In-plane Vibrations of Curved Bars Considering Shear Deformation and Rotatory Inertia*

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In this paper, the in-plane vibrations of a uniform curved bar of which the center line is a plane curve are investigated considering the bending, the extension, the shear deformation and the rotatory inertia of the bar. We obtain the Lagrangian of vibration of the curved bar in a period, and from the stationary condition of the Lagrangian, we determine the equations of vibration and the boundary conditions. As examples, we make calculations for elliptic arc bars with built-in ends and with simply supported ends, and obtain the frequencies and the mode shapes. The numerical results of the present analysis are compared with those by the classical theory and the effects of the shear deformation and the rotatory inertia are clarified.

1. Introduction

We can not neglect the effects of shear deformation and rotatory inertia when we consider the vibration of a curved bar, of which the radius of the center line is small in comparison with the cross-sectional dimension or of which the arc length of the bar is short. By an improved theory considering these effects, Morley⁽¹⁾, Seidel⁽²⁾, Rao⁽³⁾ and Nagaya⁽⁴⁾ have analyzed the in-plane vibrations of curved bars. But they have treated only the vibration problems of a circular arc bar. It seems that a treatise dealing with the vibration problems of a general curved bar such as an elliptic arc bar has not been reported yet. Recently, the authors^{(5), (6)} have analyzed the in-plane free vibrations of various curved bars of which the center lines are some plane curves by using the classical theory, in which only extension and bending of bar are considered. And they have solved the equations of vibration exactly and obtained the natural frequencies and the mode shapes. But the characteristic of vibration of the curved bar of which the radius of center line is small in comparison with the cross-sectional dimension or the arc length is short has not been studied yet.

From this point of view, they analyze, in this paper, in-plane vibrations of curved bars of which center lines are some plane curves considering the effects of bending, extension, shear deformation and rotatory inertia of bar. And they find out a general method of solution which is applicable to various curved bars and at the same time clarify the influence of them on the characteristic of vibration. They obtain the Lagrangian of a curved bar in a period and determine the equations of vibration and the

boundary conditions from the stationary condition of the Lagrangian. As numerical examples, they obtain the natural frequencies and the mode shapes of elliptic arc bars with built-in ends and with simply supported ends. Moreover, they compare the results of the classical theory with those of the present improved theory and clarify the influence of shearing deformation and rotatory inertia on the natural frequencies and the mode shapes.

2. Notations

- l : representative length
- ρ : radius of center line
- R : radius of neutral axis
- p : circular frequency
- t : time
- γ/g : density
- A : cross-sectional area
- E : modulus of longitudinal elasticity
- G : modulus of transverse elasticity
- I : moment of inertia of area
- κ : shear coefficient
- k^2 : section modulus of curved bar
- α : non-dimensional frequency, $\alpha^4 = \gamma A p^2 l^4 / E I g$
- ν : slenderness ratio, $\nu^2 = A l^2 / I$
- $U, \bar{W}$: displacements in the $\bar{y}$ and $\bar{x}$ directions
- $\bar{U}, \bar{W}$: non-dimensional displacements, $\bar{U} = U/l$, $\bar{W} = W/l$
- $\bar{u}, \bar{w}$: non-dimensional displacements towards the center of curvature and along tangent at neutral axis, respectively
- $\bar{\phi}$: slope due to bending at neutral axis
- $\bar{y}, \bar{r}$: coordinates towards the center of curvature
- y, r : non-dimensional coordinates, $y = \bar{y}/R$, $r = 1 - y$
- $\bar{s}_0$: arc length measured along center line
- $\bar{s}$: arc length measured along neutral axis
- $\bar{s}'$: arc length at a distance of $\bar{y}$ from neutral axis
- s : non-dimensional arc length, $s = \bar{s}/l$
- $\bar{N}$: resultant tangential force
- $\bar{Q}$: resultant shearing force
- $\bar{M}$: resultant moment referred to neutral axis
- f : ratio of representative length to radius of neutral axis, $f = l/R$

* Received 26th July, 1977.

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3. Theory

3.1 Lagrangian of vibration in a period and equations of vibration

Let us consider the in-plane vibrations of a uniform curved bar, of which the center line and the neutral axis are denoted by some plane curves. First, we consider a deformation of an infinitesimal portion of the curved bar, as shown in Fig.1, of which the radius of the center line is ρ and the radius of the neutral axis is R . We take the $\bar{x}$ coordinate axis in the direction of the tangent at the neutral axis, and do the $\bar{y}$ axis towards the center of the curvature on a surface comprising the neutral axis. Furthermore, we denote the displacements in $\bar{x}$ and $\bar{y}$ directions by W and U , respectively. Let A denote the cross-sectional area of the bar; $d\bar{s}_0, d\bar{s}$ and $d\bar{s}'$, the lengths of the center line, the neutral axis and the infinitesimal

Fig.1

line element at a distance of $\bar{y}$ from the neutral axis, respectively. We denote the representative length by l and use the following notations,

$$\bar{U}=U/l, \quad \bar{W}=W/l, \quad \bar{s}=\bar{s}/l, \quad \bar{y}=\bar{y}/R, \quad \bar{r}=\bar{r}/R=1-\bar{y}, \quad \bar{f}=l/R \dots (1)$$

The neutral axis is determined by the equation⁽⁷⁾

$$\int_A \frac{\bar{y}}{1-\bar{y}} dA = 0 \dots (2)$$

Denoting the normal and the shearing strains of the line element at a distance of $\bar{y}$ from the neutral axis by ϵ_s and γ_{rs} , respectively, we have

$$\epsilon_s = \frac{\partial \bar{W}}{\partial \bar{s}'} - \frac{U}{\bar{r}} = \frac{1}{\bar{r}} \left(\frac{\partial \bar{W}}{\partial \bar{s}} - \bar{f} \bar{U} \right), \quad \gamma_{rs} = \frac{\partial \bar{W}}{\partial \bar{r}} - \frac{W}{\bar{r}} - \frac{\partial U}{\partial \bar{s}'} = \bar{f} \left(\frac{\partial \bar{W}}{\partial \bar{r}} - \frac{\bar{W}}{\bar{r}} \right) - \frac{1}{\bar{r}} \frac{\partial \bar{U}}{\partial \bar{s}} \dots (3)$$

Now we assume the non-dimensional displacements $\bar{W}$ and $\bar{U}$ as follows;

$$\bar{W} = \bar{w}(s, t) - \bar{y} \bar{\phi}(s, t), \quad \bar{U} = \bar{u}(s, t) \dots (4)$$

where $\bar{w}$, $\bar{u}$ and $\bar{\phi}$ are the non-dimensional displacement along tangent, the one towards the center of curvature and the slope due to bending at the neutral axis, respectively. Substituting eq.(4) into eq.(3), we obtain

$$\epsilon_s = \frac{1}{1-\bar{y}} \left(\frac{\partial \bar{w}}{\partial \bar{s}} - \bar{f} \bar{u} - \bar{y} \frac{\partial \bar{\phi}}{\partial \bar{s}} \right), \quad \gamma_{rs} = \frac{1}{1-\bar{y}} \left\{ \bar{f} (\bar{\phi} - \bar{w}) - \frac{\partial \bar{u}}{\partial \bar{s}} \right\} \dots (5)$$

Considering eq.(2) and noting the following relations obtained from eq.(2),

$$\begin{aligned} \int_A \frac{1}{1-\bar{y}} dA &= \int_A \frac{1-\bar{y}}{1-\bar{y}} dA - \int_A \frac{\bar{y}}{1-\bar{y}} dA = A \\ \int_A \frac{\bar{y}^2}{1-\bar{y}} dA &= - \int_A \frac{\bar{y}(1-\bar{y})}{1-\bar{y}} dA + \int_A \frac{\bar{y}}{1-\bar{y}} dA = - \int_A \bar{y} dA \end{aligned}$$

the resultant tangential force over the cross-section $\bar{N}$, the resultant shearing force $\bar{Q}$ and the resultant moment referred to the neutral axis $\bar{M}$ are expressed in the following form:

$$\begin{aligned} \bar{N} &= \int_A E \epsilon_s dA = EA \left(\frac{\partial \bar{w}}{\partial \bar{s}} - \bar{f} \bar{u} \right), \quad \bar{Q} = \kappa \int_A G \gamma_{rs} dA = \kappa GA \left\{ \bar{f} (\bar{\phi} - \bar{w}) - \frac{\partial \bar{u}}{\partial \bar{s}} \right\} \\ \bar{M} &= \int_A \bar{y} E \epsilon_s dA = -E A R k^2 \frac{\partial \bar{\phi}}{\partial \bar{s}} \end{aligned} \dots (6)$$

where E , G and κ are the moduli of the longitudinal and the transverse elasticities and the shear coefficient respectively. And k^2 is the section modulus of curved bar, which is expressed as follows.

$$k^2 = - \frac{1}{A} \int_A \bar{y} dA \dots (7)$$

Next let us consider the Lagrangian of vibration of a curved bar in a period. Let g and γ denote the acceleration due to gravity and the specific weight, respectively. Denoting the kinetic energies due to U and W by T_1 and T_2 and the strain energies due to tangential and shearing forces by V_1 and V_2 , respectively, we obtain

$$\left. \begin{aligned} T_1 &= \int_{s_1}^{s_2} \left[\int_A \frac{\gamma}{2g} \left(\frac{\partial U}{\partial t} \right)^2 dA \right] ds' = \int_{s_1}^{s_2} \frac{A\gamma l^2}{2g} (1+k^2) \left(\frac{\partial \bar{u}}{\partial t} \right)^2 ds \\ T_2 &= \int_{s_1}^{s_2} \left[\int_A \frac{\gamma}{2g} \left(\frac{\partial W}{\partial t} \right)^2 dA \right] ds' = \int_{s_1}^{s_2} \frac{A\gamma l^2}{2g} \left[(1+k^2) \left(\frac{\partial \bar{w}}{\partial t} \right)^2 \right. \\ &\quad \left. + 2k_1^2 \frac{\partial \bar{w}}{\partial t} \frac{\partial \bar{\phi}}{\partial t} + k_2^2 \left(\frac{\partial \bar{\phi}}{\partial t} \right)^2 \right] ds \\ V_1 &= \int_{s_1}^{s_2} \left[\int_A \frac{E\epsilon_s^2}{2} dA \right] ds' = \int_{s_1}^{s_2} \frac{EA}{2} \left[\left(\frac{\partial \bar{w}}{\partial s} - f\bar{u} \right)^2 + k^2 \left(\frac{\partial \bar{\phi}}{\partial s} \right)^2 \right] ds \\ V_2 &= \int_{s_1}^{s_2} \left[\int_A \frac{\kappa G \gamma_s^2}{2} dA \right] ds' = \int_{s_1}^{s_2} \frac{\kappa GA}{2} \left[f(\bar{\phi} - \bar{w}) - \frac{\partial \bar{u}}{\partial s} \right]^2 ds \end{aligned} \right\} \dots\dots\dots (8)$$

where

$$Ak_1^2 = - \int_A y(1-y) dA, \quad Ak_2^2 = \int_A y^2(1-y) dA \dots\dots\dots (9)$$

We put as follows.

$$\bar{u} = u(s) \sin pt, \quad \bar{w} = w(s) \sin pt, \quad \bar{\phi} = \phi(s) \sin pt \dots\dots\dots (10)$$

where p is the circular frequency and t is the time. We define the Lagrangian of vibration in a period (τ) as follows.

$$L = \frac{1}{\tau} \int_0^\tau [T_1 + T_2 - (V_1 + V_2)] dt \dots\dots\dots (11)$$

From eqs. (8)~(11), we have

$$\begin{aligned} L / \left(\frac{l}{2} \right) &= \int_{s_1}^{s_2} \left[\frac{A\gamma p^2 l^2}{g} \{ (1+k^2)(u^2 + w^2) + 2k_1^2 w\phi + k_2^2 \phi^2 \} - EA \left(\frac{dw}{ds} - fu \right)^2 \right. \\ &\quad \left. - EAK^2 \left(\frac{d\phi}{ds} \right)^2 - \kappa GA \left\{ f(\phi - w) - \frac{du}{ds} \right\}^2 \right] ds \dots\dots\dots (12) \end{aligned}$$

To obtain the stationary condition of the Lagrangian $\delta L = 0$, let us consider the first variation of eq. (12). Then we have

$$\begin{aligned} \delta L / l &= - \int_{s_1}^{s_2} \left[\frac{dQ}{ds} - fN - \lambda(1+k^2)u \right] \delta u ds + \int_{s_1}^{s_2} \left[\frac{dN}{ds} + fQ + \lambda \{ (1+k^2)w + k_1^2 \phi \} \right] \delta w ds \\ &\quad - \int_{s_1}^{s_2} \left[\frac{d}{ds} \left(\frac{M}{R} \right) + fQ - \lambda(k_1^2 w + k_2^2 \phi) \right] \delta \phi ds + \left[Q\delta u - N\delta w + \left(\frac{M}{R} \right) \delta \phi \right]_{s_1}^{s_2} \dots\dots\dots (13) \end{aligned}$$

where

$$N = EA \left(\frac{dw}{ds} - fu \right), \quad Q = \kappa GA \left\{ f(\phi - w) - \frac{du}{ds} \right\}, \quad M/R = -EAK^2 \frac{d\phi}{ds}, \quad \lambda = A\gamma p^2 l^2 / g \dots\dots\dots (14)$$

And s_1 and s_2 are the values of s at both ends of the bar. Putting the expressions in the brackets [] of the three integral terms in the right side of eq. (13) as zero, we obtain Euler's equations (the equations of vibration) and putting the boundary terms as zero, we obtain the boundary conditions. Boundary terms consist of the products of the shearing force and the displacement towards the center of curvature, of the tangential force and the displacement along tangent, of the bending moment and the slope due to bending, respectively. The equations of vibration are

$$\left. \begin{aligned} \frac{dN}{ds} + fQ + \lambda \{ (1+k^2)w + k_1^2 \phi \} &= 0, \quad \frac{d}{ds} \left(\frac{M}{R} \right) + fQ - \lambda \{ k_1^2 w + k_2^2 \phi \} = 0 \\ \frac{dQ}{ds} - fN - \lambda(1+k^2)u &= 0 \end{aligned} \right\} \dots\dots\dots (15)$$

If we take $R = \text{const}$ and put the representative length l as R , we find $f = l/R = 1$ and equation (15) corresponds with the equations of vibration of a circular arc bar derived by Morley¹⁰. As a special case of the present analysis, we make a comparison of the Lagrangian obtained by neglecting the shear deformation, the rotatory inertia and the gap of the center line and the neutral axis in eq. (12) with the one by the classical theory in the appendix.

3.2 Solutions of equations of vibration

Let us consider a method for solving the equations of vibration (15). As evident from eqs.(7) and (9), k^2 , k_1^2 and k_2^2 in eq.(15) become the constants independent of s for a circular arc bar of which the curvature is constant but they become the functions of s for curved bars of which the curvatures vary. Calculating k^2 , k_1^2 and k_2^2 , as an example, for a uniform rectangular cross-section of height h and width b , we have

$$k^2 = -1 + \left(1 + \frac{h}{\rho}\right) \log \frac{1+h/2\rho}{1-h/2\rho} \dots\dots\dots(16)$$

$$k_1^2 = k^2(1+k^2) + \frac{1}{12} \left(\frac{h}{\rho}\right)^2 (1+k^2)^2 \dots\dots\dots(17)$$

$$k_2^2 = k^4(1+k^2) + \frac{1}{12} \left(\frac{h}{\rho}\right)^2 (1+k^2)^2 (1+3k^2) \dots\dots\dots(18)$$

Thus we have k^2 , k_1^2 and k_2^2 as the functions of (h/ρ) . It is very difficult for us to obtain a solution of eq.(15) treating them in a rigorous form. However, by expanding the right side of eq.(16) into a power series, we obtain

$$\left. \begin{aligned} k^2 &= \frac{1}{3 \times 4} \left(\frac{h}{\rho}\right)^2 + \frac{1}{5 \times 16} \left(\frac{h}{\rho}\right)^4 + \frac{1}{7 \times 64} \left(\frac{h}{\rho}\right)^6 + \dots\dots\dots \\ &= \frac{1}{\nu^2} \left(\frac{l}{\rho}\right)^2 + \frac{9}{5} \frac{1}{\nu^4} \left(\frac{l}{\rho}\right)^4 + \frac{27}{7} \frac{1}{\nu^6} \left(\frac{l}{\rho}\right)^6 + \dots\dots\dots \end{aligned} \right\} \dots\dots\dots(19)$$

where

$$\nu^2 = Al^2/I, \quad I = bh^3/12$$

And by calculating the values of k^2 , $2k^2$, k_1^2 , k_2^2 and $(1/12)(h/\rho)^2$ (the first term in the right side of eq.(19)) for various values of (h/ρ) , we find that the following relations approximately hold in the range of $h/\rho \leq 0.6$.

$$k^2 \doteq \frac{1}{12} \left(\frac{h}{\rho}\right)^2 = \left(\frac{l}{\nu\rho}\right)^2, \quad k_1^2 \doteq 2k^2, \quad k_2^2 \doteq k^2 \dots\dots\dots(20)$$

The error is within several percent and it becomes little when h/ρ becomes less than 0.6. Therefore we make the analysis according to eq.(20) hereafter. For an elliptic arc bar, which is described later, we obtain

$$\frac{h}{\rho} = \frac{\sqrt{12}}{(1-\mu^2)\nu} (1+\mu \cos 2x)^{3/2}$$

The value of (h/ρ) varies with the values of μ , ν and x (flatness, slenderness ratio, variable with a unit of angle). The maximum value is $(\sqrt{12}/\nu)\{\sqrt{1+\mu}/(1-\mu)\}$ at $x=0$ and it is 0.047 when $\mu=0.2$, $\nu=100$ and 0.592 when $\mu=0.2$, $\nu=8$. Both the maximum values are less than 0.6. In case of $\mu=0.4$, the maximum value is 0.068 when $\nu=100$ and 0.85 when $\nu=8$. In case of $\nu=8$, the maximum value is over 0.6, but all the values of h/ρ are less than 0.6 when $x \leq 0.58$.

Now let us consider the following transformation of the variable.

$$\frac{dx}{ds} = \frac{l}{R} = f(x) \dots\dots\dots(21)$$

Then, by denoting the arc length of the center line by $\bar{s}_0$, we obtain the next relation between x and $\bar{s}_0$,

$$\frac{dx}{d\bar{s}_0} = \frac{1}{\rho} \dots\dots\dots(22)$$

Rewriting eq.(12) by using eq.(21), we have

$$\begin{aligned} -L \left/ \left(\frac{l}{2}\right) \right. &= \int_{x_1}^{x_2} E A f \left(\frac{dw}{dx} - u \right)^2 + E A k^2 f \left(\frac{d\phi}{dx} \right)^2 + \kappa G A f \left(\phi - w - \frac{du}{dx} \right)^2 \\ &\quad - \lambda \{ (1+k^2)(u^2+w^2) + 2k_1^2 w\phi + k_2^2 \phi^2 \} / f \, dx \dots\dots\dots(23) \end{aligned}$$

To obtain the stationary condition of the Lagrangian $\delta L=0$, we consider the first variation of L , i.e., δL and then we obtain

$$\begin{aligned} -\delta L &= \int_{x_1}^{x_2} f \frac{dQ_0}{dx} - f N_0 - \lambda (1+k^2) u \left[\frac{\delta u}{f} dx - \int_{x_1}^{x_2} f \frac{dN_0}{dx} + f Q_0 + \lambda \{ (1+k^2)w + k_1^2 \phi \} \right] \frac{\delta w}{f} dx \\ &\quad + \int_{x_1}^{x_2} f \frac{dM_0}{dx} + f Q_0 - \lambda (k_1^2 w + k_2^2 \phi) \left[\frac{\delta \phi}{f} dx - \int_{x_1}^{x_2} Q_0 \delta u - N_0 \delta w + M_0 \delta \phi \right]_{x_1}^{x_2} \dots\dots\dots(24) \end{aligned}$$

where

$$N_0 = E A f \left(\frac{dw}{dx} - u \right), \quad Q_0 = \kappa G A f \left(\phi - w - \frac{du}{dx} \right), \quad M_0 = -E A k^2 f \frac{d\phi}{dx} \quad (25)$$

And x_1 and x_2 denote the values of x corresponding to s_1 and s_2 , in which x is a variable that shows an angle (see Reference(6)). Putting the expressions in the brackets [] of the integral terms in the right side of eq.(24) as zero, we obtain the equations of vibration and putting the boundary terms as zero, we obtain the boundary conditions. The meaning of the boundary terms is the same as in eq.(13). The equations of vibration are

$$\left. \begin{aligned} f \frac{dN_0}{dx} + f Q_0 + \lambda \{ (1+k^2)w + k_1^2 \phi \} &= 0, & f \frac{dM_0}{dx} + f Q_0 - \lambda (k_1^2 w + k_2^2 \phi) &= 0 \\ f \frac{dQ_0}{dx} - f N_0 - \lambda (1+k^2)u &= 0 \end{aligned} \right\} \quad (26)$$

We put as follows.

$$l/\rho = G_0 \Phi(x) \quad (27)$$

in which G_0 is a non-dimensional constant and $\Phi(x)$ is a function of x . We can express l/ρ as eq.(27) for all the curved bars of which the curvatures are denoted by some expressions, and we can obtain $\Phi(x)$ in a simple expression⁽⁶⁾. As an example, let us consider an ellipse of which the equation is denoted by the rectangular coordinates (ξ_1, ξ_2) as $\xi_1 = a_0 \cos \theta$, $\xi_2 = b_0 \sin \theta$ in which $2a_0$, $2b_0$ and θ are the major and the minor axes and a parameter, respectively. Using the following notations,

$$\mu = \frac{a_0^2 - b_0^2}{a_0^2 + b_0^2}, \quad l = \sqrt{\frac{a_0^2 + b_0^2}{2}}$$

we obtain, from eq.(22),

$$\tan x = \sqrt{\frac{1+\mu}{1-\mu}} \tan \theta, \quad G_0 = \frac{1}{1-\mu^2}, \quad \Phi(x) = (1 + \mu \cos 2x)^{3/2} \quad (28)$$

Using eq.(27), we find

$$f(x) = \frac{l}{\rho} (1+k^2) = \frac{l}{\rho} \left\{ 1 + \left(\frac{l}{\nu \rho} \right)^2 \right\} = G_0 \Phi \left\{ 1 + \left(\frac{G_0}{\nu} \right)^2 \Phi^2 \right\} \quad (29)$$

Substituting eq.(20) into the equations of vibration (26) and using eq.(27), we find

$$\left. \begin{aligned} f \frac{d}{dx} f \left(\frac{dw}{dx} - u \right) + \frac{\kappa G}{E} f^2 \left(\phi - w - \frac{du}{dx} \right) + \frac{\alpha^4}{\nu^2} \left\{ 1 + \left(\frac{G_0}{\nu} \right)^2 \Phi^2 \right\} w + 2 \left(\frac{G_0}{\nu} \right)^2 \Phi^2 \phi &= 0 \\ f \frac{d}{dx} f \Phi^2 \frac{d\phi}{dx} - \frac{\kappa G}{E} \left(\frac{\nu}{G_0} \right)^2 f^2 \left(\phi - w - \frac{du}{dx} \right) + \frac{\alpha^4}{\nu^2} \Phi^2 (2w + \phi) &= 0 \\ f^2 \left(\frac{dw}{dx} - u \right) - \frac{\kappa G}{E} f \frac{d}{dx} f \left(\phi - w - \frac{du}{dx} \right) + \frac{\alpha^4}{\nu^2} \left\{ 1 + \left(\frac{G_0}{\nu} \right)^2 \Phi^2 \right\} u &= 0 \end{aligned} \right\} \quad (30)$$

We put as follows.

$$\Phi^2 = \Phi_0^2 \sum_{n=0}^{\infty} \zeta_n x^{2n}, \quad f^2 = (G_0 \Phi_0)^2 \sum_{n=0}^{\infty} \eta_n x^{2n} \quad (31)$$

We can denote Φ^2 and f^2 as the above expressions for curved bars of which the curvatures are expressed as even functions of arc length. For the ellipse mentioned above, we have

$$\begin{aligned} \Phi_0^2 &= (1+\mu)^3, \quad \zeta_0 = 1, \quad \eta_0 = 1 + \frac{2(1+\mu)}{\nu^2(1-\mu)^2} + \frac{(1+\mu)^2}{\nu^4(1-\mu)^4}, \quad \zeta_n = \frac{(-1)^n \mu K_{1n}}{4(1+\mu)^3(2n)!} \\ \eta_n &= \frac{(-1)^n \mu}{4(1+\mu)^3(2n)!} K_{1n} + \frac{1}{4} \left(\frac{G_0}{\nu} \right)^2 K_{2n} + \frac{1}{64} \left(\frac{G_0}{\nu} \right)^4 K_{3n}, \quad K_{1n} = \mu^2 6^{2n} + 6\mu^4 2^{2n} + 3(\mu^2 + 1)2^{2n} \\ &\quad + 24(5\mu^4 + 20\mu^2 + 8)2^{2n} \\ K_{2n} &= \mu^3 12^{2n} + 12\mu^4 10^{2n} + 6\mu^5 (\mu^2 + 10)8^{2n} + 20\mu^2 (3\mu^2 + 8)6^{2n} + 15\mu (\mu^4 + 16\mu^2 + 16)4^{2n} \\ &\quad + 24(5\mu^4 + 20\mu^2 + 8)2^{2n} \\ K_{3n} &= \mu^3 18^{2n} + 18\mu^4 16^{2n} + 9\mu^5 (\mu^2 + 16)14^{2n} + 48\mu^3 (3\mu^2 + 14)12^{2n} + 36\mu^4 (\mu^4 + 28\mu^2 + 56)10^{2n} \\ &\quad + 504\mu^3 (\mu^4 + 8\mu^2 + 8)8^{2n} + 84\mu^2 (\mu^6 + 36\mu^4 + 120\mu^2 + 64)6^{2n} + 144\mu (7\mu^6 + 70\mu^4 + 112\mu^2 + 32)4^{2n} \\ &\quad + 18(7\mu^3 + 280\mu^5 + 1120\mu^4 + 896\mu^2 + 128)2^{2n} \end{aligned}$$

As the solutions of eq.(30), the following two kinds of solutions are obtained.

$$u = \sum_{n=0}^{\infty} A_n x^{2n+1}, \quad w = \sum_{n=0}^{\infty} B_n x^{2n}, \quad \phi = \sum_{n=0}^{\infty} C_n x^{2n} \dots (32)$$

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where A_n , B_n , and C_n are undetermined coefficients.

(i) for the solution given by eq.(32) (antisymmetric vibration)

Substituting eq.(32) into eq.(30), we have

$$\left. \begin{aligned} & \sum_{n=0}^{\infty} \left[KB_n + \sum_{p=0}^n \left[(n+p+1)\gamma_{n-p} \{ (2p+2)B_{p+1} - A_p \} + \varepsilon \gamma_{n-p} \{ C_p - B_p - (2p+1)A_p \} \right. \right. \\ & \quad \left. \left. + \left(\frac{\alpha}{\nu} \right)^4 \zeta_{n-p} (B_p + 2C_p) \right] \right] x^{2n} = 0 \\ & \sum_{n=0}^{\infty} \left[\sum_{p=0}^n \left[2(G_0 \phi_0)^2 (n+p+1)\gamma_{n-p} \sum_{q=0}^p (q+1)C_{q+1} \zeta_{p-q} - \varepsilon \nu^2 \gamma_{n-p} \{ C_p - B_p - (2p+1)A_p \} \right. \right. \\ & \quad \left. \left. + \frac{\alpha^4}{\nu^2} \zeta_{n-p} (2B_p + C_p) \right] \right] x^{2n} = 0 \\ & \sum_{n=0}^{\infty} \left[KA_n + \sum_{p=0}^n \left[\gamma_{n-p} \{ (2p+2)B_{p+1} - A_p \} - \varepsilon \{ (n+1-p)\gamma_{n-p} \{ C_p - B_p - (2p+1)A_p \} \right. \right. \right. \\ & \quad \left. \left. + (2p+2)\gamma_{n-p} \{ C_{p+1} - B_{p+1} - (2p+3)A_{p+1} \} \right] + \left(\frac{\alpha}{\nu} \right)^4 \zeta_{n-p} A_p \right] x^{2n+1} = 0 \end{aligned} \right\} \dots (34)$$

where

$$\varepsilon = \kappa G/E, \quad K = \alpha^4 / (G_0 \phi_0 \nu)^2 \dots (35)$$

If eq.(34) holds for any value of x , the expressions in the brackets [] of each equation in eq.(34) must be zero for all the values of n . From this condition, B_1 , C_1 and A_1 are obtained with A_0 , B_0 and C_0 left undetermined, and in succession, B_{n+1} , C_{n+1} and A_{n+1} ($n \geq 1$) are obtained. Thus we obtain three independent solutions. Similarly,

(ii) for the solution given by eq.(33) (symmetric vibration)

Substituting eq.(33) into eq.(30), we find

$$\left. \begin{aligned} & \sum_{n=0}^{\infty} \left[KB_n + \sum_{p=0}^n \left[(n+1-p)\gamma_{n-p} \{ (2p+1)B_p - A_p \} + (2p+2)\gamma_{n-p} \{ (2p+3)B_{p+1} - A_{p+1} \} \right. \right. \\ & \quad \left. \left. + \varepsilon \gamma_{n-p} \{ C_p - B_p - (2p+2)A_{p+1} \} + \left(\frac{\alpha}{\nu} \right)^4 \zeta_{n-p} (B_p + 2C_p) \right] \right] x^{2n+1} = 0 \\ & \sum_{n=0}^{\infty} \left[\sum_{p=0}^n \left[(G_0 \phi_0)^2 \{ (n+1-p)\gamma_{n-p} \sum_{q=0}^p (2q+1)\zeta_{p-q} C_{q+1} + (2p+2)\gamma_{n-p} \sum_{q=0}^{p+1} (2q+1)\zeta_{p+1-q} C_q \} \right. \right. \\ & \quad \left. \left. - \varepsilon \nu^2 \gamma_{n-p} \{ C_p - B_p - (2p+2)A_{p+1} \} + \frac{\alpha^4}{\nu^2} \zeta_{n-p} (2B_p + C_p) \right] \right] x^{2n+1} = 0 \\ & \sum_{n=0}^{\infty} \left[KA_n + \sum_{p=0}^n \left[\gamma_{n-p} \{ (2p+1)B_p - A_p \} - \varepsilon (n+p+1)\gamma_{n-p} \{ C_p - B_p - (2p+2)A_{p+1} \} \right. \right. \\ & \quad \left. \left. + \left(\frac{\alpha}{\nu} \right)^4 \zeta_{n-p} A_p \right] \right] x^{2n} = 0 \end{aligned} \right\} \dots (36)$$

From eq.(36), A_1 , B_1 and C_1 are obtained with A_0 , B_0 and C_0 left undetermined, and in succession, A_{n+1} , B_{n+1} and C_{n+1} ($n \geq 1$) are obtained. Thus we obtain three independent solutions as in the case (i). The solution of eq.(30) is given as an expression of a linear combination of these six independent solutions.

4. Numerical calculations

We have made calculations for symmetric elliptic arc bars with both built-in ends and with both simply supported ends. We have used 35 terms of the coefficients ξ_n and η_n and 50 terms of A_n , B_n and C_n , respectively. And we have assumed $\varepsilon (= \kappa G/E) = 0.316$ (Poisson's ratio 0.3, $\kappa = 0.822$ ⁽⁸⁾, rectangular cross section). From eq.(24), the boundary conditions are expressed at both ends ($x = \pm x_0$) as $u = w = \phi = 0$ when both ends are built-in and $u = w = d\phi/dx = 0$ when both ends are simply supported. The frequency equation for antisymmetric vibration, as an example, is obtained as follows. Combining linearly the

three independent solutions obtained from eq.(32), we have

$$\left. \begin{aligned} u &= c_1 u_1 + c_2 u_2 + c_3 u_3 \\ w &= c_1 w_1 + c_2 w_2 + c_3 w_3 \\ \phi &= c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3 \end{aligned} \right\} \dots (37)$$

where $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$ are arbitrary constants. Considering the boundary conditions mentioned above, we obtain a frequency equation in the form of the third order determinant, from which we obtain the relation between α and x_0 and the values of $\bar{c}_2/\bar{c}_1$ and $\bar{c}_3/\bar{c}_1$. We can obtain similarly a frequency equation for symmetric vibration with use of the solution in eq.(33). We have determined three independent solutions on three cases of ($A_0=1$, $B_0=C_0=0$), ($B_0=1$, $C_0=A_0=0$) and ($C_0=1$, $A_0=B_0=0$). As the coefficients A_n , B_n and C_n

contain α , μ , ν , ε , ξ_n and η_n , we have determined, at first, A_n , B_n and C_n for given values of them. And then, we have obtained the relation between α and x_0 by searching for the values of x_0 which make the frequency equation zero. In the calculations of mode shapes, we have assumed $\bar{c}_1=1$. The range of convergence of the solutions differs according to α , μ and ν , but the limit is about $x=\pi/2$.

Figure 2 represents the relation between α and x_0 for symmetric vibration in the case that both ends are built-in and $\mu=0.2$. In the figure, we compare the results by the classical theory with those by the present improved theory in case of the slenderness ratio $\nu=8$, 25 and 100. The curves are for the first mode of vibration, the second mode of vibration... in the order from below. In case of the slenderness ratio $\nu=8$, a large difference between the two theories is recognized. Thus, it is obvious that we can not neglect the effects of the shear deformation and the rotatory inertia when the cross-sectional dimension of the elliptic arc bar is large in comparison with the major and the minor axes of the ellipse. But the results of both the theories almost agree as

Fig.2 Symmetric vibration (both ends built-in, $\mu=0.2$)

the value of x_0 becomes large, that is, the arc of the bar becomes long and also as the slenderness ratio becomes large in the order of $\nu=25$, $\nu=100$. It is obvious from the figure that a transfer phenomenon is recognized even in the improved theory. That is, the frequency curves of $\nu=8$ and $\nu=25$ are fairly distant from the curves of $\nu=100$ when x_0 is small and α is large. But the curves of $\nu=8$ and 25 gradually approach the ones of $\nu=100$ as x_0 becomes large and α becomes small.

Figure 3 represents the relation between α and x_0 for the first and the second modes of antisymmetric vibration (both ends built-in, $\mu=0.2$). The tendency is almost the same as in the symmetric vibration.

Figures 4 and 5 represent the α - x_0 curves in case of $\mu=0.4$ (both ends built-in). Figure 4 corresponds to the symmetric vibration and Fig.5 does to the antisymmetric one. The tendencies in both figures are almost the same as in case of $\mu=0.2$.

Figure 6 represents the α - x_0 curves for the first and the second modes of symmetric vibration and Fig.7 does the ones of antisymmetric vibration, respectively, in the

Fig.4 Symmetric vibration (both ends built-in, $\mu=0.4$)

Fig.3 Antisymmetric vibration (both ends built-in, $\mu=0.2$)

Fig.5 Antisymmetric vibration (both ends built-in, $\mu=0.4$)

case that both ends are simply supported and $\mu=0.2$. The tendency is almost the same as in the case that both ends are built-in. That is, a large difference between the result of the classical theory and that of the improved theory is recognized when the slenderness ratio ν is small and the arc of the bar is short. But the results of both theories almost agree as the slenderness ratio becomes large and as the arc of the bar becomes long. And a transfer phenomenon is recognized when the slenderness ratio is small.

Figures 8 and 9 show the mode shapes for the first and the second modes of vibration in the case that both ends are built-in and $\mu=0.2$, $\nu=8$ and 100. Figure 8 shows the variations of mode corresponding to the points on the frequency curves (improved theory) for the symmetric vibration in Fig. 2 and Fig. 9 does the ones for the antisymmetric vibration in Fig. 3, respectively. In case of $\nu=8$, such a tendency can be seen that the frequency curves approach the ones of $\nu=100$ as x_0 becomes large and at the same time, the mode shapes of $\nu=8$ become similar to the ones of $\nu=100$. The mode shapes fairly vary according to the frequencies and especially,

they do complicatedly in a state of transformation. The tendency is almost the same as in the classical theory⁽⁶⁾, which is omitted in the figures.

Figures 10 and 11 show the mode shapes in the case that both ends are simply supported. The mode shapes vary with the same tendency as in the case that both ends are built-in.

5. Conclusions

In this paper, the authors have shown a general method of solving in-plane vibrations of a uniform curved bar of which the center line is a plane curve considering the effects of the shear deformation and the rotatory inertia. As examples, they have made calculations for the symmetric elliptic arc bars with built-in ends and with simply supported ends and examined the influences of the shear deformation and the rotatory inertia upon the natural frequencies and the mode shapes. The results can be summarized as follows;

(1) The influences of the shear deformation and the rotatory inertia upon the natural frequencies can not be ignored, when the slenderness ratio is very small as $\nu=8$ or

Fig. 6 Symmetric vibration (both ends simply supported, $\mu=0.2$)

The mode shapes corresponding to the points on the frequency curves (improved theory) in Fig. 2 are shown. The modes of ①~⑥ are the ones for $\nu=8$ and those of ⑦~⑫ are for $\nu=100$.

Fig. 8 Mode (symmetric vibration, both ends built-in, $\mu=0.2$)

Fig. 7 Antisymmetric vibration (both ends simply supported, $\mu=0.2$)

The mode shapes corresponding to the points on the frequency curves (improved theory) in Fig. 3 are shown. The modes of ①~⑥ are the ones for $\nu=8$ and those of ⑦~⑩ are for $\nu=100$.

Fig. 9 Mode (antisymmetric vibration, both ends built-in, $\mu=0.2$)

when the arc length of the bar is small in comparison with the cross-sectional dimension. But the influences decrease as the bar becomes long or as the slenderness ratio becomes large. And it can be considered that there is no effect when the slenderness ratio $\nu=100$.

(2) Even in the present improved theory, a transfer phenomenon can be seen in the frequency curve when the slenderness ratio is small.

(3) The influences of the shear deformation and the rotatory inertia upon the mode shapes are small.

The numerical calculations in this paper were carried out by using NEAC-2200 Model 700 Computer of Tohoku University.

Finally, the authors would like to thank Assistant Professor Kosuke NAGAYA, Faculty of Engineering Yamagata University, for his helpful advice in the present work and also their thanks to Mrs. YAMAGUCHI, Technical Official in Faculty of Engineering, Yamagata University for her help in the numerical calculations.

The mode shapes corresponding to the points on the frequency curves (improved theory) in Fig.6 are shown. The modes of ①~⑥ are the ones for $\nu=25$ and those of ⑦~⑫ are for $\nu=100$.

Fig.10 Mode (symmetric vibration, both ends simply supported, $\mu=0.2$)

The mode shapes corresponding to the points on the frequency curves (improved theory) in Fig.7 are shown. The modes of ①~⑥ are the ones for $\nu=25$ and those of ⑦~⑩ are for $\nu=100$.

Fig.11 Mode (antisymmetric vibration, both ends simply supported, $\mu=0.2$)

Appendix

When the shear deformation is neglected in eq.(12) we have

$$f\dot{\phi} = \frac{du}{ds} + fw$$

And then, by using the relation

$$f \frac{d\phi}{ds} = \frac{d}{ds}(f\dot{\phi}) - \dot{\phi} \frac{df}{ds}$$

we find

$$\begin{aligned} E A k^2 \left(\frac{d\phi}{ds} \right)^2 &= E A \left(\frac{k}{f} \right)^2 \left(f \frac{d\phi}{ds} \right)^2 \\ &= \frac{E A k^2 R^2}{f^2} \left[\frac{d}{ds} \left\{ \frac{du}{ds} + \frac{lw}{R} \right\} \right]^2 \\ &\quad - R \frac{d}{ds} \left(\frac{1}{R} \right) \left\{ \frac{du}{ds} + \frac{lw}{R} \right\}^2 \end{aligned}$$

When the gap between the center line and the neutral axis is neglected we have

$$R = \rho, \quad E A k^2 R^2 \approx E I$$

Therefore, by using again lS , lU and lW by s_0 , u_0 and w_0 (arc length measured along center line, displacements towards the center of curvature and along tangent on the center line) and also by neglecting the rotatory inertia, equation(12) becomes

$$\begin{aligned} L &= \frac{EI}{2} \int_{s_0}^L \left[\left(\frac{\alpha}{l} \right)^2 (u_0'^2 + w_0'^2) - \left(\frac{\nu}{l} \right)^2 \right. \\ &\quad \times \left(\frac{dw_0}{ds_0} - \frac{u_0}{\rho} \right)^2 - \left. \left[\frac{d}{ds_0} \left(\frac{du_0}{ds_0} + \frac{w_0}{\rho} \right) \right. \right. \right. \\ &\quad \left. \left. \left. - \rho \frac{d}{ds_0} \left(\frac{1}{\rho} \right) \left(\frac{du_0}{ds_0} + \frac{w_0}{\rho} \right) \right]^2 \right] ds_0 \end{aligned}$$

When we compare the above Lagrangian with the one by the classical theory⁽⁵⁾, we find that it is difficult in that

$$- \rho \frac{d}{ds_0} \left(\frac{1}{\rho} \right) \left(\frac{du_0}{ds_0} + \frac{w_0}{\rho} \right)$$

is added to the third term on the right side of the above expression. This is due to the approximation, we surmise, assuming y was independent of s in obtaining eqs.(3)~(5) of this paper.

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