621.822.9

Paper No. 174-17

Characteristics of Outwardly Pumping Spherical Spiral Groove Bearings

(Part 1. Load Capacity)*

By Yuichi SATO**, Akiyoshi TAMURA***

The static characteristics of outwardly pumping spherical spiral groove bearings are investigated. Solving the differential equation for a bearing which has an infinite number of grooves, we obtain the axial load capacity and the total liquid flow under radially concentric operating conditions. The effects of the bearing parameters on the load capacity and the liquid flow are examined. The experimental results are in good agreement with the theoretical ones. Although the axial load capacity of an outwardly pumping spherical spiral groove bearing is a fraction of that of an inwardly pumping one, an outwardly pumping spherical spiral groove bearing has enough load capacity.

1. Introduction

Spherical spiral groove bearings (SSGB's) are suitable as bearings to support vertical rotors since they combine radial and axial load capacities. So far most of the investigations have been done into the characteristics of inwardly pumping SSGB's, and few investigations have been done into those of outwardly pumping SSGB's. In outwardly pumping SSGB's the bearing surface, which extends from θ_0 to $\pi/2$, is only grooved from θ_0 to θ_1 (Fig. 1). The ungrooved part, i.e. $\theta_1 < \theta < \pi/2$, acts as a resistance through which the spiral grooves pump up the oil. Oil at ambient pressure enters the bearing clearance through a small hole at the bottom of the bearing.

In this report, we study the static characteristics of outwardly pumping SSGB's theoretically and experimentally.

2. Nomenclature

- C_r : mean radial clearance in ridge region mm
 f : axial load capacity, weight of test rotor kgf
 $F_z = f_z / (r_0^3 p^*)$
 h_g : depth of groove only mm
 $H_g = h_g / C_r$
 h : local film thickness in ridge region mm
 $H = h / C_r$, $H = H_0 - \varepsilon_r \cos \varphi \sin \theta$, $H_0 = 1 - \varepsilon_z \cos \theta$
 $H_0 = H_g + 1$, $H_n = H + H_g$, $H_T = H_0 + H_g$
 N : rotor speed rps
 p : smoothed pressure distribution kgf/mm²

$$P = p / p^*$$

$$p^* = 12 \mu \omega r_0^2 / C_r^2 \quad \text{kgf/mm}^2$$

$$q_\theta : \text{liquid flow in } \theta\text{-direction} \quad \text{mm}^2/\text{s}$$

$$Q_\theta = q_\theta / (C_r r_0 \omega)$$

$$r_0 : \text{bearing radius} \quad \text{mm}$$

$$\alpha : \text{groove angle} \quad \text{deg}$$

$$\tilde{\theta} : \text{fraction of width of groove-ridge pair that is occupied by groove}$$

$$\varepsilon_z = z / C_r : \text{dimensionless axial displacement}$$

$$\varepsilon_r = \sqrt{x^2 + y^2} / C_r : \text{dimensionless radial eccentricity}$$

$$\theta_0, \theta_1 : \text{(Fig. 1)} \quad \text{rad}$$

$$\mu : \text{lubricant viscosity} \quad \text{kgf s/mm}^2$$

$$\dot{\psi} : \text{total liquid flow} \quad \text{mm}^3/\text{s}$$

$$\Psi = \dot{\psi} / (C_r \omega r_0^2)$$

$$\omega : \text{angular velocity of rotor} \quad \text{rad/s}$$

$$\omega_1, \omega_2 : \text{angular velocities of grooved member and smoothed member, respectively} \quad \text{rad/s}$$

$$Q_i = \omega_i / \omega \quad (i=1, 2)$$

Subscripts

' indicates the quantity in the ungrooved part

Fig. 1 Outwardly pumping spherical spiral groove bearing

* Received 20th October, 1977.

** Student of Graduate Course, Tokyo Institute of Technology, Meguro-ku Tokyo.

*** Professor, Tokyo Institute of Technology.

3. Analysis

3.1. Load capacity

We will treat spherical spiral groove bearings which have an infinite number of rectangular grooves and are lubricated with an incompressible liquid. As shown in Fig. 1 grooves are cut in the range of θ from θ_0 to θ_1 , and no grooves are cut in the range from θ_1 to $\pi/2$. Lubricant enters the bearing clearance at the bottom of the bearing and flows out at $\theta=\pi/2$. The dimensionless liquid flow in θ direction is, referring to Fig. 2, expressed as follows,

$$\sin \theta Q_\theta = -\sin \theta \left\{ \eta_1 \frac{\partial P}{\partial \theta} + \eta_2 \frac{1}{\sin \theta} \frac{\partial P}{\partial \varphi} + (\mathcal{Q}_2 - \mathcal{Q}_1) \lambda_c \eta_4 \sin \theta \right\} \quad (1)$$

where η_1 , η_2 , η_4 , and λ_c are defined in Appendix 1.

In the further analysis, radially unloaded bearings will be treated. In other words, a shaft axis coincides with the z-axis (i.e., $\varepsilon_r = \varepsilon_\theta = \varepsilon_\varphi = 0$) in a static equilibrium state. At this radially concentric position (subscript 0), the smoothed pressure distribution P_0 , liquid flow $Q_{\theta 0}$ and η_{i0} ($i=1,2,4$) are functions of θ only and do not depend on φ . Thus we may rewrite Eq.(1) as

$$-\sin \theta Q_{\theta 0} = \eta_{10} \sin \theta \frac{\partial P_0}{\partial \theta} + \lambda_c \eta_{40} (\mathcal{Q}_2 - \mathcal{Q}_1) \sin^2 \theta \quad (2)$$

And the continuity equation is given by

$$\sin \theta \frac{\partial}{\partial \theta} (-\sin \theta Q_{\theta 0}) = 0 \quad (3)$$

Substituting Eq.(2) into Eq.(3), we get a differential equation for P_0 . However, it is very difficult to solve the equation when $\varepsilon_z \neq 0$, since η_{10} and η_{40} are functions of θ . Therefore, we will obtain approximate solutions for P_0 and $Q_{\theta 0}$, valid for small axial displacements, by perturbation expansion of P_0 and $Q_{\theta 0}$ in terms of ε_z . We assume that P_0 , $Q_{\theta 0}$, etc., can be expanded in a series in ε_z in the following form:

Fig. 2

$$P_0 = P_{00} + \varepsilon_z P_{01} \quad (4)$$

$$Q_{\theta 0} = \bar{Q}_{\theta 0} + \varepsilon_z \bar{Q}_{\theta 0}' \quad (5)$$

$$\eta_{10} = \bar{\eta}_{10} + \varepsilon_z \cos \theta \bar{\eta}_{10}' \quad (6)$$

$$\eta_{40} = \bar{\eta}_{40} + \varepsilon_z \cos \theta \bar{\eta}_{40}' \quad (7)$$

$$\eta_{10}' = 1 - 3 \cos \theta \varepsilon_z \quad (8)$$

$$\eta_{40}' = 0 \quad (9)$$

where the mark " ' " indicates the quantity in the ungrooved part, i.e., $\theta_1 < \theta < \pi/2$, and

$$\bar{\eta}_{10} = \frac{(H_a^3 - 1)^2 \delta (1 - \delta) \sin^2 \alpha + H_a^3}{(1 - \delta) H_a^3 + \delta} \quad (10)$$

$$\begin{aligned} \bar{\eta}_{10}' = & -3[H_a^2[(1 - \delta)H_a^4 + \delta] + \delta(1 - \delta) \sin^2 \alpha \\ & \times (H_a^3 - 1)[(1 - \delta)H_a^2(H_a^3 - 2H_a + 1) \\ & - \delta(1 - 2H_a^2 + H_a^3)]] / [(1 - \delta)H_a^3 + \delta]^2 \end{aligned} \quad (11)$$

$$\bar{\eta}_{40} = (H_a^3 - 1) / [(1 - \delta)H_a^3 + \delta] \quad (12)$$

$$\bar{\eta}_{40}' = 3H_a H_a^2 / [(1 - \delta)H_a^3 + \delta]^2 \quad (13)$$

$$H_a = H_z + 1 \quad (14)$$

Substituting Eqs.(4) through (9) into Eq.(2) and collecting the terms of order zero and order ε_z , we get

$$\begin{aligned} \varepsilon_0 : & -\sin \theta \bar{Q}_{\theta 0} = \bar{\eta}_{10} \sin \theta \frac{\partial P_{00}}{\partial \theta} \\ & + \lambda_c \bar{\eta}_{40} (\mathcal{Q}_2 - \mathcal{Q}_1) \lambda \sin^2 \theta \\ & - \sin \theta \bar{Q}_{\theta 0}' = \sin \theta \frac{\partial P_{00}'}{\partial \theta} \end{aligned} \quad (15)$$

$$\begin{aligned} \varepsilon_1 : & -\sin \theta \bar{Q}_{\theta 0} = \bar{\eta}_{10} \cos \theta \sin \theta \frac{\partial P_{00}}{\partial \theta} \\ & + \bar{\eta}_{10} \sin \theta \frac{\partial P_{01}}{\partial \theta} + \lambda_c \bar{\eta}_{40} (\mathcal{Q}_2 - \mathcal{Q}_1) \\ & \times \cos \theta \sin^2 \theta \\ & - \sin \theta \bar{Q}_{\theta 0}' = -3 \cos \theta \sin \theta \frac{\partial P_{00}'}{\partial \theta} \\ & + \sin \theta \frac{\partial P_{01}'}{\partial \theta} \end{aligned} \quad (16)$$

Substituting Eq.(5) into Eq.(3) and integrating it once with respect to θ , we obtain

$$\begin{aligned} \bar{Q}_{\theta 0} = -c_1 / \sin \theta, \quad Q_{\theta 0} = -c_2 / \sin \theta \\ \bar{Q}_{\theta 0}' = -c_1' / \sin \theta, \quad \bar{Q}_{\theta 0}' = -c_2' / \sin \theta \end{aligned} \quad (17)$$

where c_1 , c_1' , etc., are integration constants. Substituting Eq.(17) into Eqs.(15) and (16), we get four differential equations. Integrating them once with respect to θ and applying the boundary conditions

$$(*) \begin{cases} \theta = \theta_0 : P_{00} = 0, P_{01} = 0 \\ \theta = \theta_1 : P_{00} = P_{00}', P_{01} = P_{01}' \\ \bar{Q}_{\theta 0} = \bar{Q}_{\theta 0}', \bar{Q}_{\theta 0} = \bar{Q}_{\theta 0}' \\ \theta = \pi/2 : P_{00}' = 0, P_{01}' = 0 \end{cases}$$

we obtain

$$P_{00} = \frac{c_1}{\bar{\eta}_{10}} \ln \left| \tan \frac{\theta}{2} \right| + \frac{\lambda_c \bar{\eta}_{40} (\mathcal{Q}_2 - \mathcal{Q}_1)}{\bar{\eta}_{10}} \cos \theta + D_1 \quad (18)$$

$$P_{00}' = c_1 \ln \left| \tan \frac{\theta}{2} \right| \quad (19)$$

$$\begin{aligned} P_{01} = & \frac{c_2}{\bar{\eta}_{10}} \ln \left| \tan \frac{\theta}{2} \right| - \frac{\bar{\eta}_{10}}{(\bar{\eta}_{10})^2} c_1 \ln |\sin \theta| \\ & + \frac{1}{2} \lambda_c (\mathcal{Q}_2 - \mathcal{Q}_1) \sin^2 \theta \left[\frac{\bar{\eta}_{40} \bar{\eta}_{10}}{(\bar{\eta}_{10})^2} - \frac{\bar{\eta}_{40}}{\bar{\eta}_{10}} \right] + D_2 \end{aligned} \quad (20)$$

$$P_{01}' = c_2 \ln \left| \tan \frac{\theta}{2} \right| + 3c_1 \ln |\sin \theta| \quad (21)$$

where

$$c_1 = \frac{\lambda_c \bar{\eta}_{40} (\mathcal{Q}_2 - \mathcal{Q}_1) (\cos \theta_1 - \cos \theta_0)}{(\bar{\eta}_{10} - 1) \ln |\tan (\theta_1/2)| + \ln |\tan (\theta_0/2)|} \quad (22)$$

$$D_1 = -\frac{\lambda_c \bar{\eta}_{40}}{\bar{\eta}_{10}} (\mathcal{Q}_2 - \mathcal{Q}_1) \frac{(\bar{\eta}_{10} - 1) \cos \theta_0 \ln |\tan(\theta_1/2)| + \cos \theta_1 \ln |\tan(\theta_0/2)|}{(\bar{\eta}_{10} - 1) \ln |\tan(\theta_1/2)| + \ln |\tan(\theta_0/2)|} \dots (23)$$

$$c_2 = \frac{1}{(\bar{\eta}_{10} - 1) \ln |\tan(\theta_1/2)| + \ln |\tan(\theta_0/2)|} \left\{ -c_1 \left[3\bar{\eta}_{10} \ln |\sin \theta_1| + \frac{\bar{\eta}_{10}}{\bar{\eta}_{10}} (\ln |\sin \theta_1| - \ln |\sin \theta_0|) \right] + \frac{1}{2} \lambda_c (\mathcal{Q}_2 - \mathcal{Q}_1) (\sin^2 \theta_1 - \sin^2 \theta_0) \left[\frac{\bar{\eta}_{40} \bar{\eta}_{10}}{\bar{\eta}_{10}} - \bar{\eta}_{40} \right] \right\} \dots (24)$$

$$D_2 = -\frac{c_2}{\bar{\eta}_{10}} \ln \left| \tan \frac{\theta_0}{2} \right| + \frac{\bar{\eta}_{10}}{(\bar{\eta}_{10})^2} c_1 \ln |\sin \theta_0| + \frac{1}{2} \lambda_c (\mathcal{Q}_2 - \mathcal{Q}_1) \sin^2 \theta_0 \left(\frac{\bar{\eta}_{40}}{\bar{\eta}_{10}} - \frac{\bar{\eta}_{40} \bar{\eta}_{10}}{(\bar{\eta}_{10})^2} \right) \dots (25)$$

And the liquid flows $\bar{Q}_{\theta 0}$, $\bar{Q}_{\theta 0}'$, etc., are obtained as follows:

$$\sin \theta \bar{Q}_{\theta 0} = \sin \theta \bar{Q}_{\theta 0}' = -c_1, \quad \sin \theta \bar{Q}_{\theta 0} = \sin \theta \bar{Q}_{\theta 0}' = -c_2 \dots (26)$$

The axial load capacity F_z is calculated from the following expression:

$$F_z = F_{z0} + \varepsilon_z F_{z1} \dots (27)$$

where

$$\left. \begin{aligned} F_{z0} &= \int_0^{2\pi} d\varphi \left\{ \int_{\theta_0}^{\theta_1} P_{00} \sin \theta \cos \theta d\theta + \int_{\theta_1}^{\pi/2} P_{00}' \sin \theta \cos \theta d\theta \right\} \\ F_{z1} &= \int_0^{2\pi} d\varphi \left\{ \int_{\theta_0}^{\theta_1} P_{01} \sin \theta \cos \theta d\theta + \int_{\theta_1}^{\pi/2} P_{01}' \sin \theta \cos \theta d\theta \right\} \end{aligned} \right\} \dots (28)$$

Substituting Eqs. (18) through (21) into Eqs. (28), we get

$$F_{z0} = 2\pi \left\{ \frac{\lambda_c \bar{\eta}_{40}}{\bar{\eta}_{10}} (\mathcal{Q}_2 - \mathcal{Q}_1) J_1 + D_1 J_2 + \frac{c_1}{\bar{\eta}_{10}} J_3 + c_1 J_4 \right\} \dots (29)$$

$$F_{z1} = 2\pi \left\{ D_2 J_2 + \frac{c_2}{\bar{\eta}_{10}} J_3 + c_2 J_4 - \frac{\bar{\eta}_{10}}{(\bar{\eta}_{10})^2} c_1 J_5 + 3c_1 J_6 + \frac{1}{2} (\mathcal{Q}_2 - \mathcal{Q}_1) \lambda_c \left[\frac{\bar{\eta}_{40} \bar{\eta}_{10}}{(\bar{\eta}_{10})^2} - \frac{\bar{\eta}_{40}}{\bar{\eta}_{10}} \right] J_7 \right\} \dots (30)$$

where

$$\left. \begin{aligned} J_1 &= (\cos^3 \theta_0 - \cos^3 \theta_1)/3, & J_2 &= (\cos^2 \theta_0 - \cos^2 \theta_1)/2 \\ J_3 &= \left(\sin^2 \theta_1 \ln \left| \tan \frac{\theta_1}{2} \right| - \sin^2 \theta_0 \ln \left| \tan \frac{\theta_0}{2} \right| + \cos \theta_1 - \cos \theta_0 \right) / 2 \\ J_4 &= - \left(\sin^2 \theta_1 \ln \left| \tan \frac{\theta_1}{2} \right| + \cos \theta_1 \right) / 2 \\ J_5 &= \frac{1}{4} \sin^2 \theta_1 (2 \ln |\sin \theta_1| - 1) - \frac{1}{4} \sin^2 \theta_0 (2 \ln |\sin \theta_0| - 1) \\ J_6 &= -\frac{1}{4} - \frac{1}{4} \sin^2 \theta_1 (2 \ln |\sin \theta_1| - 1), & J_7 &= (\sin^4 \theta_1 - \sin^4 \theta_0)/4 \end{aligned} \right\} \dots (31)$$

Using Eqs. (26), we obtain the total liquid flow Ψ as follows:

$$\Psi = \int_0^{2\pi} \bar{Q}_{\theta 0} \sin \theta d\varphi = \int_0^{2\pi} (\bar{Q}_{\theta 0} + \varepsilon_z \bar{Q}_{\theta 0}') \sin \theta d\varphi = \Psi_0 + \varepsilon_z \Psi_1 \dots (32)$$

where $\Psi_0 = -2\pi c_1$, $\Psi_1 = -2\pi c_2$

3.2. Calculated results

Fig. 3 shows the axial load capacity F_z as a function of α and H_z when the other parameters, δ , θ_0 , and θ_1 , are set at those values which give maximum load capacity at $\varepsilon_z = 0$. As can be seen, F_z reaches a maximum when $\alpha = 19^\circ$, $H_z = 3.45$, $\delta = 0.7$, $\theta_0 = 17^\circ$, and $\theta_1 = 63^\circ$. The value is $F_z = 0.0186$. In Figs. 4 and 5 is shown the dependence of axial load capacity and total liquid flow on α and H_z . The values of α and H_z which give maximum load capacity F_{z0} and maximum total liquid flow Ψ_0 are almost equal. F_{z1} is a quantity which indicates the magnitude of change in F_z with axial displacement, i.e., ε_z . In other words, if F_{z1} is large, the decrease in load capacity is great for values of $\varepsilon_z < 0$. The groove parameters α and H_z at which F_{z0} reaches a maximum, are almost equal to those at which F_{z1} reaches a maximum. Figs. 4 and 5 show that F_{z1} is sensitive to the change of α and H_z . Thus, when we use the bearing under the condition of $\varepsilon_z < 0$, it

Fig. 3 Load capacity as a function of α and H_z when δ , θ_0 , and θ_1 are set at those values which give maximum load capacity

is desirable that the values of α and H_0 be a little larger than those which give a maximum load capacity at $\epsilon_0 = 0$.

We have already analyzed the characteristics of inwardly pumping SSGB's.⁽¹⁾⁽²⁾ In Table 1 are listed the maximum load capacities of outwardly and inwardly pumping SSGB's. We note that the maximum load capacity of outwardly pumping SSGB is about one fifth of that of inwardly pumping SSGB. Therefore, from the viewpoint of load capacity inwardly pumping SSGB's are better than outwardly pumping ones.

4. Experiments

4.1. Apparatus and procedure

Two outwardly pumping SSGB's having three grooves were evaluated. Grooves are cut on the surface of a stator. The bearing parameters are

$$\alpha = 15^\circ, \quad h_g = 80 \mu\text{m}, \quad \delta = 0.5, \\ \theta_0 = 15^\circ, \quad \theta_1 = 60^\circ.$$

One bearing has a radial clearance C_r of 10 μm and a bearing radius r_0 of 5.164 mm and another has a C_r of 20 μm and a r_0 of 5.174 mm. Four oils having different viscosities were used as lubricants (Fig. 6). The experimental apparatus is shown in Fig. 7. A test rotor ② is positioned vertically so that gravity forces do not act on the bearings radially. The test rotor is mounted on a test bearing, i.e., an outwardly pumping SSGB ① and an externally pressurized air bearing ③, and is driven by the turbine-drive ④. One rotor is used and its weight f_r is 1.4 kgf. The radius and the radial clearance of the externally pressurized air bearing ③ are 15 mm and 50 μm , respectively. And the hydrostatic pressures are between 1 and 6 kgf/cm². Oil entered at the bottom of the test bearings.

The measurement of the axial displacement and the total liquid flow was made under the whirl onset speed. The test rotor speed was increased in increments ranging from 20 rps to 30 rps up to 340 rps. The axial rotor displacement was measured by a vertical capacitance probe ⑤. The total liquid flow was calculated from the time required for oil of 0.2 cm³ to enter the test bearing. When we arranged the experimental data, we assumed that the temperature of oil was almost equal to that of the test bearing.

Table 1 Optimum conditions and dimensionless load capacity when $\epsilon_0 = 0$

	Outwardly pumping	Inwardly pumping
α	19°	16°
H_0	3.45	2.6
δ	0.7	0.5
θ_0	17°	—
θ_1	63°	—
F_z	0.0186	0.0955

Fig. 4 Load capacity and total liquid flow

Fig. 5 Load capacity and total liquid flow

Fig. 6 Viscosity as a function of temperature

Fig. 7 Test apparatus

Fig. 8 Comparison of theoretical and experimental load-capacity data

Fig. 9 Comparison of theoretical and experimental load-capacity data

4.2. Experimental results

Figs. 8 and 9 show experimental-theoretical data comparisons of axial displacement. Theoretical curves are obtained from Eq.(33) which is derived from Eq.(27).

$$\frac{f_z}{r_0^2 \mu N} = \frac{24 \pi r_0^2}{C_r^2} F_z \quad (33)$$

Numerical solutions (the mark "•") are also plotted and are obtained by numerical integration of Eqs.(2) and (3) (see Appendix 2). The number of grooves being taken into account, the theoretical results with terms of ε^2 and higher ones neglected are in fairly good agreement with the experimental ones in the range of ε from +0.5 to -0.5. Numerical solutions show good agreement qualitatively and quantitatively. Figs. 10 and 11 show experimental-theoretical data comparisons of total liquid flow. At $\varepsilon \approx 0$, the theoretical results are in good agreement with experimental ones. When the axial displacement from the bottom of the bearing is large, i.e., $\varepsilon < 0$, ψ does not so much increase with an increase in rotor speed.

Fig. 10 Comparison of theoretical and experimental total liquid flow data

Fig. 11 Comparison of theoretical and experimental total liquid flow data

5. Conclusion

As the first step to investigation of the characteristics of outwardly pumping SSGB's, the axial load capacities are studied theoretically and experimentally. The conclusion is that outwardly pumping spherical spiral groove bearings have enough axial load capacities though their axial load capacities are a fraction of those of inwardly pumping spherical spiral groove bearings.

Acknowledgement

Thanks are due to Mr. Y. Kubota and Mr. H. Sakakida, Toshiba Corporation, who ran most of the tests which produced the data for this paper.

Appendix

(1)

$$\lambda_c = \frac{1}{2} H_s (1 - \delta) \delta \sin \alpha \cos \alpha$$

$$\eta_1 = [(H_m^3 - H^3)^2 \delta (1 - \delta) \sin^2 \alpha + H_m^3 H^3] / \eta_B$$

$$\eta_2 = -(H_m^3 - H^3)^2 \delta (1 - \delta) \sin \alpha \cos \alpha / \eta_B$$

$$\eta_4 = (H_m^3 - H^3) / \eta_B$$

where $\eta_B = (1 - \delta) H_m^3 + \delta H^3$

η_{10} (i=1,2,4) are obtained by substituting H_0 and H_T instead of H and H_m respectively into η_i (i=1,2,4).

(2) Using Eqs. (2) and (3) and applying the boundary conditions:

$$\begin{aligned} P_0 &= 0 & \text{at} & \theta = \theta_0 \\ P_0 &= P_0' & Q_{\theta 0} &= Q_{\theta 0}' & \text{at} & \theta = \theta_1 \\ P_0' &= 0 & \text{at} & \theta = \pi/2 \end{aligned}$$

we obtain

$$\begin{aligned} F_z &= \pi \lambda_c (Q_1 - Q_2) \left\{ \int_{\theta_0}^{\theta_1} \int_{\theta}^{\pi/2} \left[\frac{\eta_{40}(\xi) \sin \xi}{\eta_{10}(\xi)} \right. \right. \\ &\quad \left. \left. - \frac{c_0}{\eta_{10}(\xi) \sin \xi} \right] \sin 2\theta d\xi d\theta \right. \\ &\quad \left. + c_0 \int_{\theta_1}^{\pi/2} \int_{\theta}^{\pi/2} \frac{\sin 2\theta}{H_0^3(\xi) \sin \xi} d\xi d\theta \right\} \dots \dots (34) \end{aligned}$$

$$\Psi = 2\pi \lambda_c (Q_1 - Q_2) c_0 \dots \dots (35)$$

where

$$c_0 = \frac{\int_{\theta_0}^{\theta_1} \frac{\eta_{40} \sin \theta}{\eta_{10}} d\theta}{\int_{\theta_0}^{\theta_1} \frac{d\theta}{\eta_{10} \sin \theta} + \int_{\theta_1}^{\pi/2} \frac{d\theta}{H_0^3 \sin \theta}} \dots \dots (36)$$

Equations (34), (35), and (36) are valid not only for $\varepsilon_z = 0$ but also for $\varepsilon_z \neq 0$.

References

- (1) Sato, Y., and Tamura, A., Bull. JSME, Vol. 22, No.164(1979-2), p.174.
- (2) Sato, Y., and Tamura, A., Bull. JSME, Vol. 22, No.170(1979-8), p.1059.