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Stresses in a Circular Cylinder Having a Spherical Inclusion Under Transverse Bending*

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This paper gives a three-dimensional solution for the stresses and displacements around an elastic spherical inclusion perfectly bonded to an infinite circular cylinder, when the cylinder is under transverse bending. The analysis is based upon Dougall's stress function approach referred to the cylindrical and spherical harmonics. The boundary conditions on the surface of the cylinder and the contact surface are well satisfied by using the relations between the cylindrical and spherical functions. Numerical results are given for various ratios of the elastic moduli of the inclusion and the cylinder and four different radii of the inclusion.

1. Introduction

When foreign inclusions exist in an elastic matrix, their presence leads to an intensification of stress in their vicinity, which plays a dominant role in controlling the mechanical behaviour of the whole material. The degree of this stress concentration depends on all sorts of influences, such as their size, the disparity in the elastic properties, the shapes or forms, of the inclusions and the matrix, their existing locations and the sort of applied loads and others. And the evaluation of the effect of inclusions on the strength of materials is important in engineering technology, especially in connection with fatigue and brittle fracture.

To clarify the effect of these factors, the authors have studied the problem of a circular cylinder with a spherical inclusion under loading conditions corresponding to pure torsion⁽¹⁾ and uniform axial tension.⁽²⁾ This paper considers an asymmetric problem of an infinite circular cylinder having a symmetrically located spherical inclusion under transverse bending. In general, in comparison with the axisymmetric cylinder problem of torsion and tension, the asymmetric problem is difficult in finding the suitable stress functions, so there are few results.^{(3)~(7)}

In the present analysis, the solution is obtained by the use of Dougall's three harmonic stress functions $\varphi_0, \varphi_1, \lambda_3$ ⁽⁸⁾ and is expressed by the cylindrical and spherical harmonics.

The boundary conditions both on the surface of the cylinder and at the interface between the inclusion and surrounding

matrix are well satisfied with the aid of relations between the cylindrical and spherical harmonic functions.

The method of solution is first described. Then, as an illustration, numerical examples are shown graphically for the stresses in the neighborhood of the inclusion.

2. Method of solution

Consider an infinite elastic circular cylinder with a symmetrically located spherical inclusion, which is subjected to a constant bending moment M as shown in Fig.1.

Denote as usual the cylindrical and spherical coordinates of a point by (r, θ, z) and (R, θ, φ) respectively. They are connected to each other by

$$r = R \sin \varphi, \quad z = R \cos \varphi.$$

For convenience, r, z and R are regarded as dimensionless quantities each measured by a unit of the radius of the cylinder.

The axis of the cylinder will be taken as the axis of z and the center of the inclusion of radius a as the origin.

Following Dougall, the general solution for the displacement equations of equilibrium in the absence of body forces, is given by the sum of the three displacement fields:

Fig.1 Coordinate system.

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$$\left. \begin{aligned} 2Gu_R &= \frac{\partial \psi_0}{\partial R} - \sin \varphi \frac{\partial \psi_4}{\partial \varphi} \\ &\quad - 4(1-\nu)\varphi_4 \cos \varphi + \frac{2}{R} \frac{\partial \lambda_3}{\partial \theta} \\ 2Gv_\theta &= \frac{1}{R \sin \varphi} \frac{\partial \psi_0}{\partial \theta} \\ &\quad - 2 \left[\sin \varphi \frac{\partial \lambda_3}{\partial R} + \frac{\cos \varphi}{R} \frac{\partial \lambda_3}{\partial \varphi} \right] \\ \text{and } 2Gw_\varphi &= \frac{1}{R} \frac{\partial \psi_0}{\partial \varphi} + R \sin \varphi \frac{\partial \psi_4}{\partial R} \\ &\quad + 4(1-\nu)\varphi_4 \sin \varphi + \frac{2}{R} \cot \varphi \frac{\partial \lambda_3}{\partial \theta} \end{aligned} \right\} \dots (1)$$

where

$$\begin{aligned} \nabla^2 \psi_0 &= \nabla^2 \psi_4 = \nabla^2 \lambda_3 = 0 \\ \nabla^2 &= \frac{\partial^2}{\partial R^2} + \frac{2}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2} \\ &\quad + \frac{\cot \varphi}{R^2} \frac{\partial}{\partial \varphi} + \frac{1}{R^2 \sin^2 \varphi} \frac{\partial^2}{\partial \theta^2} \end{aligned}$$

ν is Poisson's ratio and G is the shear modulus of the elastic medium.

When a circular cylinder is subjected to a constant bending moment M around the y -axis, and a perfect bond is assumed between the cylinder and the inclusion, the boundary conditions for the problem may be written as:

- (i) On the lateral surface of the cylinder $r=1$

$$(\sigma_r)_{r=1} = (\tau_{rz})_{r=1} = (\tau_{r\theta})_{r=1} = 0 \dots (2)$$

- (ii) On the interface between the inclusion and the surrounding matrix $R=a$

$$\left. \begin{aligned} (u_R)_{R=a} &= (\bar{u}_R)_{R=a}, & (v_\theta)_{R=a} &= (\bar{v}_\theta)_{R=a} \\ (w_\varphi)_{R=a} &= (\bar{w}_\varphi)_{R=a}, & (\sigma_R)_{R=a} &= (\bar{\sigma}_R)_{R=a} \\ (\tau_{R\varphi})_{R=a} &= (\bar{\tau}_{R\varphi})_{R=a}, & (\tau_{R\theta})_{R=a} &= (\bar{\tau}_{R\theta})_{R=a} \end{aligned} \right\} \dots (3)$$

- (iii) At infinity of $|z| \rightarrow \infty$
 $\sigma_z = k r \cos \theta$, other stress components vanish. (4)

where $k=4/\pi M$

The letters overscored "—" designate quantities relating to the inclusion.

And k designates a constant depending on the moment M and the cross-section of the cylinder.

In the absence of the inclusion, a uniform prescribed stress distribution $\sigma_z = kx$ occurring in the cylinder is given by a basic stress function [I]. (7)

$$[I] \left\{ \begin{aligned} \psi_0 &= -\frac{3-4\nu}{24(1+\nu)} k R^3 P_3'(\mu) \cos \theta \\ \psi_4 &= -\frac{1}{6(1+\nu)} k R^3 P_2'(\mu) \cos \theta \\ \lambda_3 &= \frac{1-4\nu}{48(1+\nu)} k R^3 P_3'(\mu) \sin \theta \end{aligned} \right\} \dots (5)$$

where $p_n^m(\mu)$ represent associated Legendre functions of the first kind of order n and degree m , and $\mu = \cos \varphi$.

Equation (5) gives no traction on the lateral surface of the cylinder but yields the following displacements and stresses at the spherical surface of radius R :

$$\left. \begin{aligned} \frac{2Gu_R}{k \cos \theta} &= \frac{R^2}{30(1+\nu)} \{3(1-4\nu)P_1'(\mu) + 2(1+\nu)P_3'(\mu)\} \\ \frac{2Gv_\theta}{k \sin \theta} &= \frac{R^2}{30(1+\nu) \sin \varphi} \{3(1-4\nu)P_1'(\mu) + 2(1+\nu)P_3'(\mu)\} \\ \frac{2Gw_\varphi}{k \cos \theta} &= \frac{R^2}{30(1+\nu)} \{3(1-4\nu)(P_1''(\mu) + 2(1+\nu)P_3''(\mu) - 20(1+\nu)P_3'(\mu))\} \\ \frac{\sigma_R}{k \cos \theta} &= \frac{R}{15} \{3P_1'(\mu) + 2P_3'(\mu)\}, & \frac{\sigma_\varphi}{k \cos \theta} &= \frac{2R}{15} \{6P_1'(\mu) - P_3'(\mu)\} \\ \frac{\tau_{R\varphi}}{k \cos \theta} &= -\frac{R}{3} P_2'(\mu) \sin \varphi, & \sigma_\theta = \tau_{R\theta} = \tau_{\theta\varphi} &= 0 \end{aligned} \right\} \dots (6)$$

Thus, the method of satisfying the boundary conditions when an inclusion is present within a cylinder is to construct auxiliary stress functions together with the stress function [I] by using suitable harmonic functions.

We now choose the required auxiliary stress functions in the following forms:

In the matrix ($R > a$)

$$\begin{aligned} [II] \left\{ \begin{aligned} \varphi_0 &= k \sum_{m=0}^{\infty} A_m \frac{P_{2m+1}'(\mu)}{R^{2m+2}} \cos \theta \\ \varphi_4 &= k \sum_{m=0}^{\infty} B_m \frac{P_{2m+2}'(\mu)}{R^{2m+3}} \cos \theta \\ \lambda_3 &= k \sum_{m=0}^{\infty} C_m \frac{P_{2m+1}'(\mu)}{R^{2m+2}} \sin \theta \end{aligned} \right\} \dots (7) \\ [III] \left\{ \begin{aligned} \varphi_0 &= k \int_0^\infty \psi_1(\lambda) I_1(\lambda r) \cos \lambda z \cos \theta d\lambda \\ \varphi_4 &= k \int_0^\infty \lambda \psi_2(\lambda) I_1(\lambda r) \sin \lambda z \cos \theta d\lambda \\ \lambda_3 &= k \int_0^\infty \psi_3(\lambda) I_1(\lambda r) \cos \lambda z \sin \theta d\lambda \end{aligned} \right\} \dots (8) \end{aligned}$$

In the inclusion ($R < a$)

$$[IV] \left\{ \begin{aligned} \varphi_0 &= k \sum_{n=0}^{\infty} \bar{A}_n R^{2n+1} P_{2n+1}'(\mu) \cos \theta \\ \varphi_4 &= k \sum_{n=0}^{\infty} \bar{B}_n R^{2n+2} P_{2n+2}'(\mu) \cos \theta \\ \lambda_3 &= k \sum_{n=0}^{\infty} \bar{C}_n R^{2n+3} P_{2n+3}'(\mu) \sin \theta \end{aligned} \right\} \dots (9)$$

where A_m , B_m , C_m and $\bar{A}_n$, $\bar{B}_n$, $\bar{C}_n$ are constants and $\psi_1(\lambda)$, $\psi_2(\lambda)$ and $\psi_3(\lambda)$ are arbitrary functions to be determined from the boundary conditions $I_n(\lambda r)$ are modified Bessel functions of the first kind of order n .

In a particular case, when the inclusion is much smaller than the cylinder, that is to say, when the inclusion is isolated in an infinite solid, these coefficients can be determined as follows:

$$\left. \begin{aligned} \frac{A_0}{a^3} &= \frac{7(\Gamma-1)\{4(2+7\nu) + (27-28\nu)\Gamma\}}{30(4+\Gamma)M_1} \\ &\quad + \frac{(1-\nu)(1+\nu)\Gamma}{6(1+\nu)M_2} + \frac{2-7\Gamma}{30(4+\Gamma)} \\ \frac{A_1}{a^2} &= -\frac{\Gamma-1}{3M_1}, & \frac{B_0}{a^3} &= \frac{7}{6} \frac{(\Gamma-1)}{M_1} \\ \frac{C_0}{a^3} &= \frac{(\Gamma-1)\{4(2+7\nu) + (27-28\nu)\Gamma\}}{12(4+\Gamma)M_1} \end{aligned} \right\}$$

$$\begin{aligned}
\frac{\bar{A}_0}{a^2} &= \frac{\Gamma}{6(1+\nu)} - \frac{5(1-\nu)\Gamma}{6(1+\nu)M_2} \\
\bar{A}_1 &= \frac{(59-49\nu+2(23-28\nu)\Gamma)\Gamma}{12(4+\Gamma)M_1} \\
&\quad - \frac{7(1-\nu)(3-2\nu)\Gamma}{24(1+\nu)M_2} \\
\bar{B}_0 &= -\frac{5(1-\nu)\Gamma}{6(1+\nu)M_2} \\
\bar{C}_0 &= \frac{5(1-\nu)(3-2\nu)\Gamma}{48(1+\nu)M_2} - \frac{5\Gamma}{24(4+\Gamma)} \\
A_m &= \bar{A}_m = 0 \quad (m \geq 2)
\end{aligned}
\left. \vphantom{\begin{aligned} \bar{A}_0 \\ \bar{A}_1 \\ \bar{B}_0 \\ \bar{C}_0 \\ A_m \end{aligned}} \right\} \begin{aligned} B_m &= C_m = \bar{B}_m = \bar{C}_m = 0 \quad (m \geq 1) \\ M_1 &= 13-7\nu+2(11-14\nu)\Gamma \\ M_2 &= 2(2-3\nu)+(1+\nu)\Gamma \end{aligned} \quad (10)$$

The ratio $\Gamma = \bar{G}/G$ describes the relative rigidity of inclusion to matrix, it being zero when the inclusion is a spherical cavity and infinitely large for a perfectly rigid inclusion.

In order to adjust the boundary conditions on the lateral surface of a cylinder, it is convenient to transform the stress function [II] into cylindrical coordinates, with the aid of the relations:⁽⁹⁾

$$\begin{aligned}
\frac{P_{n+2s+1}(\mu)}{R^{n+2s+1}} &= \frac{2}{\pi} \frac{(-1)^s}{(2s)!} \int_0^\infty \lambda^{n+2s} K_n(\lambda r) \cos \lambda z d\lambda \\
\frac{P_{n+2s+1}(\mu)}{R^{n+2s+2}} &= \frac{2}{\pi} \frac{(-1)^s}{(2s+1)!} \int_0^\infty \lambda^{n+2s+1} K_n(\lambda r) \sin \lambda z d\lambda
\end{aligned} \quad (11)$$

The stress function [II] is rewritten in the form:

$$[V] \left\{ \begin{aligned} \varphi_0 &= k \sum_{n=0}^{\infty} a_n \int_0^\infty \lambda^{2n+1} K_1(\lambda r) \cos \lambda z \cos \theta d\lambda \\ \varphi_1 &= k \sum_{n=0}^{\infty} b_n \int_0^\infty \lambda^{2n+2} K_1(\lambda r) \sin \lambda z \cos \theta d\lambda \\ \lambda_3 &= k \sum_{n=0}^{\infty} c_n \int_0^\infty \lambda^{2n+1} K_1(\lambda r) \cos \lambda z \sin \theta d\lambda \end{aligned} \right\} \quad (12)$$

$$\text{where} \quad a_n = \frac{2}{\pi} \frac{(-1)^n}{(2n)!} A_n, \quad b_n = \frac{2}{\pi} \frac{(-1)^n}{(2n+1)!} B_n, \quad c_n = \frac{2}{\pi} \frac{(-1)^n}{(2n)!} C_n$$

and $K_m(\lambda r)$ are modified Bessel functions of the second kind of order m .

Using Eq. (2) and appropriate stress-displacement relations for the stress functions [I], [III] and [V], the following expressions are given under the condition of zero traction on the surface of the cylinder:

$$\begin{aligned}
\left(\frac{\sigma_r}{k \cos \theta} \right)_{r=1} &= \int_0^\infty \lambda^2 \left[\left\{ I_1(\lambda) - \frac{I_2(\lambda)}{\lambda} \right\} \phi_1(\lambda) + \{ 2(1-\nu)I_1(\lambda) + \lambda I_2(\lambda) \} \phi_2(\lambda) + \frac{2}{\lambda} I_2(\lambda) \phi_3(\lambda) \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \left[\left\{ K_1(\lambda) + \frac{K_2(\lambda)}{\lambda} \right\} a_n + \{ 2(1-\nu)K_1(\lambda) - \lambda K_2(\lambda) \} b_n - \frac{2}{\lambda} K_2(\lambda) c_n \right] \lambda^{2n+1} \right] \cos \lambda z d\lambda = 0
\end{aligned} \quad (13)$$

$$\begin{aligned}
-\left(\frac{\tau_{rz}}{k \cos \theta} \right)_{r=1} &= \int_0^\infty \lambda^2 \left[I_1'(\lambda) \phi_1(\lambda) + \left\{ 2(1-\nu)I_1'(\lambda) + \left(\frac{1}{2\lambda} + \lambda \right) I_1(\lambda) \right\} \phi_2(\lambda) + \frac{I_1(\lambda)}{\lambda} \phi_3(\lambda) \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \left[K_1'(\lambda) a_n + \left\{ 2(1-\nu)K_1'(\lambda) + \left(\frac{1}{2\lambda} + \lambda \right) K_1(\lambda) \right\} b_n + \frac{K_1(\lambda)}{\lambda} c_n \right] \lambda^{2n+1} \right] \sin \lambda z = 0
\end{aligned} \quad (14)$$

$$\begin{aligned}
-\left(\frac{\tau_{r\theta}}{k \sin \theta} \right)_{r=1} &= \int_0^\infty \lambda^2 \left[\frac{I_2(\lambda)}{\lambda} \phi_1(\lambda) + \frac{I_1(\lambda)}{2} \phi_2(\lambda) + I_2'(\lambda) \phi_3(\lambda) \right. \\
&\quad \left. - \sum_{n=0}^{\infty} \left[\frac{K_2(\lambda)}{\lambda} a_n - \frac{K_1(\lambda)}{2} b_n + K_2'(\lambda) c_n \right] \lambda^{2n+1} \right] \cos \lambda z d\lambda = 0
\end{aligned} \quad (15)$$

in which the prime denotes differentiation with respect to λ .

By the application of Fourier transform to Eqs. (13)-(15), the arbitrary functions $\phi_1(\lambda) \sim \phi_3(\lambda)$ are found as follows:

$$\left. \begin{aligned} \phi_1(\lambda) &= \sum_{n=0}^{\infty} [a_n f_1(\lambda) + b_n f_2(\lambda) + c_n f_3(\lambda)] \lambda^{2n+1} \\ \phi_2(\lambda) &= \sum_{n=0}^{\infty} [a_n f_4(\lambda) + b_n f_5(\lambda) + c_n f_6(\lambda)] \lambda^{2n+1} \\ \phi_3(\lambda) &= \sum_{n=0}^{\infty} [a_n f_7(\lambda) + b_n f_8(\lambda) + c_n f_9(\lambda)] \lambda^{2n+1} \end{aligned} \right\} \quad (16)$$

where

$$\begin{aligned}
F_0(\lambda) f_1(\lambda) &= f_0(\lambda) - (1+2\nu)\lambda I_1(\lambda) + \{\lambda^2 - 2(2-\nu)\} I_2(\lambda) \\
F_0(\lambda) f_2(\lambda) &= -\{\lambda^2 - 2(1-\nu)(1-2\nu)\} \lambda I_1(\lambda) + \{2\lambda^2 - (1-2\nu)(5-4\nu)\} I_2(\lambda) \\
F_0(\lambda) f_3(\lambda) &= -2\lambda I_1(\lambda) - 2(1-2\nu) I_2(\lambda), \quad F_0(\lambda) f_4(\lambda) = -\lambda I_1(\lambda) + 2I_2(\lambda) \\
F_0(\lambda) f_5(\lambda) &= f_0(\lambda) - (5-2\nu)\lambda I_1(\lambda) + \{\lambda^2 + 3(1-2\nu)\} I_2(\lambda) \\
F_0(\lambda) f_6(\lambda) &= 2I_2(\lambda), \quad F_0(\lambda) f_7(\lambda) = -\frac{\lambda}{2} I_1(\lambda) - (1-2\nu) I_2(\lambda) \\
F_0(\lambda) f_8(\lambda) &= \frac{1}{2} (1-2\nu)\lambda I_1(\lambda) + \frac{1}{2} \{\lambda^2 - (1-2\nu)(5-4\nu)\} I_2(\lambda) \\
F_0(\lambda) f_9(\lambda) &= f_0(\lambda) - 2\lambda I_1(\lambda) - 3I_2(\lambda) \\
F_0(\lambda) &= -\lambda^3 I_1^3(\lambda) - 2\lambda^2 I_2^3(\lambda) + 4\lambda^2 I_1^2(\lambda) I_2(\lambda) + \{\lambda^2 - 2(1+\nu)\} \lambda I_1(\lambda) I_2^2(\lambda) \\
f_0(\lambda) &= \lambda^2 I_1^2(\lambda) K_1(\lambda) - 2\lambda^2 I_2^2(\lambda) K_2(\lambda) + 4\lambda^2 I_1^2(\lambda) K_2(\lambda) - \{\lambda^2 - 2(1+\nu)\} \lambda K_1(\lambda) I_2^2(\lambda)
\end{aligned} \quad (17)$$

which are the conditions of zero traction on the surface of the cylinder. And the boundary condition (11) is satisfied since the stress fields derived from [II] and [III] vanish at infinity of $|z| \rightarrow \infty$.

To satisfy the remaining boundary conditions at the interface of the spherical inclusion, it is necessary to express the stress function [III] in spherical coordinates and the following relations are useful.

$$\left. \begin{aligned} I_{2m+1}(\lambda r) \sin \lambda z &= \sum_{n=0}^{\infty} (-1)^{n+m} \frac{(\lambda R)^{2n+2}}{(2n+2m+3)!} P_{2n+2m+1}(\mu) \\ I_{2m+1}(\lambda r) \cos \lambda z &= \sum_{n=0}^{\infty} (-1)^{n+m} \frac{(\lambda R)^{2n+1}}{(2n+2m+2)!} P_{2n+1}(\mu) \end{aligned} \right\} \dots\dots\dots (18)$$

Substituting Eq. (18) into Eq. (8) yields

$$[VI] \left\{ \begin{aligned} \varphi_0 &= k \sum_{n=0}^{\infty} \alpha_n R^{2n+1} P_{2n+1}(\mu) \cos \theta \\ \varphi_4 &= k \sum_{n=0}^{\infty} \beta_n R^{2n+2} P_{2n+2}(\mu) \cos \theta \\ \lambda_3 &= k \sum_{n=0}^{\infty} \gamma_n R^{2n+1} P_{2n+1}(\mu) \sin \theta \end{aligned} \right\} \dots\dots\dots (19)$$

$$\text{where } \alpha_n = \int_0^{\infty} \phi_1(\lambda) \frac{(-1)^n}{(2n+2)!} \lambda^{2n+1} d\lambda, \quad \beta_n = \int_0^{\infty} \phi_2(\lambda) \frac{(-1)^n}{(2n+3)!} \lambda^{2n+3} d\lambda, \quad \gamma_n = \int_0^{\infty} \phi_3(\lambda) \frac{(-1)^n}{(2n+2)!} \lambda^{2n+1} d\lambda$$

$$\text{and } \left\{ \begin{aligned} \alpha_n &= \sum_{m=0}^{\infty} [{}^m\alpha_{n1}A_m + {}^m\alpha_{n2}B_m + {}^m\alpha_{n3}C_m] \\ \beta_n &= \sum_{m=0}^{\infty} [{}^m\beta_{n1}A_m + {}^m\beta_{n2}B_m + {}^m\beta_{n3}C_m] \\ \gamma_n &= \sum_{m=0}^{\infty} [{}^m\gamma_{n1}A_m + {}^m\gamma_{n2}B_m + {}^m\gamma_{n3}C_m] \end{aligned} \right\} \dots\dots\dots (21)$$

$$\left\{ \begin{aligned} {}^m\alpha_{n1} &= \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+2)!(2m)!} \int_0^{\infty} \lambda^{2n+2m+2} f_1(\lambda) d\lambda, \quad {}^m\alpha_{n2} = \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+2)!(2m+1)!} \int_0^{\infty} \lambda^{2n+2m+2} f_2(\lambda) d\lambda \\ {}^m\alpha_{n3} &= \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+2)!(2m)!} \int_0^{\infty} \lambda^{2n+2m+3} f_3(\lambda) d\lambda, \quad {}^m\beta_{n1} = \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+3)!(2m)!} \int_0^{\infty} \lambda^{2n+2m+4} f_4(\lambda) d\lambda \\ {}^m\beta_{n2} &= \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+3)!(2m+1)!} \int_0^{\infty} \lambda^{2n+2m+4} f_5(\lambda) d\lambda, \quad {}^m\beta_{n3} = \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+3)!(2m)!} \int_0^{\infty} \lambda^{2n+2m+4} f_6(\lambda) d\lambda \\ {}^m\gamma_{n1} &= \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+2)!(2m)!} \int_0^{\infty} \lambda^{2n+2m+2} f_7(\lambda) d\lambda, \quad {}^m\gamma_{n2} = \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+2)!(2m+1)!} \int_0^{\infty} \lambda^{2n+2m+2} f_8(\lambda) d\lambda \\ {}^m\gamma_{n3} &= \frac{2}{\pi} \frac{(-1)^{n+m}}{(2n+2)!(2m)!} \int_0^{\infty} \lambda^{2n+2m+2} f_9(\lambda) d\lambda \end{aligned} \right\} \dots\dots\dots (22)$$

By deriving the stresses and displacements from the stress functions [I],[II],[VI] and [IV] and making use of the recurrence relations for the associated Legendre function, the conditions of continuity of the surface stresses and displacements at the interface will be satisfied as follows:

$$\begin{aligned} (u_R)_{R=a} &= (\bar{u}_R)_{R=a} ; \\ \sum_{n=0}^{\infty} \left[-\frac{A_n}{a^{2n+5}} + \frac{(2n+3)(2n-1+4\nu)}{4n+5} \frac{B_n}{a^{2n+5}} - \frac{4n(n+2-2\nu)}{4n+1} \frac{B_{n-1}}{a^{2n+3}} \right. \\ &\quad + \frac{2C_n}{a^{2n+5}} + (2n+1)\alpha_n a^{2n-2} + \frac{(2n+3)(2n-1+4\nu)}{4n+5} \beta_n a^{2n} - \frac{4n(n+2-2\nu)}{4n+1} \beta_{n-1} a^{2n-2} \\ &\quad + 2\gamma_n a^{2n-2} - \frac{1}{\Gamma} \left\{ (2n+1)\bar{A}_n a^{2n-2} + \frac{(2n+3)(2n-1+4\nu)}{4n+5} \bar{B}_n a^{2n} - \frac{4n(n+2-2\nu)}{4n+1} \bar{B}_{n-1} a^{2n-2} \right. \\ &\quad \left. \left. + 2\bar{C}_{n-1} a^{2n-2} \right\} \right] P_{2n+1}(\mu) = -\frac{1-4\nu}{10(1+\nu)} P_1(\mu) - \frac{1}{15} P_3(\mu) \dots\dots\dots (23) \end{aligned}$$

$$\begin{aligned} (v_\theta)_{R=a} &= (\bar{v}_\theta)_{R=a} ; \\ \sum_{n=0}^{\infty} \left[-\frac{A_n}{a^{2n+5}} + \frac{2(4n^2+10n+7)}{4n+5} \frac{C_n}{a^{2n+5}} - \frac{4n(2n-1)}{4n+1} \frac{C_{n-1}}{a^{2n+3}} \right. \\ &\quad - \alpha_n a^{2n-2} - \frac{2(4n^2+2n+1)}{4n+1} \gamma_n a^{2n-2} + \frac{2(2n+3)(2n+4)}{4n+5} \gamma_{n+1} a^{2n} \\ &\quad \left. + \frac{1}{\Gamma} \left\{ \bar{A}_n a^{2n-2} + \frac{2(4n^2+2n+1)}{4n+1} \bar{C}_{n-1} a^{2n-2} - \frac{2(2n+3)(2n+4)}{4n+5} \bar{C}_n a^{2n} \right\} \right] P_{2n+1}(\mu) \\ &= -\frac{1-4\nu}{10(1+\nu)} P_1(\mu) - \frac{1}{15} P_3(\mu) \dots\dots\dots (24) \end{aligned}$$

$$\begin{aligned} (w_r)_{R=a} &= (\bar{w}_r)_{R=a} ; \\ \sum_{n=0}^{\infty} \left[-\frac{A_n}{a^{2n+5}} + \frac{2(4n^2+10n+7)}{4n+5} \frac{C_n}{a^{2n+5}} - \frac{4n(2n-1)}{4n+1} \frac{C_{n-1}}{a^{2n+3}} \right. \\ &\quad - \alpha_n a^{2n-2} - \frac{2(4n^2+2n+1)\gamma_n}{4n+1} a^{2n-2} + \frac{2(2n+3)(2n+4)}{4n+5} \gamma_{n+1} a^{2n} \\ &\quad \left. + \frac{1}{\Gamma} \left\{ \bar{A}_n a^{2n-2} + \frac{2(4n^2+2n+1)}{4n+1} \bar{C}_{n-1} a^{2n-2} - \frac{2(2n+3)(2n+4)}{4n+5} \bar{C}_n \right\} \right] P_{2n+1}(\mu) \\ &\quad + \sum_{n=0}^{\infty} \left[-(2n-1+4\nu) \frac{B_n}{a^{2n+5}} + 2(2n+1)(2n+3) \frac{C_n}{a^{2n+5}} + 2(n+3-2\nu) \beta_n a^{2n} \right. \\ &\quad \left. + 2(2n+2)(2n+4) \gamma_{n+1} a^{2n} - \frac{1}{\Gamma} \left\{ 2(n+3-2\nu) \bar{B}_n a^{2n} + 2(2n+2)(2n+4) \bar{C}_n a^{2n} \right\} \right] P_{2n+2}(\mu) \\ &= -\frac{1-4\nu}{10(1+\nu)} P_1(\mu) - \frac{1}{15} P_3(\mu) + \frac{2}{3} P_5(\mu) \dots\dots\dots (25) \end{aligned}$$

$$\begin{aligned} (\sigma_R)_{R=a} &= (\bar{\sigma}_R)_{R=a} ; \\ \sum_{n=0}^{\infty} \left[(2n+2)(2n+3) \frac{A_n}{a^{2n+5}} - \frac{(2n+3)^2(2n-1+4\nu)}{4n+5} \frac{B_n}{a^{2n+5}} \right. \\ &\quad + \frac{4n(2n^2+5n+2-2\nu)}{4n+1} \frac{B_{n-1}}{a^{2n+3}} - 2(2n+3) \frac{C_n}{a^{2n+5}} + 2n(2n+1)\alpha_n a^{2n-2} \\ &\quad + \frac{2(2n+3)(2n^2+n-1-\nu)}{4n+5} \beta_n a^{2n} - \frac{8n^2(n+2-2\nu)}{4n+1} \beta_{n-1} a^{2n-2} + 4n\gamma_n a^{2n-2} \\ &\quad - \left\{ 2n(2n+1)\bar{A}_n a^{2n-2} + \frac{2(2n+3)(2n^2+n-1-\nu)}{4n+5} \bar{B}_n a^{2n} - \frac{8n^2(n+2-2\nu)}{4n+1} \bar{B}_{n-1} a^{2n-2} \right. \\ &\quad \left. \left. + 4n\bar{C}_{n-1} a^{2n-2} \right\} \right] P_{2n+1}(\mu) = -\frac{1}{5} P_1(\mu) - \frac{2}{15} P_3(\mu) \dots\dots\dots (26) \end{aligned}$$

$$\begin{aligned}
(\tau_{R\varphi})_{R=n} &= (\bar{\tau}_{R\varphi})_{R=n} ; \\
\sum_{n=0}^{\infty} & \left[(2n+3) \frac{A_n}{a^{2n+5}} - \frac{(2n+3)(2n-1+4\nu)}{2(4n+5)} \frac{B_n}{a^{2n+5}} + \frac{4n(n+2-2\nu)}{2(4n+1)} \frac{B_{n-1}}{a^{2n+5}} \right. \\
& - \frac{(2n+3)(4n^2+12n+11)}{4n+5} \frac{C_n}{a^{2n+5}} + \frac{2n(2n-1)(2n+2)}{4n+1} \frac{C_{n-1}}{a^{2n+5}} - 2n\alpha_n a^{2n-2} \\
& - \frac{(2n+3)(2n-1+4\nu)}{2(4n+5)} \beta_n a^{2n} + \frac{4n(n+2-2\nu)}{2(4n+1)} \beta_{n-1} a^{2n-2} - \frac{4n(2n^2+1)}{4n+1} \gamma_n a^{2n-2} \\
& + \frac{(2n+1)(2n+3)(2n+4)}{4n+5} \gamma_{n+1} a^{2n} + \left[2n\bar{A}_n a^{2n-2} + \frac{(2n+3)(2n-1+4\nu)}{2(4n+5)} \bar{B}_n a^{2n} \right. \\
& - \frac{4n(n+2-2\nu)}{2(4n+1)} \bar{B}_{n-1} a^{2n-2} + \frac{4n(2n^2+1)}{4n+1} \bar{C}_{n-1} a^{2n-2} \\
& \left. - \frac{(2n+1)(2n+3)(2n+4)}{4n+5} \bar{C}_n a^{2n} \right] P_{2n+1}'(\mu) + \sum_{n=0}^{\infty} \left[(n+2)(2n-1+4\nu) \frac{B_n}{a^{2n+5}} \right. \\
& - (2n+1)(2n+3)(2n+4) \frac{C_n}{a^{2n+5}} + (2n+1)(n+3-2\nu) \beta_n a^{2n} \\
& + (2n+1)(2n+2)(2n+4) \gamma_{n+1} a^{2n} - ((2n+1)(n+3-2\nu) \bar{B}_n a^{2n} \\
& \left. + (2n+1)(2n+2)(2n+4) \bar{C}_n a^{2n}) \right] P_{2n+2}'(\mu) = \frac{1}{3} P_2'(\mu) \dots\dots\dots (27)
\end{aligned}$$

$$\begin{aligned}
(\tau_{R\theta})_{R=n} &= (\bar{\tau}_{R\theta})_{R=n} ; \\
\sum_{n=0}^{\infty} & \left[(2n+3) \frac{A_n}{a^{2n+5}} - \frac{(2n+3)(2n-1+4\nu)}{2(4n+5)} \frac{B_n}{a^{2n+5}} + \frac{4n(n+2-2\nu)}{2(4n+1)} \frac{B_{n-1}}{a^{2n+5}} \right. \\
& - \frac{(2n+3)(4n^2+12n+11)}{4n+5} \frac{C_n}{a^{2n+5}} + \frac{2n(2n-1)(2n+2)}{4n+1} \frac{C_{n-1}}{a^{2n+5}} - 2n\alpha_n a^{2n-2} \\
& - \frac{(2n+3)(2n-1+4\nu)}{2(4n+5)} \beta_n a^{2n} + \frac{4n(n+2-2\nu)}{2(4n+1)} \beta_{n-1} a^{2n-2} - \frac{4n(2n^2+1)}{4n+1} \gamma_n a^{2n-2} \\
& + \frac{(2n+1)(2n+3)(2n+4)}{4n+5} \gamma_{n+1} a^{2n} + \left[2n\bar{A}_n a^{2n-2} + \frac{(2n+3)(2n-1+4\nu)}{2(4n+5)} \bar{B}_n a^{2n} \right. \\
& - \frac{4n(n+2-2\nu)}{2(4n+1)} \bar{B}_{n-1} a^{2n-2} + \frac{4n(2n^2+1)}{4n+1} \bar{C}_{n-1} a^{2n-2} - \frac{(2n+1)(2n+3)(2n+4)}{4n+5} \bar{C}_n a^{2n} \left. \right] \\
& \times P_{2n+1}'(\mu) = 0 \dots\dots\dots (28)
\end{aligned}$$

where the prime denotes differentiation with respect to μ .

Equating the coefficients of $p_{2n+1}^1(\mu)$ in Eqs.(23),(24),(26) and (28), and of $p_{2n+1}^{1'}(\mu)$ and $p_{2n+1}^1(\mu)$ in Eqs.(25) and (27), to zero, a system of linear equations is obtained. It can be seen that the relations obtained by equating the coefficients of $p_{2n+1}^{1'}(\mu)$ in Eqs.(25),(27) to zero are the same as the ones obtained from Eqs.(24) and (28) respectively.

For solving the linear equations, out of the terms which indicate the rigid body motions, α_0 and γ_0 are equated to zero but $\bar{A}_0$ is retained as a necessary term for fixing the coordinates in the cylinder and of the inclusion which is to displace with the

transformation of the matrix. And it should be noticed that three relations obtained by putting $n=0$ in the relations obtained from Eqs.(26) ~ (28) are not independent of each other, so we eliminate one equation out of them.

Solving the linear equations for the unknown constants A_n , B_n , C_n , $\bar{A}_n$, $\bar{B}_n$ and $\bar{C}_n$, we get a solution. All displacements and stresses in the matrix are derived from a linear combination of the stress functions [I],[II],[III], and those in the inclusion are obtained from [IV].

Fig.2 Variation of σ_R , $\bar{\sigma}_R$ on the spherical interface with φ for $\theta=0$.

Fig.3 Variation of maximum values of σ_R , $\bar{\sigma}_R$ on the spherical interface with a .

3. Numerical examples

Numerical calculations have been carried out for four values of the radius of the inclusion, $a=0.2, 0.4, 0.6$ and 0.8 , and seven values of the stiffness ratio: $\Gamma=0, 0.1, 0.5, 1, 2, 10$ and ∞ , while keep-

Fig.4 Variation of σ_φ on the spherical interface with φ for $\theta=0$.

Fig.5 Variation of $\bar{\sigma}_\varphi$ on the spherical interface with φ for $\theta=0$.

ing Poisson's ratios of the matrix and inclusion $\nu=\bar{\nu}=0.3$. The convergence of the infinite series becomes slower as the radius of the inclusion approaches that of the cylinder. For the largest value of $a=0.8$, the coefficients $A_n \sim C_n$ and $\bar{A}_n \sim \bar{C}_n$ are required up to $n=8$ to make the values of the stress components accurate to within three significant figures.

In Fig.2 we plot, versus φ , the stress σ_R or $\bar{\sigma}_R$ at the interface for $a=0.2$ and 0.8 . In this and the following figures (Figs. 4, 6, 7, 8, 9, 10, 11), we use the nominal stress $\sigma_z=ka$ which represents the stress arising at the corresponding point in a cylinder with no inclusion. For the case of $\Gamma=1$ in Fig.2, curves for $a=0.2$ and $a=0.8$ coincide perfectly and are shown by the broken line.

The maximum tensile value of σ_R occurs between 30° and 40° for all values of Γ , and the stress concentration is evident in

Fig.6 Variation of σ_φ at point A ($\phi=90^\circ$, $\theta=0$) with a .

Fig.7 Variation of $\bar{\sigma}_\varphi$ at point A ($\phi=90^\circ$, $\theta=0$) with a .

hard inclusion ($\Gamma > 1$). It can be seen that variation of normalized stress σ_R/ka or $\bar{\sigma}_R/ka$ against the nominal stress $\sigma = ka$ for $a=0.2$ are similar to the ones for $a=0.8$ for all values of Γ .

Figure 3 shows the variation of the maximum value of σ_R shown in Fig.2 versus a . In this figure and Figs.5,6,7 and 12, we use the nominal stress $(\sigma_z)_{x=1=k}$, which is the maximum tensile stress occurring on the surface of a cylinder without an inclusion under transverse bending. The values of $(\sigma_R, R=a)_{\max}$ increase monotonously as a increases and their rate of increase tends to fall for $\Gamma < 1$ (soft inclusion) and to rise for $\Gamma > 1$ (hard inclusion) as a exceeds 0.6.

Fig.8 Variation of σ_θ on the spherical interface with φ for $\theta=0$.

Fig.9 Variation of $\bar{\sigma}_\theta$ on the spherical interface with φ for $\theta=0$.

Figs.4 and 5 show the variations of σ_r and $\bar{\sigma}_r$ with the angle φ for $a=0.2, 0.8$. The stress distributions of σ_r and $\bar{\sigma}_r$ are strongly affected by the stiffness ratio Γ .

In the case of a hard inclusion ($\Gamma > 1$), stress concentration occurs in $\bar{\sigma}_r$ at the inclusion, whereas in the case of a soft inclusion ($\Gamma < 1$), it occurs in σ_r at the matrix.

From the above figures, it is seen that significant stresses are those occurring at point A ($\varphi=90^\circ, R=a$). Thus, variations of σ_r and $\bar{\sigma}_r$ with a occurring at point A are shown in Figs.6 and 7. The effect of a free surface on those stresses is marked when a exceeds 0.5 and this tendency is also seen for the torsion problem.⁽¹⁾

Fig.10 Variation of $\tau_{R\varphi}, \bar{\tau}_{R\varphi}$ on the spherical interface with φ for $\theta=0$.

Fig.11 Variation of $\tau_{R\theta}, \bar{\tau}_{R\theta}$ on the spherical interface with φ for $\theta=\pi/2$.

Fig.12 Variation of σ_z with r on the plane $z=0$ for $\theta=0$.

Figures 8 and 9 show the variations of σ_θ and $\bar{\sigma}_\theta$ versus ϕ for $\theta=0$. In the case of a hard inclusion ($\Gamma>1$), the stress σ_θ has maximum tensile value at $30^\circ<\phi<40^\circ$, whereas in the case of a soft inclusion ($\Gamma<1$) it has maximum compressive value.

On the contrary, the stress $\bar{\sigma}_\theta$ is compressive for $\Gamma>1$ and tensile for $\Gamma<1$. Figures 10 and 11 show the variations of the shear stresses $\tau_{R\theta}$, $\bar{\tau}_{R\theta}$ and $\tau_{R\theta}$, $\bar{\tau}_{R\theta}$

for $\theta=0$ and $\pi/2$ at the interface versus ϕ , respectively.

Figure 12 shows the variations of the stress for $\theta=0$ occurring at the section where $z=0$ for the cases $\alpha=0.4$ and 0.8 .

Each curve is discontinuous at the interface and then approaches gradually the line $\sigma_z=kx$ as x becomes unity. When $\Gamma<1$ and $\Gamma>1$, stress concentration occurs at the interface. In the matrix and in the inclusion, respectively.

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