration.

3.5.5 Acceleration

Although iterative methods are very effective for solving nonlinear simultaneous equations, especially with multiple variables, the rate of convergence is small in general. So, many acceleration techniques have been developed⁽¹³⁾. In the iterative method proposed, the acceleration technique shown in Eq.(6) is applicable, but the rate of acceleration should be less if initial approximate solutions are not sufficiently accurate.

3.5.6 Others

Although computed examples of solution in this report are all based on simultaneous replacement, for cases with more variables it is advantageous to iterate the computation by dividing variables into some blocks from a viewpoint of I/O time for disks or MT of computers.

The computations were executed on the HITAC M-180 [Gibson MIX 0.21μs with HSA] at Tokyo Institute of Technology. The results are based on double precision computation by FORTRAN language and the volume of maximum space used (at the type I.O. in Table 2) is 970kB 1000kB for virtual space and 2280kB for temporary disks.

4. Conclusions

Systems of nonlinear equations can be solved by the simultaneous iterative method proposed in this report and the following characters are found.

- (1) Solutions can be obtained starting from relatively rough approximate solutions.
- (2) The character for convergence does scarcely depend on parameters included in the original equations.
- (3) Relatively high rate of convergence can be obtained.
- (4) It is possible to use an acceleration parameter for rapid convergence.

[Appendix 1] Local convergence

To find a spectrum radius ρ of a matrix $(F + \epsilon A)^{-1} \{F^m + \epsilon(\partial H^m / \partial x^i)_{x_0}\}$,

let the characteristics of the matrix be λ . Then

$$\{F^i + \epsilon(\partial H^i / \partial x^i)_{x_0} - \lambda(F^i + \epsilon A^i)\} = 0 \quad \dots (19)$$

If H^i is assumed to be twice-differentiable and continuous (C^2 -class for real functions with real variables) or to be regular, then

$$(\partial H^i / \partial x^i)_{x_0} = (\partial H^i / \partial x^i)_{x^*} + (x_0^m - x^{*m})(\partial^2 H^i / \partial x^m \partial x^i) \dots \dots \dots (20)$$

where the term including a symbol — corresponds to a remainder of Lagrange and its argument is omitted. In case of a regular function, it is replaced by a remainder with a type of complex integral.

In the case of $\det(F) \neq 0$, λ can be expanded near $\epsilon=0$ (except in very few cases) as

$$\lambda = 1 + \epsilon f_1 + \epsilon^2 f_2 + \dots \quad (21)$$

For f_1 , from Eqs. (7), (19), and (20),

$$-(1+f_1)F'_j + (x_0^m - x^{*m}) \times (\partial^2 H / \partial x^m \partial x^j) = 0 \quad (22)$$

Hence

$$f_1 = -1 + O(\|x^* - x_0\|) \quad (23)$$

Hence in case that initial approximate solutions are relatively accurate, for a small parameter ϵ [$R_e(\epsilon) > 0$] the following holds according to Eqs. (23) and (21):

$$\rho = \max |\lambda| < 1 \quad (24)$$

This indicates that the sequences by iteration converge locally. According to Eqs. (21) and (23), all characteristics are expressed as

$$\lambda \approx 1 - \epsilon \quad (25)$$

if x^* is in relatively good accuracy, and it is found the residuals behave as Eq. (8) in case of having no multiple roots.

[Appendix 2] Estimation for a rate of acceleration

Acceleration due to Eq. (6) is based on the original relation (25) for characteristics. So the character for convergence and stability depends on the degree of realization of Eq. (25), including the effect of accuracy of initial approximate solutions. For this reason, in case that the absolute value of $\partial^2 H^i / \partial x^m \partial x^j$ is comparable to or greater than that of the element F_j^i (e.g. in case of Table 3 with a function logarithm), it is considered that deviation from Eq. (25) may become greater, and so in such a case it is recommended that the rate of acceleration is less, e.g. every 2 to 5 times.

[Appendix 3] Coefficient of acceleration : ϕ

For cases of treating real numbers, a parameter ϵ is normally recommended to be $0.05 \sim 0.2$ and in this case $\phi \approx 1.0$. But in case that ϵ should be assumed such that $0 < \epsilon \leq 0.05$, for stability ϕ should be positive and smaller than unity to avoid amplifying residuals.

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