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Dynamic Stability of Cylindrical Shells Taking into Account In-Plane Inertia and In-Plane Disturbance*

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This paper is concerned with the dynamic stability of cylindrical shells subjected to periodic axial load or periodic external pressure. The equations of motion taking into account the in-plane inertia and the in-plane disturbance are derived. Applying the Bolotin method, the effects of the in-plane inertia and the in-plane disturbance on the principal instability regions of the cylindrical shells are examined. Some of the results show that their effects are remarkable in the case of periodic axial load at low frequencies and in the case of periodic external pressure at high frequencies. This study also includes the comparison of the results for the theories due to authors such as Mizoguchi, Flügge and Donnell.

1. Introduction

The problems on dynamic stability of cylindrical shells subjected to periodic loads have been dealt with by the authors such as Bolotin⁽¹⁾, Hsu⁽²⁾, Yao⁽³⁾, Koval⁽⁴⁾ and others^{(5)~(7)}. Some of these studies, however, have been carried out by using the dynamic equations considering only the inertia force in the radial direction, that is, neglecting the inertia forces in the axial and circumferential directions called the in-plane inertia. Koval, taking into account the in-plane inertia, examined the effect of longitudinal inertia on the width of instability regions by employing Donnell's theory. It may be at once anticipated that the effect of in-plane inertia is rather slight in the case when a cylindrical shell is periodically excited by an external pressure at low frequency, because the bending mode of oscillation is predominant. On the other hand, when we wish to discuss the dynamic stability for an axial load and/or at high frequency, it seems quite insufficient to conclude whether neglect of the in-plane inertia is pertinent or not. And furthermore it is taken for granted that the dynamic stability at high frequencies corresponding to the longitudinal or the torsional vibration can not be discussed without introducing the in-plane inertia.

In the present study, the fundamental equations of dynamic stability considering the in-plane inertia for a cylindrical shell is derived from the equations of forced vibration and an analysis is made by applying the Bolotin method to the fundamental equations. In deriving the fundamental equations some additional terms caused by deformation from an initial membrane state should be

introduced. In this investigation not only an additional term in the radial direction as done by Bolotin, Yao, Koval, etc., but also the additional terms in the axial and circumferential directions called the in-plane disturbance are taken into account based on the stability theory by Timoshenko⁽⁸⁾. As for shell theories, since Donnell's equation adopted by Koval is known to have some limitations in practical applications, three theories based on Mizoguchi, Flügge and Donnell are employed to examine the differences among the theories.

The purpose of the present paper is to discuss the effects of the in-plane inertia and the in-plane disturbance on the principal instability regions at low frequencies as well as at high ones corresponding to longitudinal or torsional vibration of cylindrical shells under periodic axial load or periodic external pressure.

2. Fundamental equations

Applying the stability theory of Timoshenko, the fundamental equations of dynamic stability taking account of the in-plane inertia is deduced.

The equation of motion in terms of a single radial displacement w for the forced vibration of a cylindrical shell has already been described in references (9) and (10), and is given as follows when the relations $L_{ij} = L_{ji}$ hold:

$$\begin{aligned} & [2L_{12}L_{13}L_{23} + L_{11}L_{22}L_{33} - L_{13}^2L_{22} - L_{11}L_{23}^2 - L_{12}^2L_{33}]w \\ & = [L_{13}L_{22} - L_{12}L_{23}]f_t + [L_{11}L_{23} - L_{12}L_{13}]q_t + [L_{11}L_{22} - L_{12}^2]p_t \\ & - \rho h(\partial^2/\partial t^2)[L_{11}L_{22} - L_{22}L_{33} - L_{11}L_{33} - L_{12}^2 + L_{13}^2 + L_{23}^2]w \\ & + (\rho h)^2(\partial^4/\partial t^4)[L_{11} + L_{22} - L_{33}]w - (\rho h)^2\partial^2 w/\partial t^2 \\ & - \rho h(\partial^2/\partial t^2)[(L_{11} + L_{22})p_t + L_{13}f_t + L_{23}q_t] + (\rho h)^2\partial^4 p_t/\partial t^4 \end{aligned} \quad (1)$$

where f_t , q_t and p_t are, respectively, the axial, circumferential and radial components of dynamic surface force, t time, ρ density of shell material, h thickness of shell wall and L_{ij} are differential operators, by which the equilibrium equations of statics

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in terms of displacement components can be represented in symbolic form as

$$L_{ij}u_j = p_i \quad (i, j=1, 2, 3) \quad (2)$$

where $u^T = [u, v, w]$ are, respectively, the axial, circumferential and radial components of displacement, $p^T = [-f, -q, p]$ are those of static surface force.

The differential operators L_{ij} , being symmetric for each theory of Mizoguchi⁽¹¹⁾, Flügge⁽¹²⁾ and Donnell⁽¹³⁾, are written down as follows:

Mizoguchi;

$$[L_{ij}] = \frac{12D}{h^2} \begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2a^2} \frac{\partial^2}{\partial \theta^2} & \frac{1+\nu}{2a} \frac{\partial^2}{\partial x \partial \theta} & \frac{\nu}{a} \frac{\partial}{\partial x} \\ \text{symmetric} & \frac{(1-\nu)A}{2} \frac{\partial^2}{\partial x^2} + \frac{B}{a^2} \frac{\partial^2}{\partial \theta^2} & -\frac{(2-\nu)\epsilon^2}{12} \frac{\partial^3}{\partial x^2 \partial \theta} - \frac{\epsilon^2}{12a^2} \frac{\partial^3}{\partial \theta^3} + \frac{\partial}{a^2 \partial \theta} \\ & & \frac{h^2}{12} \nabla^4 + \frac{1}{a^2} \end{bmatrix} \quad (3)$$

Flügge;

$$[L_{ij}] = \frac{12D}{h^2} \begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{(1-\nu)B}{2a^2} \frac{\partial^2}{\partial \theta^2} & \frac{1+\nu}{2a} \frac{\partial^2}{\partial x \partial \theta} & \frac{(1-\nu)\epsilon^2}{24a} \frac{\partial^3}{\partial x \partial \theta^2} - \frac{a\epsilon^2}{12} \frac{\partial^3}{\partial x^3} + \frac{\nu}{a} \frac{\partial}{\partial x} \\ \text{symmetric} & \frac{(1-\nu)A'}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{a^2 \partial \theta^2} & -\frac{(3-\nu)\epsilon^2}{24} \frac{\partial^3}{\partial x^2 \partial \theta} + \frac{\partial}{a^2 \partial \theta} \\ & & \frac{h^2}{12} \nabla^4 + \frac{\epsilon^2}{6a^2} \frac{\partial^2}{\partial \theta^2} + \frac{\epsilon^2}{12a^2} + \frac{1}{a^2} \end{bmatrix} \quad (4)$$

Donnell;

$$[L_{ij}] = \frac{12D}{h^2} \begin{bmatrix} \frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2a^2} \frac{\partial^2}{\partial \theta^2} & \frac{1+\nu}{2a} \frac{\partial^2}{\partial x \partial \theta} & \frac{\nu}{a} \frac{\partial}{\partial x} \\ \text{symmetric} & \frac{1-\nu}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{a^2 \partial \theta^2} & \frac{\partial}{a^2 \partial \theta} \\ & & \frac{h^2}{12} \nabla^4 + \frac{1}{a^2} \end{bmatrix} \quad (5)$$

where a is radius of cylindrical shell, $\epsilon = h/a$, ν Poisson's ratio, $D = Eh^3/12(1-\nu^2)$ flexural rigidity, E Young's modulus of elasticity, $[A, A', B] = 1 + \epsilon^2[1/3, 1/4, 1/12]$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial \theta^2$.

Assuming that a state just before the dynamic instability occurs may be given by the buckling formula based on Timoshenko⁽⁸⁾, the components of the dynamic surface force p_{ti} ($i=1, 2, 3$) can be replaced by the following expressions:

$$\left. \begin{aligned} f_i &\rightarrow -\frac{N_\theta}{a} \left(\frac{\partial^2 v}{\partial x \partial \theta} + \frac{\partial w}{\partial x} \right), & q_i &\rightarrow -N_x \frac{\partial^2 v}{\partial x^2} \\ p_i &\rightarrow N_x \frac{\partial^2 w}{\partial x^2} + \frac{N_\theta}{a^2} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right) \end{aligned} \right\} \quad (6)$$

where N_x and N_θ , being functions of time t , are the tensile membrane stresses in the axial and circumferential directions, respectively.

Substituting expressions (6) into Eq. (1) an equation which contains other displacement components than a radial displacement w is obtained. In order to eliminate these displacement components d'Alembert's principle ($p_i - p_{ti} - \rho h \partial^2 u_i / \partial t^2$) is applied to Eqs. (2). If we drop the infinitesimal quantities of the square and the product of N_x and N_θ in the equation thus obtained, we can obtain a fundamental equation for dynamic stability of cylindrical shell in terms of a single displacement w .

$$\begin{aligned} \text{The r.h.s. of Eq. (1)} &= -\rho h \frac{\partial^2}{\partial t^2} [L_{11}L_{22} - L_{22}L_{33} - L_{11}L_{33} - L_{12}^2 + L_{13}^2 + L_{23}^2] w \\ &+ (\rho h)^2 \frac{\partial^4}{\partial t^4} [L_{11} + L_{22} - L_{33}] w - (\rho h)^2 \frac{\partial^4 w}{\partial t^4} + \left[N_x \frac{\partial^2}{\partial x^2} \right. \\ &+ \frac{N_\theta}{a^2} \left(\frac{\partial^2}{\partial \theta^2} + 1 \right) \left. \right] [L_{11}L_{22} - L_{12}^2] w - \rho h \frac{\partial^2}{\partial t^2} [L_{11} + L_{22}] \left[N_x \frac{\partial^2 w}{\partial x^2} \right. \\ &+ \frac{N_\theta}{a^2} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right) \left. \right] + (\rho h)^2 \frac{\partial^4}{\partial t^4} \left[N_x \frac{\partial^2 w}{\partial x^2} + \frac{N_\theta}{a^2} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right) \right] \\ &- N_x \frac{\partial^2}{\partial x^2} [L_{13}^2 - L_{11}L_{33}] w + N_x \rho h \frac{\partial^4}{\partial t^2 \partial x^2} (L_{11}w) - \rho h \frac{\partial^2}{\partial t^2} \left[\right. \\ &\quad \left. N_x \frac{\partial^2}{\partial x^2} (L_{33}w) \right] - (\rho h)^2 \frac{\partial^2}{\partial t^2} \left(N_x \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) \\ &- \frac{N_\theta}{a} \frac{\partial}{\partial x} \left[\frac{\partial}{\partial \theta} (L_{12}L_{33} - L_{13}L_{23}) + (L_{13}L_{22} - L_{12}L_{23}) \right] w \\ &- \frac{N_\theta}{a} \rho h \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x \partial \theta} (L_{12}w) + \rho h \frac{\partial^2}{\partial t^2} \left[\frac{N_\theta}{a} \frac{\partial}{\partial x} (L_{13}w) \right] \quad (7) \end{aligned}$$

In the above expression the terms put between a mark $\lfloor \rightarrow \leftarrow \rfloor$ correspond to the correction ones due to the in-plane disturbance.

Substitution of L_{ij} given by Eq. (3) through Eq. (5) into Eq. (7) leads to the fundamental equations in the form of concrete representation for each theory, which can be written in the case of $l \gg \varepsilon^2$ as

$$\begin{aligned}
 H(w) = & \frac{\rho h}{D} \frac{\partial^2}{\partial t^2} \left[\frac{(3-\nu)h^2}{12(1-\nu)} \nabla^2 w - \nabla^4 w + \frac{3+2\nu}{a^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{a^4 \partial \theta^2} \right] \\
 & - \frac{\rho^2 h^4}{12(1-\nu)D^2} \frac{\partial^4}{\partial t^4} \left[\frac{h^2}{6} \nabla^4 w - (3-\nu) \nabla^2 w + \frac{2}{a^2} w \right] - \frac{\rho^3 h^7}{72(1-\nu)D^3} \frac{\partial^6 w}{\partial t^6} \\
 & + \frac{N_z}{D} \frac{\partial^2}{\partial x^2} \nabla^4 w + \frac{N_\theta}{a^2 D} \left(\frac{\partial^2}{\partial \theta^2} + 1 \right) \nabla^4 w \\
 & - \frac{\rho h^3}{12(1-\nu)D^2} \left[(3-\nu) \frac{\partial^2}{\partial t^2} \nabla^2 - \frac{\rho h^2}{6D} \frac{\partial^4}{\partial t^4} \right] \left[N_z \frac{\partial^2 w}{\partial x^2} + \frac{N_\theta}{a^2} \left(\frac{\partial^2 w}{\partial \theta^2} + w \right) \right] \\
 & + \frac{N_z}{D} \frac{\partial^2}{\partial x^2} \left[\frac{h^2}{6(1-\nu)} \left(\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2a^2} \frac{\partial^2}{\partial \theta^2} \right) \nabla^4 w + \frac{2(1+\nu)}{a^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{a^4 \partial \theta^2} \right] \\
 & + N_z \frac{\rho h^3}{6(1-\nu)D^2} \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} + \frac{1-\nu}{2a^2} \frac{\partial^2 w}{\partial \theta^2} \right) \\
 & - \frac{\rho h^3}{6(1-\nu)D^2} \frac{\partial^2}{\partial t^2} \left[N_z \frac{\partial^2}{\partial x^2} \left(\frac{h^2}{12} \nabla^4 w + \frac{w}{a^2} \right) - \frac{\rho^2 h^6}{72(1-\nu)D^3} \frac{\partial^2}{\partial t^2} \left(N_z \frac{\partial^4 w}{\partial t^2 \partial x^2} \right) \right] \\
 & - \frac{N_\theta}{aD} \frac{\partial^2}{\partial x^2} \left[\frac{(1+\nu)h^2}{12(1-\nu)a} \frac{\partial^2}{\partial \theta^2} \nabla^4 w + \frac{(1+3\nu)(2-\nu)\varepsilon^2}{12(1-\nu)a} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{(1+3\nu)\varepsilon^2}{12(1-\nu)a^3} \frac{\partial^4 w}{\partial \theta^4} \right] \\
 & + \frac{\nu}{a} \frac{\partial^2 w}{\partial x^2} \left] - \frac{N_\theta}{a} \frac{(1+\nu)\rho h^3}{12(1-\nu)D^2 a} \frac{\partial^2}{\partial t^2} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} \right. \\
 & \left. + \frac{\rho h^3}{6(1-\nu)D^2} \frac{\partial^2}{\partial t^2} \left[\frac{N_\theta}{a} \frac{\nu}{a} \frac{\partial^2 w}{\partial x^2} \right] \right] \dots \dots \dots (8)
 \end{aligned}$$

$H(w)$ can be written for each theory as follows:

Mizoguchi;

$$\begin{aligned}
 H(w) = & \nabla^2 w + \frac{8-2\nu^2}{a^4} \frac{\partial^6 w}{\partial x^4 \partial \theta^2} + \frac{8}{a^6} \frac{\partial^6 w}{\partial x^2 \partial \theta^4} + \frac{2}{a^8} \frac{\partial^6 w}{\partial \theta^6} \\
 & + \frac{12(1-\nu^2)}{a^2 h^2} \frac{\partial^4 w}{\partial x^4} + \frac{4}{a^6} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{\partial^4 w}{a^8 \partial \theta^4} \dots \dots \dots (9)
 \end{aligned}$$

Flügge;

$$\begin{aligned}
 H(w) = & \nabla^2 w + \frac{2\nu}{a^2} \frac{\partial^4 w}{\partial x^2} + \frac{6}{a^4} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{2(4-\nu)}{a^6} \frac{\partial^4 w}{\partial x^2 \partial \theta^4} + \frac{2}{a^8} \frac{\partial^4 w}{\partial \theta^4} \\
 & + \frac{12(1-\nu^2)}{a^2 h^2} \frac{\partial^4 w}{\partial x^4} + \frac{2(2-\nu)}{a^6} \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{\partial^4 w}{a^8 \partial \theta^4} \dots \dots \dots (10)
 \end{aligned}$$

Donnell;

$$H(w) = \nabla^2 w + \frac{12(1-\nu^2)}{a^2 h^2} \frac{\partial^4 w}{\partial x^4} \dots \dots \dots (11)$$

If the in-plane inertia and the in-plane disturbance are neglected, only $H(w)$ and the underlined parts in the right hand side of expression (8) remain.

3. Theoretical analysis

Let us consider a finite cylindrical shell subjected to periodic axial load or periodic external pressure. Applying the Bolotin method the systems of equation determining the instability regions are derived. The cylindrical shell is assumed to be supported simply at both ends.

First, a cylindrical shell with length l is treated when it is excited by a periodic axial load with circular frequency ω . Putting

$$N_z(t) = -N_1 \cos \omega t, \quad w(t) = f(t) \sin m\pi x / l \cos n\theta \dots \dots \dots (12)$$

(m is the number of axial half-waves and n the number of circumferential waves) and substituting expressions (12) into Eqs. (8) through (11), we obtain

$$\begin{aligned}
 \frac{d^6 f}{dt^6} + & \left[\left(\frac{D}{\rho h^3 a^2} \right) \xi - \frac{N_1 \cos \omega t}{\rho h a^2} \mu^2 (1-\nu) \right] \frac{d^4 f}{dt^4} + \frac{N_1 \omega \sin \omega t}{\rho h a^2} \mu^2 (4-2) \frac{d^3 f}{dt^3} \\
 & + \left[\left(\frac{D}{\rho h^3 a^2} \right)^2 \eta - \frac{6DN_1 \cos \omega t}{\rho^2 h^4 a^4} \mu^2 \left\{ (3-\nu)(\mu^2 + n^2) - 2 \left(\mu^2 + \frac{1-\nu}{2} n^2 \right) \right\} \right. \\
 & \left. - 2 \left[\frac{\varepsilon^2}{12} (\mu^2 + n^2)^2 + 1 \right] \right] + \frac{N_1 \omega^2 \cos \omega t}{\rho h a^2} \mu^2 (6-1) \left] \frac{d^2 f}{dt^2}
 \end{aligned}$$

$$\begin{aligned}
& + \left[\frac{12DN_i\omega \sin \omega t}{\rho^2 h^4 a^4} \mu^2 \left\{ (3-\nu)(\mu^2+n^2) - 2 \left[\frac{\varepsilon^2}{12} (\mu^2+n^2)^2 + 1 \right] \right\} \right. \\
& - \frac{4N_i\omega^3 \sin \omega t}{\rho h a^2} \mu^2 \left. \frac{df}{dt} + \left[\left(\frac{D}{\rho h^3 a^2} \right)^3 \lambda - \frac{72(1-\nu)D^2 N_i \cos \omega t}{\rho^3 h^3 a^3} \mu^2 \right. \right. \\
& \times \left\{ (\mu^2+n^2)^2 - \frac{\varepsilon^2}{6(1-\nu)} \left(\mu^2 + \frac{1-\nu}{2} n^2 \right) (\mu^2+n^2)^2 - 2(1+\nu)\mu^2 - n^2 \right\} \\
& + \frac{6DN_i\omega^2 \cos \omega t}{\rho^2 h^4 a^4} \mu^2 \left. \left\{ (3-\nu)(\mu^2+n^2) - 2 \left[\frac{\varepsilon^2}{12} (\mu^2+n^2)^2 + 1 \right] \right\} \right. \\
& \left. - \frac{N_i\omega^4 \cos \omega t}{\rho h a^2} \mu^2 \right] f=0 \dots\dots\dots (13)
\end{aligned}$$

where

$$\begin{aligned}
\mu &= m\pi a/l \\
\xi &= \varepsilon^2(\mu^2+n^2)^2 + 6(3-\nu)(\mu^2+n^2) + 12 \\
\eta &= 72(1-\nu)[(3-\nu)\varepsilon^2(\mu^2+n^2)^3/12(1-\nu) + (\mu^2+n^2)^2 + (3+2\nu)\mu^2 + n^2] \\
\lambda &= 72(1-\nu)\varepsilon^2 H_{mn}
\end{aligned}$$

H_{mn} can be written for each theory as

$$H_{mn} = \begin{cases} (\mu^2+n^2)^4 - (8-2\nu^2)\mu^4 n^2 - n^2(4\mu^2+n^2)(2n^2-1) \\ \quad + 12(1-\nu^2)\mu^4/\varepsilon^2 & \text{(Mizoguchi)} \\ (\mu^2+n^2)^4 - 2\nu\mu^6 - 6\mu^4 n^2 - 2(4-\nu)\mu^2 n^4 - 2n^6 \\ \quad + 12(1-\nu^2)\mu^4/\varepsilon^2 + 2(2-\nu)\mu^2 n^2 + n^4 & \text{(Flügge)} \\ (\mu^2+n^2)^4 + 12(1-\nu^2)\mu^4/\varepsilon^2 & \text{(Donnell)} \end{cases}$$

The parts of $\sim$ in Eq. (13) represent the correction terms for the in-plane disturbance.

If both in-plane inertia and in-plane disturbance are dropped, Eq. (13) yields

$$\frac{d^2 f}{dt^2} + \left[\omega_0^2 - \frac{\mu^2}{\rho h a^2} N_i \cos \omega t \right] f = 0 \dots\dots\dots (14)$$

This is a well-known Mathieu equation, and ω_0 is the natural frequency corresponding to bending vibration which is given by

$$\omega_0^2 = \frac{D}{\rho h a^4} \frac{[(\mu^2+n^2)^4 + 12(1-\nu^2)/\varepsilon^2]}{(\mu^2+n^2)^2} \dots\dots\dots (15)$$

In order to determine the instability regions the Bolotin method is applied⁽¹⁾. We assume the solutions of Eq. (13) or Eq. (14) with a period of $4\pi/\omega$ or $2\pi/\omega$ in the following form:

$$\left. \begin{aligned} f(t) &= \sum_{k=0,1,2,\dots}^{\infty} \left(a_{2k+1} \cos \frac{2k+1}{2} \omega t + b_{2k+1} \sin \frac{2k+1}{2} \omega t \right) \\ f(t) &= \sum_{k=0,1,2,\dots}^{\infty} (a_{2k} \cos k \omega t + b_{2k} \sin k \omega t) \end{aligned} \right\} \dots\dots\dots (16)$$

Substitution of these expressions into Eq. (13) or Eq. (14) leads to two systems of homogeneous simultaneous equation for determining the coefficients a_k and b_k . From the conditions that these coefficients do not vanish simultaneously we obtain the solutions for the frequency equation which determine the boundaries of the stable and unstable regions.

For the coefficients $a_{1,3,5,\dots}$ and $b_{1,3,5,\dots}$ they may be written for Eq. (13) as

$$\begin{vmatrix} \pm k_0 \varepsilon \frac{N_i}{N_{cr}} [\alpha(0) + \beta(0)] & k_0 \varepsilon \frac{N_i}{N_{cr}} [\alpha(1) + \beta(1)] & 0 & \dots\dots \\ + \left(\frac{Q}{2} \right)^6 - \left(\frac{Q}{2} \right)^4 \xi + \left(\frac{Q}{2} \right)^2 \eta - \lambda & & & \\ k_0 \varepsilon \frac{N_i}{N_{cr}} [\alpha(0) - \beta(0)] & \left(\frac{3Q}{2} \right)^6 - \left(\frac{3Q}{2} \right)^4 \xi + \left(\frac{3Q}{2} \right)^2 \eta - \lambda & k_0 \varepsilon \frac{N_i}{N_{cr}} [\alpha(2) + \beta(2)] & \dots\dots \\ 0 & k_0 \varepsilon \frac{N_i}{N_{cr}} [\alpha(1) - \beta(1)] & \left(\frac{5Q}{2} \right)^6 - \left(\frac{5Q}{2} \right)^4 \xi + \left(\frac{5Q}{2} \right)^2 \eta - \lambda & \dots\dots \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots \end{vmatrix} = 0 \dots\dots\dots (17)$$

The plus sign in the element of the first row and the first column corresponds to the coefficient $a_{1,3,5,\dots}$ and minus one to $b_{1,3,5,\dots}$. $N_{cr} = E h^2 / \sqrt{3(1-\nu^2)} a$ is the classical axial buckling load, $k_0 = 2\sqrt{3(1-\nu^2)}$, $\Omega^2 = \rho h^3 a^2 \omega^2 / D$ the dimensionless frequency and $\alpha(k)$, $\beta(k)$ are given as follows:

$$\begin{aligned}
\alpha(k) = & \mu^2 \left\{ (1-\nu) \left(\frac{2k+1}{2} Q \right)^4 - 6 \left\{ (3-\nu) (\mu^2 + n^2) - 2 \left(\mu^2 + \frac{1-\nu}{2} n^2 \right) \right. \right. \\
& \left. \left. - 2 \left[\frac{\epsilon^2}{12} (\mu^2 + n^2)^2 + 1 \right] \right\} \left(\frac{2k+1}{2} Q \right)^2 + (6-\nu) \left(\frac{2k+1}{2} \right)^2 Q^4 + 72(1-\nu) \right. \\
& \times \left\{ (\mu^2 + n^2)^2 - \frac{\epsilon^2}{6(1-\nu)} \left(\mu^2 + \frac{1-\nu}{2} n^2 \right) (\mu^2 + n^2)^2 - 2(1-\nu) \mu^2 - n^2 \right\} \\
& \left. - 6 \left\{ (3-\nu) (\mu^2 + n^2) - 2 \left[\frac{\epsilon^2}{12} (\mu^2 + n^2)^2 + 1 \right] \right\} Q^2 + Q^4 \right\} \\
\beta(k) = & \mu^2 \left(\frac{2k+1}{2} Q \right)^2 \left\{ -(4-2) \left(\frac{2k+1}{2} Q \right)^2 + 12 \left\{ (3-\nu) (\mu^2 + n^2) \right. \right. \\
& \left. \left. - 2 \left[\frac{\epsilon^2}{12} (\mu^2 + n^2)^2 + 1 \right] \right\} - 4Q^2 \right\}
\end{aligned}$$

The parts of $\sim$ represent the term for the in-plane disturbance.

An equation neglecting the in-plane inertia and the in-plane disturbance can be obtained by removing the terms including Ω^4 and Ω^6 in Eq. (17) and putting $\alpha(k) \rightarrow 72(1-\nu)\mu^2 \times (\mu^2 + n^2)^2$, $\beta(k) \rightarrow 0$ after replacing $\eta \rightarrow 72(1-\nu)(\mu^2 + n^2)^2$.

Similar types of equations are also obtained for $a_{2,4,6,\dots}$ and $b_{2,4,6,\dots}$, but they are not described here.

Solutions for a periodic external pressure can be deduced by a similar procedure as a periodic axial load if we put $N_0(t) = -P_t \cos \omega t$ in Eq. (12) (P_t is equal to $p_0 a$ when the pressure is denoted by p_0).

Turning our attention to the principal instability regions, solutions due to the first approximation are given for an axial load and an external pressure as follows:

$$\begin{aligned}
\frac{N_t}{N_{cr}} = & \mp \frac{1}{2\sqrt{3(1-\nu^2)}\epsilon[\alpha(0) + \beta(0)]} \left[\left(\frac{Q}{2} \right)^6 - \left(\frac{Q}{2} \right)^4 \xi + \left(\frac{Q}{2} \right)^2 \eta - \lambda \right] \\
\frac{P_t}{P_{cr}} = & \mp \frac{1}{1.5\epsilon^2[\alpha'(0) + \beta'(0)]} \left[\left(\frac{Q}{2} \right)^6 - \left(\frac{Q}{2} \right)^4 \xi + \left(\frac{Q}{2} \right)^2 \eta - \lambda \right]
\end{aligned} \quad (18)$$

$P_{cr} = Eh^3/4(1-\nu^2)a^2$ is the classical buckling pressure for an infinitely long cylindrical shell. $\alpha(0)$ and $\beta(0)$ are obtained by putting $k=0$ in the above mentioned expressions. $\alpha'(0)$ and $\beta'(0)$ are omitted here for the sake of brevity.

4. Numerical examples

Numerical examples are presented for cylindrical shells of a radius to thickness ratio a/h equal to 50 and Poisson's ratio ν 0.3. The effects of the in-plane inertia and the in-plane disturbance on the principal instability regions at low and high frequencies, with comparison of the results for each theory, are examined together with the variations due to the length to radius ratio l/a . In the later discussion the numerical results of the first approximation (18) will be shown, since much discrepancies in the results between Eqs. (17) and (18) are not recognized. The mode shapes adopted in the computations are $n \geq 2$ for the circumferential wave and $m=1$ for the axial half wave.

Figure 1 shows the feature of the principal instability regions at low frequencies (bending vibration) based on the approximation method of the fundamental equation of a cylindrical shell under a periodic axial load. Dimensionless excited frequencies Ω for $N_t/N_{cr} = 0$ correspond to the frequencies twice as high as the natural ones. From the figures we notice that neglect of the in-plane inertia, often done, which makes the natural frequencies somewhat higher, results in an increase of the width of the principal instability

regions. In other words, the in-plane inertia makes the width of the principal instability regions narrow. The introduction of the in-plane disturbance also tends to contract these regions, the tendency being more remarkable when l/a and n are small. The feature of the principal instability regions for wide range of parameters will be described in more detail if we use the property that the boundaries dividing the stable and the unstable regions make nearly a straight line, as seen in Fig. 2.

The relation between the width of principal instability regions for $N_t/N_{cr} = 0.5$ and l/a , n is shown in Fig. 2, from which how the in-plane terms influence the instability regions can be at once understood. We should note that for a small value of n the neglect of the in-plane effects results in an overestimation of the principal instability regions.

The principal instability regions for each theory of Mizoguchi, Flügge and Donnell are represented, in a manner similar to Fig. 2, in Fig. 3, which is drawn for the case when the in-plane inertia and the p_t term in expression (6) are included. The values by Mizoguchi's and Flügge's theory almost coincide with each other and the difference between the two may not appear in the figure. Donnell's theory makes the instability regions more narrow and this tendency is conspicuous for a small value of n .

Figure 4 illustrates the principal instability regions in the range of high frequencies corresponding to the longitudinal or the torsional vibration of a cylindrical shell under a periodic axial

Fig. 1 Principal instability regions at low frequencies in case of periodic axial load (Mizoguchi's theory)

Fig. 2 Relation between width of principal instability regions and l/a at low frequencies in case of periodic axial load (Mizoguchi's theory) (Comparison by approximation method of equation)

Fig. 3 Relation between width of principal instability regions and l/a at low frequencies in case of periodic axial load (Comparison based on shell theories)

load. Utilizing the property of linearity with regard to the boundaries between the stable and the unstable regions, which also holds even in the range of high frequencies, the relation between the width of the instability regions and l/a , n is illustrated in this figure, which shows that the in-plane disturbance exerts little influence upon the instability regions at high frequencies and a difference in the results by theories of Mizoguchi, Flügge and Donnell is not perceptible.

Figure 5 shows the relation between the width of the principal instability regions for $P_t/P_{cr} = 5$ at low frequencies and l/a in the case when a cylindrical shell is periodically excited by an external pressure. The in-plane disturbance in the case of an external pressure, as might be expected, does not affect the width of the instability regions, which means that the application of an exact equation in the complicated form may be needless in the discussion of dynamic stability at low frequency of a cylindrical shell under an external pressure.

The principal instability regions in the range of high frequencies are shown in Fig. 6. In this case, unlike the case of low frequencies, these regions are affected remarkably by the in-plane disturbance, which suggests that when we deal with an instability problem at high frequencies in the case of external pressure, we should use a shell theory developed exactly without relying on the simple approximation theory.

Fig. 4 Relation between width of principal instability regions and l/a at high frequencies in case of periodic axial load

Fig. 5 Relation between width of principal instability regions and l/a at low frequencies in case of periodic external pressure (Mizoguchi's theory)

From Fig. 2 through Fig. 6 the figures similar to Fig. 1, which are usually used in the discussion of the instability regions, can be drawn without any difficulty; the utilization of the given width of the principal instability regions and of the property of linearity of the boundaries between the stable and the unstable regions enables us to depict the desired figures after having the natural frequencies Ω by equating the numerator in the right hand side of Eq. (18) to zero.

5. Conclusions

Some of the results in this study are as follows:

(1) In the range of low frequencies the in-plane inertia and the in-plane disturbance exhibit an effect of making the principal instability regions of a cylindrical shell subjected to a periodic axial load narrow and this tendency is remarkable when the length to radius ratio l/a and the number of circumferential waves n are small. On the other hand, in the case of periodic external pressure the effects on the prin-

Fig. 6 Relation between width of principal instability regions and l/a at high frequencies in case of periodic external pressure (Mizoguchi's theory)

cipal instability regions are very slight, as might be expected.

(2) In the range of high frequencies corresponding to the longitudinal or the torsional vibration the in-plane disturbance has an effect of widening the principal instability regions in the case of periodic external pressure, while it does not affect the width of the instability regions in the case of periodic axial load.

(3) The comparison of the results by each theory in the case of periodic axial load shows that at low frequencies Donnell's theory makes the principal instability regions more narrow than those of Mizoguchi and Flügge, but at high frequencies differences among them are not perceptible.

The numerical calculations were carried out with the aid of a TOSBAC 5600 at University of Osaka Prefecture.

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