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A Study on Characteristics of Externally Pressurized Gas Thrust Bearing with Surface-restriction Compensation*

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The "surface-restriction" compensation in an externally pressurized gas bearing is a type of hydrostatic step compensation, and is composed of many narrow grooves on the bearing surface instead of shallow recess to form a restricting step.

In this paper, two kinds of flow models are introduced to analyze the hydrostatic performances of externally pressurized gas thrust bearing with this compensation yielding the theoretical pressure distribution, load capacity and bearing stiffness.

The experimental results agreed well with theoretical ones for pressure distribution and load capacity, while they showed a little larger values than theory for bearing stiffness.

A design criterion is also given for this type of the thrust bearing.

1. Introduction

In the design of an externally pressurized gas bearing, there must be arranged a hydraulic resistance, called restrictor, in the gas supply in order to produce the bearing stiffness. Orifice compensation or inherent compensation is used generally as the restrictor, and also step compensation is used in some special applications. This is composed of a comparatively large gas supply hole without special hydraulic resistance but with shallow recess. The compensation takes advantage of the fact that the change of flow resistance to the change of bearing clearance at the recess region differs from that at the outer land region.

The bearing with surface-restriction compensation analyzed in this paper is one of the step compensation bearings. An externally pressurized gas thrust bearing with this compensation is illustrated in Fig. 1. A number of shallow grooves are designed on the bearing surface radially from central supply port, along which the supplied gas flows out, and the outer end of the grooves acts as the restricting step.

The principle of generation of bearing stiffness is similar to that of the

common step compensation bearing. This can also be presumed from Fig. 6, which shows the change of pressure distributions to the change of bearing clearance.

The surface-restriction bearings are advantageous in that a fine machining of the supply holes, especially for a small bearing clearance case, is not necessary for them, and that they have larger angular stiffness, as compared with the familiar orifice or inherent compensation bearing.

In this paper, the hydrostatic characteristics, such as pressure distribution, load capacity and static stiffness, of gas thrust bearing with surface-restriction compensation will be investigated. Two kinds of flow models will be introduced; in model A, an equivalent recessed bearing is considered neglecting the circumferential flow in the bearing clearance, while the effect of this secondary flow is simply evaluated in model B.

2. Nomenclature

A_0 : throat area of inherent compensation effect
 C_D : discharge coefficient
 $C_1 - C_6, C_1' - C_6', \bar{C}_1 - \bar{C}_6$: constants
 G_g, G_l : constants

Fig. 1 Externally pressurized gas thrust bearing with surface-restriction compensation

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H : dimensionless clearance ($=h/h_l$)
 h : bearing clearance
 h_{eq} : equivalent clearance
 h_{g0} : depth of groove
 h_0 : reference clearance
 K : dimensionless bearing stiffness
 k : bearing stiffness
 m : mass flow rate
 P : dimensionless pressure ($=p/p_a$)
 p : pressure
 p_a : ambient pressure
 p_m : pressure just after supply port
 p_s : supply pressure
 R : dimensionless radial coordinate ($=r/r_0$)
 R : gas constant
 r : radial coordinate
 r_s : radius of supply port
 r_0 : bearing radius
 r_1 : radius to groove end
 T : temperature
 W : dimensionless load capacity
 w : load capacity
 Γ_0 : feeding parameter
 γ : angular fraction of groove
 δH_l : increment of dimensionless clearance
 θ : angular coordinate
 θ_0 : half of center angle of a pair of groove and land
 κ : adiabatic index
 λ_1, λ_2 : constants
 μ : viscosity
 ρ : density
 Suffix and superfix
 g : groove
 l : land
 I, II : regions I and II, respectively
 $-$: average pressure

3. Flow model A (equivalent clearance model)

The thrust bearing analyzed here is shown schematically in Fig. 2, in which some symbols used are also noted.

The supply port is r_s in radius. The shallow grooves with h_{g0} in depth are designed on the bearing surface to radial position r_1 (region I), and no grooves are in the outer region ($r_1 - r_0$) (region II). The grooves have a constant angular width,

Fig. 2 Schematic diagram of the bearing

$2\gamma\theta_0$, where γ is fraction of groove part in one angular section of a pair of groove and land. The manufacture of the grooves may be rather easy if the groove width is constant, but, in this paper, the above mentioned type of groove configuration is considered because the analytical formulation becomes simplified. The latter type can be analyzed similarly considering the change of equivalent clearance in radial direction.

Now, we consider half of an angular section, θ_0 . When the groove is sufficiently narrow, we can assume that the pressure in groove, p_g , is identical to that in land, p_l , in region I, that is,

$$p_g(r) = p_l(r) = p(r). \quad \dots\dots\dots(1)$$

Then, the radial mass flow rate is given by

$$m = m_g + m_l = -\frac{h_g^3}{12\mu} \rho \frac{dp}{dr} r \gamma \theta_0 - \frac{h_l^3}{12\mu} \rho \frac{dp}{dr} r (1-\gamma) \theta_0. \quad \dots\dots\dots(2)$$

We represent the flow in this bearing by one in an assumed bearing with recess, which gives the same pressure distribution, $p(r)$, and the same flow rate, m , as the former one. We denote the clearance at the recess of this assumed bearing by h_{eq} (we call it equivalent clearance), then we have

$$m = -\frac{h_{eq}^3}{12\mu} \rho \frac{dp}{dr} r \theta_0. \quad \dots\dots\dots(3)$$

From Eqs. (2) and (3),

$$h_{eq} = \{\gamma h_g^3 + (1-\gamma) h_l^3\}^{1/3}. \quad (\text{in region I}) \quad \dots\dots\dots(4)$$

In region II, the clearance is uniformly h_l , then,

$$h_{eq} = h_l. \quad (\text{in region II}) \quad \dots\dots\dots(5)$$

The Reynolds equations are derived from continuity condition with the assumption of isothermal flow as

$$\left. \begin{aligned} \text{Region I} : \frac{d}{dR} (H_{eq}^3 R \frac{dP_I^2}{dR}) &= 0 \\ \text{Region II} : \frac{d}{dR} (H_l^3 R \frac{dP_{II}^2}{dR}) &= 0 \end{aligned} \right\} \quad \dots\dots\dots(6)$$

in which the variables are normalized by

$$\left. \begin{aligned} P &= p/p_a \quad (p_a : \text{ambient pressure}), \\ R &= r/r_0, \quad H = h/h_l. \end{aligned} \right\} \quad \dots\dots\dots(7)$$

We now consider the boundary conditions.

At the supply port ($R = R_s$), the flow is restricted due to the inherent compensation effect as done in a familiar hydrostatic gas bearing. The pressure reduces to p_m from supply pressure p_s due to this effect, which is evaluated by the formula for the flow through an orifice with throat area A_0 . Then we have

$$\frac{C D A_0}{\sqrt{RT}} P_s \phi = -\frac{2\pi r_s h_{eq}^3}{12\mu RT} \frac{dp^2}{dr} \Big|_{r=r_s}$$

or in normalized expression,

$$-R_s \frac{dP^2}{dR} \Big|_{R=R_s} = \frac{H_{eq}^3}{H_l^3} \Gamma_0 P_s \phi \quad \dots\dots\dots(8)$$

where

$$A_0 = 2\pi r_s \{ \gamma h_g + (1-\gamma) h_l \} \\ \equiv 2\pi r_s h_l H_{en} \dots\dots\dots (9) \\ \Gamma_0 = \frac{12\mu C_D \cdot 2r_s \sqrt{RT}}{p_a h_l^2} \quad (\text{feeding parameter}) \dots\dots\dots (10)$$

$$\left. \begin{aligned} \phi &= \left(\frac{2\kappa}{\kappa-1} \right)^{1/2} \left(\frac{p_m}{p_s} \right)^{1/\kappa} \\ &\quad \times \left(1 - \left(\frac{p_m}{p_s} \right)^{(\kappa-1)/\kappa} \right)^{1/2}; \\ &\quad \frac{p_m}{p_s} \geq \left(\frac{2}{\kappa+1} \right)^{\kappa/(\kappa-1)} \\ \phi &= \left(\frac{2\kappa}{\kappa+1} \right)^{1/2} \left(\frac{2}{\kappa+1} \right)^{1/(\kappa-1)}; \\ &\quad \frac{p_m}{p_s} < \left(\frac{2}{\kappa+1} \right)^{\kappa/(\kappa-1)} \end{aligned} \right\} \dots\dots (11)$$

At the outer end of groove ($R=R_1$), the continuity conditions of pressure and flow rate give

$$P_I = P_{II} \dots\dots\dots (12)$$

$$H_{eq}^3 \cdot dP_I^2/dR = dP_{II}^2/dR. \dots\dots\dots (13)$$

At the outer periphery of bearing ($R=1$),

$$P_{II} = 1. \dots\dots\dots (14)$$

We can obtain the pressure distribution by solving Eq. (6) under the boundary condition (8) and (12) ~ (14).

4. Flow model B

In the above equivalent clearance model, the circumferential flow in the bearing clearance is neglected, and the

Fig. 3 Flow model B

pressure drop at the supply port is evaluated by averaging for the groove and the land part. In order to improve the flow model, the circumferential flow effect is taken into consideration in a simple manner.

Referring to Fig. 3, we consider the mean pressures $\bar{p}_g(r)$ and $\bar{p}_l(r)$ in groove and land part, respectively. Then, the mass flow rate in each part is given by

$$\left. \begin{aligned} \text{Region I:} \\ m_g &= -\frac{\bar{p}_g h_g^3}{12\mu} \frac{d\bar{p}_g}{dr} r \gamma \theta_0 \\ m_l &= -\frac{\bar{p}_l h_l^3}{12\mu} \frac{d\bar{p}_l}{dr} r (1-\gamma) \theta_0 \\ \text{Region II:} \\ m_g &= -\frac{\bar{p}_g h_l^3}{12\mu} \frac{d\bar{p}_g}{dr} r \gamma \theta_0 \\ m_l &= -\frac{\bar{p}_l h_l^3}{12\mu} \frac{d\bar{p}_l}{dr} r (1-\gamma) \theta_0 \end{aligned} \right\} \dots\dots (15)$$

In region II, the clearance is uniform, but we extend the groove and the land part of region I to region II.

In general, the pressure $\bar{p}_g$ is not equal to $\bar{p}_l$, hence the circumferential flow, m_θ , is induced by a difference between these pressures. In order to evaluate the flow m_θ , we assume the pressure distribution such that p^2 varies quadratically in circumferential direction as shown in Fig. 3 (b). One reason for it is that the expressions become simple, and the other is that the circumferential flow rate may possibly increase or decrease linearly in θ -direction.

Under this assumption, the relation of flow rate across the boundary of groove and land leads to the following equation,

$$\begin{aligned} m_\theta &= -\frac{\rho_g h_g^3}{12\mu} \frac{dp_g}{r d\theta} \bigg|_{\theta=\theta_g} = -\frac{RT h_g^3}{24\mu} \frac{2(\bar{p}_g^2 - p_e^2)}{r \gamma \theta_0} \\ &= \frac{RT h_g^3}{24\mu} \frac{3(\bar{p}_g^2 - p_e^2)}{r \gamma \theta_0} \\ &= -\frac{\rho_l h_l^3}{12\mu} \frac{dp_l}{r d\theta} \bigg|_{\theta=\theta_g} = -\frac{RT h_l^3}{24\mu} \frac{2(p_e^2 - \bar{p}_l^2)}{r (1-\gamma) \theta_0} \\ &= \frac{RT h_l^3}{24\mu} \frac{3(p_e^2 - \bar{p}_l^2)}{r (1-\gamma) \theta_0} \dots\dots\dots (16) \end{aligned}$$

where $\bar{p}_g$ and $\bar{p}_l$ are the pressures on the center line of groove and land, respectively, and p_e is the pressure on their boundary.

The continuity relations of flow between r and $r+dr$ are given by

$$\begin{aligned} m_g &= m_\theta dr + \left(m_g + \frac{dm_g}{dr} dr \right) \\ m_l &= -m_\theta dr + \left(m_l + \frac{dm_l}{dr} dr \right). \end{aligned}$$

Substituting Eqs. (15) and (16) into the above equations, the following Reynolds equations can be derived,

$$\left. \begin{aligned} R \frac{d}{dR} \left(R \frac{d\bar{p}_g}{dR} \right) &= G g j (\bar{p}_g^2 - \bar{p}_l^2) \\ R \frac{d}{dR} \left(R \frac{d\bar{p}_l}{dR} \right) &= -G l j (\bar{p}_g^2 - \bar{p}_l^2) \end{aligned} \right\} \dots\dots (17)$$

$$j = I, II$$

where

$$\left. \begin{aligned} G_{gI} &= \frac{3h_l^3}{\{(1-\gamma)h_g^3 + \gamma h_l^3\}\gamma\theta_0^2} \\ G_{lI} &= \frac{3h_g^3}{\{(1-\gamma)h_g^3 + \gamma h_l^3\}(1-\gamma)\theta_0^2} \\ G_{gII} &= \frac{3}{\gamma\theta_0^2}, \quad G_{lII} = \frac{3}{(1-\gamma)\theta_0^2} \end{aligned} \right\} \cdot (18)$$

The boundary conditions for Eq. (17) are given by similar expressions to Eqs. (8) and (12) ~ (14) respectively for groove and land part.

The dimensionless load capacity is obtained from the solved pressure distribution as

$$\begin{aligned} W &= w/\pi r_0^2 p_a \\ &= 2\gamma \left(\int_{R_8}^{R_1} \bar{P}_{gI} R dR + \int_{R_1}^1 \bar{P}_{gII} R dR \right) \\ &\quad + 2(1-\gamma) \left(\int_{R_8}^{R_1} \bar{P}_{lI} R dR + \int_{R_1}^1 \bar{P}_{lII} R dR \right). \end{aligned} \quad \dots\dots (19)$$

We consider a perturbation of bearing clearance from the equilibrium state and denote the increment as δH_l , that is, $H_l = 1 + \delta H_l$, then the increments of the variables can be written as

$$\left. \begin{aligned} \bar{P}_{ij} &= \bar{P}_{ij0} + \delta H_l \cdot \bar{P}_{ij1} \\ G_{ij} &= G_{ij0} + \delta H_l \cdot G_{ij1} \end{aligned} \right\} \dots\dots (20)$$

$$(i = g, l, \quad j = I, II).$$

By substituting them into Eqs. (17) and (18) and rearranging, we can obtain the fundamental equations to be solved. The solution of $\bar{P}_{ij0}$ corresponding to the 0-th order of δH_l yields the load capacity mentioned above, and $\bar{P}_{ij1}$ corresponding to the first order of δH_l gives the bearing stiffness. The stiffness is given by the following equation in dimensionless form,

$$\begin{aligned} K &= \frac{kh_l}{\pi r_0^2 p_a} \cdot \frac{h_0}{h_l} \\ &= - \left\{ 2\gamma \left(\int_{R_8}^{R_1} \bar{P}_{gI1} R dR + \int_{R_1}^1 \bar{P}_{gII1} R dR \right) \right. \\ &\quad \left. + 2(1-\gamma) \left(\int_{R_8}^{R_1} \bar{P}_{lI1} R dR + \int_{R_1}^1 \bar{P}_{lII1} R dR \right) \right\} \frac{h_0}{h_l} \end{aligned} \quad \dots\dots (21)$$

It should be noted that this is slightly modified from the usual definition of dimensionless stiffness by multiplying with h_0/h_l . This is because this K corresponds to dimensional stiffness proportionally when the clearance h_l varies so that we may see the bearing performance more directly. h_0 is the reference clearance and it is set at $h_0 = 10$ microns in the later calculation.

The fundamental equation (17) is linear so that it can be solved analytically, the solutions of which are given in Appendix.

5. Experimental and theoretical results

The experimental apparatus for measurement of pressure distribution is shown schematically in Fig. 4. The surface plate and the test bearing compose the bearing clearance, into which the compressed gas is supplied from the central supply port. The bearing clearance is set up by three micrometers and an electric capacitance micrometer. The pressure measuring hole is bored in the surface plate, which can be moved in a plane by fine adjustments, while the test bearing can rotate to adjust the angular position. The diameter of pressure measuring hole is 0.15 mm, through which the pressure is introduced to a fine pressure gage with the minimum division of 0.01 kg/cm².

The test apparatus for measurement of load capacity and bearing stiffness is

Fig. 4 Experimental apparatus for measurement of pressure distribution

Fig. 5 Experimental apparatus for measurement of bearing stiffness

shown schematically in Fig. 5. In this apparatus, the bearing is fixed on the upper frame. The rotor is supported by this test thrust bearing, externally pressurized gas journal bearing and a back pressure of lower volume. The bearing clearance is set by adjusting the back pressure, and the same time, the bearing load is obtained from a balance of the applied forces. The bearing clearance is measured by a test indicator with the minimum division of 1 micron, which is moved away during the experiments for bearing stiffness.

The bearing stiffness is obtained by the following method. An impulse load is given to the shaft, while the response to it is measured by a noncontact type transducer, and analyzed by a fast Fourier analyzer yielding the amplitude and frequency of shaft vibration. The mean logarithmic damping ratio δ_0 is obtained from three data of amplitude ratios after the transient state, from which the dimensionless bearing stiffness is calculated by the following equation using frequency f from Fourier analyzer,

$$K = 4\pi^2 f^2 m_0 \left\{ 1 + \left(\frac{\delta_0}{2\pi} \right)^2 \right\} \frac{h_l}{\pi r_0^2 p_a} \frac{h_0}{h_l} \quad (22)$$

where m_0 is mass of rotor and the reference clearance h_0 is taken to be 10 microns mentioned before.

The design parameters of the test bearings are changed variously in the experiments, but in the following only a typical case will be dealt with; that is, $r_0 = 20$ mm, $R_1 = 0.8$, $R_s = 0.2$, $\gamma = 0.2$, $\theta_0 = 12^\circ$ (number of grooves = 15), $p_s - p_a = p_{sg} = 1.0$ kg/cm², $h_{g0} = 11$ and 55 microns. The grooves are produced by photo-etching method.

Examples of radial pressure distributions are shown in Fig. 6. Near the inlet port and the end of groove, the pressure differences can be seen between in groove and in land part, but there is practically no difference in the middle of regions I and II. Speaking precisely, the pressure is higher in the land part than in the groove near the inlet port, which means that the gas flows from land to groove, and the reverse is the case near the outer end of groove. The results with flow model B (solid curves) very well prove this fact. The flow model A (broken curves) coincides well with model B, though it does not distinguish the flows in groove and in land. Both results of theoretical models agree well with experimental ones.

An experimental example of circumferential pressure distribution at the end of groove is shown in Fig. 7 together with theoretical results with model B. Considering that the gas flow may be disturbed by the step effect near the end of groove, it may be concluded that we can assume the circumferential pressure distribution in a simple quadratic form as done with model B.

Results of load capacity and bearing

stiffness are shown in Fig. 8. As to the load capacity, theoretical results with both models A and B agree well with each other though a slight difference can be seen at a small bearing clearance, and both of them coincide well with experimental ones.

As to the bearing stiffness, the bearing is considered fundamentally to have a characteristic similar to a bearing with restrictions in series with a shallow recess. When the grooves are comparatively deep, there occurs the maximum bearing stiffness at a specific value of clearance just as in the familiar hole-admission gas bearing. When the grooves are shallow, the bearing has a notable feature of characteristic curve, that is, there exists the maximum stiffness at a small clearance of about 5 microns in the figure, and a saddle point of stiffness curve can be

Fig. 6 Radial pressure distributions

seen at about 40~60 microns. This seems to be explained by the superposed characteristics of surface-restriction compensation and inherent compensation effect; the former is predominant at a small clearance, and the latter occurs at a comparatively large clearance. For the case of $h_{g0} = 11$ microns in Fig. 8, the feeding parameter Γ_0 is calculated to be 2.04 for $h = 50$ microns. This value is near the optimum value of maximum stiffness⁽¹⁾ for the familiar inherent compensation gas bearing.

The experimental results of bearing stiffness agree qualitatively well with the theoretical ones, but quantitatively exceed the latter. Considering that the static test results of load capacity coin-

Fig. 7 Circumferential pressure distributions

Fig. 8 Load capacity and bearing stiffness

cide well with theoretical ones, the discrepancy may be attributed to the dynamic measurement of bearing stiffness. We note here that, for $h = 30$ microns, the squeeze number defined by $\lambda = (12\mu\omega/p_a) \cdot (r_o/h)^2$ (where ω = angular frequency) is calculated to be 0.60 for $h_{g0} = 11$ microns, while $\lambda = 0.98$ for $h_{g0} = 55$ microns. This agrees approximately with the results of Sato et al.⁽²⁾, who concluded theoretically that the bearing stiffness would come to be affected by squeeze number when λ increases up to 0.5~1.0 for the bearing with a shallow recess, though the configuration of the bearing is a little different from the one discussed in this paper.

It is noted that there are few experimental data for a very small clearance, which is due to the fact that the indicial response was overdamped there so that the data could not be obtained by the method employed in this study.

Figure 9 illustrates the theoretical results of bearing stiffness with flow model B. In the figure, the first peak of curves in the region of comparatively small clearances corresponds to the surface-restriction compensation. Considering the bearing characteristics in this region, it can be concluded that we can design a bearing with very high stiffness when we arrange the thin and shallow grooves extending close to outer periphery of the bearing and choose a bearing clearance just matching the dimensions of grooves.

Fig. 9 Theoretical bearing stiffness

6. Concluding remarks

In this paper, an externally pressurized gas thrust bearing with surface-restriction compensation was analyzed. Two kinds of flow models were introduced for the analysis, yielding pressure distribution, load capacity and static stiffness. The experimental results agreed well with theoretical ones for pressure distribution and load capacity, and qualitatively well for bearing stiffness though they showed a little quantitative discrepancy due to the dynamic effect encountered in experimental measurements. A recommendable design of this type of bearing was derived from the theoretical results.

It would be necessary for establishment of design method of the bearings to study the bearing performances such as the dynamic stiffness, the damping characteristics, the angular stiffness and the application to journal bearings.

Appendix

Solutions for 0-th order of δH_I

Considering the boundary conditions of Eq. (14), we obtain

$$\bar{P}_{gI0}^2 = \frac{G_{gI}}{\lambda_1^2} (C_1 e^{\lambda_1 \log R} + C_2 e^{-\lambda_1 \log R}) + C_3 \log R + C_4$$

$$\bar{P}_{lI0}^2 = -\frac{G_{lI}}{\lambda_1^2} (C_1 e^{\lambda_1 \log R} + C_2 e^{-\lambda_1 \log R}) + C_3 \log R + C_4$$

$$\bar{P}_{gII0}^2 = \frac{C_5 G_{gII}}{\lambda_2^2} (e^{\lambda_2 \log R} - e^{-\lambda_2 \log R}) + C_6 \log R + 1$$

$$\bar{P}_{lII0}^2 = -\frac{C_5 G_{lII}}{\lambda_2^2} (e^{\lambda_2 \log R} - e^{-\lambda_2 \log R}) + C_6 \log R + 1$$

$$\lambda_1^2 = G_{gI} + G_{lI}, \quad \lambda_2^2 = G_{gII} + G_{lII}$$

where $C_1 \sim C_6$ are integration constants determined by the boundary conditions. We cannot express these constants in a closed form because of non-linearity of Eq. (8). Solutions for first order of δH_I

Solutions can be given by the following equations considering the condition (14),

$$\begin{aligned} (\bar{P}_{g0} \bar{P}_{g1})_I &= \frac{G_{gI}}{\lambda_1^2} (C_1' e^{\lambda_1 \log R} \\ &+ C_2' e^{-\lambda_1 \log R}) + C_3' \log R + C_4' \\ &+ (\bar{C}_1 e^{\lambda_1 \log R} + \bar{C}_2 e^{-\lambda_1 \log R}) \log R \\ &+ \bar{C}_3 e^{\lambda_1 \log R} + \bar{C}_4 e^{-\lambda_1 \log R} \end{aligned}$$

$$\begin{aligned} (\bar{P}_{l0} \bar{P}_{l1})_I &= -\frac{G_{lI}}{\lambda_1^2} (C_1' e^{\lambda_1 \log R} \\ &+ C_2' e^{-\lambda_1 \log R}) + C_3' \log R + C_4' \\ &+ (\bar{C}_5 e^{\lambda_1 \log R} + \bar{C}_6 e^{-\lambda_1 \log R}) \log R \\ &+ \bar{C}_3 e^{\lambda_1 \log R} + \bar{C}_4 e^{-\lambda_1 \log R} \end{aligned}$$

$$(\bar{P}_{g0} \bar{P}_{g1})_{II} = \frac{C_5' G_{gII}}{\lambda_2^2} (e^{\lambda_2 \log R} - e^{-\lambda_2 \log R}) + C_6' \log R$$

$$(\bar{P}_{l0} \bar{P}_{l1})_{II} = -\frac{C_5' G_{lII}}{\lambda_2^2} (e^{\lambda_2 \log R} - e^{-\lambda_2 \log R}) + C_6' \log R$$

where $C_1' \sim C_6'$ are integration constants determined by the boundary conditions. $\bar{C}_1 \sim \bar{C}_6$ are constants involving C_1 and C_2 , and given by the solutions of the following simultaneous equations,

$$\begin{bmatrix} G_{lI} & G_{gI} & 0 \\ 2\lambda_1 & 0 & \lambda_1^2 \\ 0 & 2\lambda_1 & \lambda_1^2 \end{bmatrix} \times \begin{bmatrix} \bar{C}_1 & -\bar{C}_2 \\ \bar{C}_5 & -\bar{C}_6 \\ \bar{C}_3 & \bar{C}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ G_{gI} C_1/2 & G_{gI} C_2/2 \\ -G_{lI} C_1/2 & -G_{lI} C_2/2 \end{bmatrix}$$

where

$$\begin{aligned} G_{gI} &= \frac{9(1-\gamma)}{\gamma \theta_0^2} \cdot \frac{H_{g0}^3 - H_{g0}^2}{\{(1-\gamma)H_{g0}^3 + \gamma\}^2} \\ G_{lI} &= -\frac{9\gamma}{\gamma \theta_0^2} \cdot \frac{H_{g0}^3 - H_{g0}^2}{\{(1-\gamma)H_{g0}^3 + \gamma\}^2} \end{aligned}$$

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