539.384.6:539.373

Paper No. 177-2

Elastoplastic Analysis of Unsymmetrical Bending of Axisymmetric Shells*

By Minoru HAMADA**, Masataka TANAKA*** and Kazutaroh NAGASUNA****

This paper is concerned with a numerical method of solution by the finite difference technique for unsymmetrical bending of thin elastoplastic shells of revolution. That is, the basic relations in Sanders' linear theory of thin elastic shells are expressed in terms of increment, and the constitutive relation based on the incremental theory of plasticity is applied. A set of the governing differential equations are derived by using a Fourier series expansion in the circumferential direction of the shells, and they are solved by the finite difference technique. The solutions at an arbitrary deformation state are obtained by numerical integration of the increments. Some examples for a cylindrical shell and a circular plate are computed, whereby the effectiveness and the appropriateness of the proposed method are demonstrated.

1. Introduction

In accordance with the rapid development of high-speed electronic computers with huge memories, it has become possible to analyze with sufficient accuracy the linear and also the nonlinear behavior of any structure by means of the finite difference or the finite element technique. As for the thin shell problems, the solution procedures for linear elastic problems were first explored and then those for nonlinear problems are at present being investigated extensively. In references [1] and [2], for example, we can find a number of such investigations.

The authors have been so far concerned with the elastoplastic problems of shells of revolution subjected to axisymmetrical loadings and have established a solution procedure using the finite difference method from the stand point of the incremental theory of plasticity. They [3-6] analyzed some problems of circular plates, pressure vessels and bellows, and applied the results obtained to practical design. However, on the elastoplastic problems of thin shells of revolution subjected to asymmetrical loadings there are few investigations in spite of their importance, partly because their solution procedures are considerably cumbersome and also there is little experimental work which is available in the literature. As far as the authors know, there are only one experiment and one corresponding finite element analysis concerning the elastoplastic problem of a circular plate

subjected to asymmetrical loading [7]. It is desirable, however, also on the circular plate problem to carry out analytical investigation from another point of view.

In this paper, we try to extend the numerical method of solution for linear elastic problems based on Sanders' shell theory [8], which was proposed by Budiansky et al. [9], to elastoplastic problems. That is, we express in terms of increment all the basic equations except the constitutive relations which were previously derived by the authors [4-6] for the incremental theory of plasticity. Furthermore, the resulting variables are expanded into appropriate Fourier series in the circumferential direction of the shell, and then a set of ordinary governing differential equations are derived with respect to the independent variable in the meridional direction. These equations are solved numerically by means of the finite difference method. Numerical integration of the increments thus obtained determines the deformed shape of a shell subjected to a certain magnitude of external load. Some examples of a cylindrical shell and a circular plate are analyzed. Especially, for the latter case the results obtained are compared with the corresponding experimental ones and also the finite element solutions, whereby the effectiveness and the appropriateness of the proposed method of solution are demonstrated.

2. Theory

2.1 Basic relations concerning shell geometry

As shown in Fig.1, we denote by r the distance from the axis of symmetry to an arbitrary point P on the middle surface of the shell. The shape of the middle surface can be given by the following equation:

$$r = r(s) \quad (1)$$

where s is the distance along the shell meridian measured from a boundary point. Let the principal radii at point P be R_s and

* Received 2nd September, 1977.

** Professor, Department of Mechanical Engineering, Faculty of Engineering, Osaka University.

*** Research Associate, Department of Mechanical Engineering, Faculty of Engineering, Osaka University, Yamada-Kami, Suita.

**** Dead.

R_θ , and put

$$\xi = s/a, \quad \rho = r/a, \quad \omega_\xi = a/R_s, \quad \omega_\theta = a/R_\theta \quad (2)$$

Then the following relations hold:

$$\omega_\theta = [1 - (\rho')^2]^{1/2} / \rho, \quad \omega_\xi = -(\gamma' + \gamma^2) / \omega_\theta \quad (3)$$

where

$$\gamma = \rho' / \rho, \quad ()' = d() / d\xi \quad (4)$$

Here a denotes the reference length of the shell. We have also

$$\omega_\theta' = \gamma(\omega_\xi - \omega_\theta), \quad \rho'' / \rho = -\omega_\xi \omega_\theta \quad (5)$$

An arbitrary point of the shell can be expressed in the orthogonal coordinate system (ξ, θ, ζ) , where θ denotes an angle along the circumference and ζ the distance from the middle surface of the shell.

Fig.1 Coordinate system and notations of axisymmetric shell

2.2 Incremental expressions of basic relations

2.2.1 Equilibrium equations

We define resultant forces, resultant moments and external loads acting on an infinitesimally small element of the middle surface as shown in Fig.2. N_θ , N_ξ , $N_{\theta\xi}$, $N_{\xi\theta}$, and M_θ , M_ξ , $M_{\theta\xi}$, $M_{\xi\theta}$ are the resultant forces and the resultant moments, respectively, which are measured per unit length of the middle surface. P_θ , P_ξ and P_ζ are distributed loads per unit area of the middle surface acting in the directions θ , ξ and ζ , respectively. We regard each quantity indicated by an arrow in Fig.2 as positive. Denoting an increment by Δ , we can express the equilibrium equations in the following incremental form:

$$a[\partial(\rho\Delta N_\xi)/\partial\xi + \partial\Delta N_{\xi\theta}/\partial\theta - \rho'\Delta N_\theta] + \omega_\xi[\partial(\rho\Delta M_\xi)/\partial\xi + \partial\Delta M_{\xi\theta}/\partial\theta - \rho'\Delta M_\theta] + (1/2)(\omega_\xi - \omega_\theta)\partial\Delta N_{\xi\theta}/\partial\theta + \alpha^2\rho\Delta P_\xi = 0 \quad (6.a)$$

$$a[\partial\Delta N_\theta/\partial\theta + \partial(\rho\Delta N_{\xi\theta})/\partial\xi + \rho'\Delta N_{\xi\theta}] + \omega_\theta[\partial\Delta M_\theta/\partial\theta + \partial(\rho\Delta M_{\xi\theta})/\partial\xi + \rho'\Delta M_{\xi\theta}] + (\rho/2)\partial[(\omega_\theta - \omega_\xi)\Delta N_{\xi\theta}]/\partial\xi + \alpha^2\rho\Delta P_\theta = 0 \quad (6.b)$$

Fig.2 Resultant forces, resultant moments and distributed loads

Fig.3 Displacements and rotations

$$\begin{aligned} & \partial[\partial(\rho\Delta M_\xi)/\partial\xi + \partial\Delta M_{\xi\theta}/\partial\theta - \rho'\Delta M_\theta]/\partial\xi + (1/\rho)\partial[\partial\Delta M_\theta/\partial\theta + \partial(\rho\Delta M_{\xi\theta})/\partial\xi + \rho'\Delta M_{\xi\theta}]/\partial\theta \\ & - \alpha\rho(\omega_\xi\Delta N_\xi + \omega_\theta\Delta N_\theta) + \alpha^2\rho\Delta P_\zeta = 0 \end{aligned} \quad (6.c)$$

We call $\Delta N_{\xi\theta}$ and $\Delta N_{\theta\xi}$ the modified twisting moment and the modified in-plane shearing force, respectively. They are defined in Sanders' shell theory as follows:

$$\begin{aligned} \Delta N_{\xi\theta} &= (1/2)(\Delta M_{\xi\theta} + \Delta M_{\theta\xi}) \\ \Delta N_{\theta\xi} &= (1/2)(\Delta N_{\xi\theta} + \Delta N_{\theta\xi}) + (1/4a)(\omega_\theta - \omega_\xi) \times (\Delta M_{\xi\theta} - \Delta M_{\theta\xi}) \end{aligned} \quad (7)$$

2.2.2 Displacements, rotations and strains

The rotations ϕ_ξ and ϕ_θ shown in Fig.3 are related to the displacements U_ξ , U_θ and W in the incremental form by the following equations:

$$\begin{aligned} \Delta\phi_\xi &= (1/a)(-\partial\Delta W/\partial\xi + \omega_\xi\Delta U_\xi) \\ \Delta\phi_\theta &= (1/a)[-(1/\rho)\partial\Delta W/\partial\theta + \omega_\theta\Delta U_\theta] \end{aligned} \quad (8)$$

The membrane strain increments $\Delta\epsilon_{\xi m}$, $\Delta\epsilon_{\theta m}$ and $\Delta\gamma_{\xi\theta m}$ are expressed in terms of displacement increment as follows:

$$\begin{aligned} \Delta\epsilon_{\xi m} &= (1/a)(\partial\Delta U_\xi/\partial\xi + \omega_\xi\Delta W) \\ \Delta\epsilon_{\theta m} &= (1/a)[(1/\rho)\partial\Delta U_\theta/\partial\theta + \gamma\Delta U_\xi + \omega_\theta\Delta W] \\ \Delta\gamma_{\xi\theta m} &= (1/a)[(1/\rho)\partial\Delta U_\xi/\partial\theta + \partial\Delta U_\theta/\partial\xi - \gamma\Delta U_\theta] \end{aligned} \quad (9)$$

where $\Delta\gamma_{\xi\theta m}$ is the shearing strain conventionally used in engineering. The incremental changes in curvatures of the middle surface of the shell are given by

$$\begin{aligned} \Delta\kappa_\xi &= (1/a)\partial\Delta\phi_\xi/\partial\xi \\ \Delta\kappa_\theta &= (1/a)[(1/\rho)\partial\Delta\phi_\theta/\partial\theta + \gamma\Delta\phi_\xi] \\ \Delta\kappa_{\xi\theta} &= (1/a)[(1/\rho)\partial\Delta\phi_\xi/\partial\theta + \partial\Delta\phi_\theta/\partial\xi - \gamma\Delta\phi_\theta] \\ & + (1/2a)(\omega_\xi - \omega_\theta)\{ (1/\rho)\partial\Delta U_\xi/\partial\theta - \partial\Delta U_\theta/\partial\xi - \gamma\Delta U_\theta \} \end{aligned} \quad (10)$$

Using the Kirchhoff-Love hypothesis and

assuming that ζ/R_s and ζ/R_θ are sufficiently small compared with unity, we obtain for the strain increments of a point which is separated from the middle surface by ζ

$$\{\Delta\epsilon\} = \{\Delta\epsilon_m\} + \zeta\{\Delta\kappa\} \quad (13)$$

where

$$\begin{aligned} \{\Delta\epsilon\} &= \{\Delta\epsilon_\xi \Delta\epsilon_\theta \Delta\gamma_{\xi\theta}\} \\ \{\Delta\epsilon_m\} &= \{\Delta\epsilon_{\xi m} \Delta\epsilon_{\theta m} \Delta\gamma_{\xi\theta m}\} \\ \{\Delta\kappa\} &= \{\Delta\kappa_\xi \Delta\kappa_\theta \Delta\kappa_{\xi\theta}\} \end{aligned} \quad (12)$$

Here we denote a column vector by $\{ \}$.

2.2.3 Constitutive relations

The constitutive relations based on the incremental theory of plasticity which were previously derived by the authors [3-6] by taking into consideration the isotropic hardening, kinematic hardening and also their combination can be expressed as follows:

$$\{\Delta\sigma\} = ([D^e] - \rho^* [D^p]) \{\Delta\epsilon\} \quad (13)$$

where $\rho^* = 1$ for plastic loading state, otherwise $\rho^* = 0$. The above relations can be reduced to the thin shell problems discussed here, and if we assume Hooke's law for elastic deformation, we obtain

$$\{\Delta\sigma\} = \{\Delta\sigma_\xi \Delta\sigma_\theta \Delta\sigma_{\xi\theta}\} \quad (14)$$

$$[D^e] = E \begin{bmatrix} 1/(1-\nu^2) & & \text{SYM.} \\ \nu/(1-\nu^2) & 1/(1-\nu^2) & \\ \nu/(1-\nu^2) & \nu/(1-\nu^2) & 1/(2(1+\nu)) \end{bmatrix} \quad (15)$$

$$[D^p] = \frac{1}{3} \begin{bmatrix} S_1^2 & \text{SYM.} \\ S_1 S_2 & S_2^2 \\ S_1 S_3 & S_2 S_3 & S_3^2 \end{bmatrix} \quad (16)$$

where

$$\left. \begin{aligned} S &= 4E_p \sigma_e^2 / 9 + S_{\xi D} S_1 + S_{\theta D} S_2 + 2S_{\xi\theta} S_3 \\ \sigma_e^2 &= S_\xi^2 - S_\xi S_\theta + S_\theta^2 + 3S_{\xi\theta}^2 \\ S_1 &= E (S_{\xi D} + \nu S_{\theta D}) / (1-\nu^2) \\ S_2 &= E (\nu S_{\xi D} + S_{\theta D}) / (1-\nu^2) \\ S_3 &= E S_{\xi\theta} / (1+\nu) \\ S_{\xi D} &= \{2(\sigma_\xi - \alpha_\xi) - (\sigma_\theta - \alpha_\theta)\} / 3 \\ S_{\theta D} &= \{-(\sigma_\xi - \alpha_\xi) + 2(\sigma_\theta - \alpha_\theta)\} / 3 \\ S_{\xi\theta} &= \sigma_{\xi\theta} - \alpha_{\xi\theta} \end{aligned} \right\} \quad (17)$$

Here ν , E and E_p denote Poisson's ratio, Young's modulus and the plastic tangent modulus, respectively. α_ξ , α_θ and $\alpha_{\xi\theta}$ indicate the coordinates of the center of the yield surface.

Resulting forces and moments are now obtained from the following relations:

$$\begin{aligned} \{\Delta N\} &= \int_{-h/2}^{h/2} \{\Delta\sigma\} d\zeta \\ \{\Delta M\} &= \int_{-h/2}^{h/2} \{\Delta\sigma\} \zeta d\zeta \end{aligned} \quad (18)$$

where h is the shell thickness which is assumed to be constant in this study, and

$$\begin{aligned} \{\Delta N\} &= \{\Delta N_\xi \Delta N_\theta \Delta N_{\xi\theta}\} \\ \{\Delta M\} &= \{\Delta M_\xi \Delta M_\theta \Delta M_{\xi\theta}\} \end{aligned} \quad (19)$$

Substituting Eqs. (11) and (13) into Eq. (18) we have

$$\begin{Bmatrix} \Delta N \\ \Delta M \end{Bmatrix} = [B^e] \begin{Bmatrix} \Delta\epsilon_m \\ \Delta\kappa \end{Bmatrix} - \begin{Bmatrix} \Delta N^p \\ \Delta M^p \end{Bmatrix} \quad (20)$$

where

$$[B^e] = \int_{-h/2}^{h/2} \begin{bmatrix} [D^e] & \zeta [D^e] \\ \zeta [D^e] & \zeta^2 [D^e] \end{bmatrix} d\zeta \quad (21)$$

$$\begin{Bmatrix} \Delta N^p \\ \Delta M^p \end{Bmatrix} = \int_{-h/2}^{h/2} \rho^* \begin{bmatrix} [D^p] & \zeta [D^p] \\ \zeta [D^p] & \zeta^2 [D^p] \end{bmatrix} d\zeta \begin{Bmatrix} \Delta\epsilon_m \\ \Delta\kappa \end{Bmatrix} \quad (22)$$

in which

$$\{\Delta N^p\} = \{\Delta N_\xi^p \Delta N_\theta^p \Delta N_{\xi\theta}^p\}, \quad \{\Delta M^p\} = \{\Delta M_\xi^p \Delta M_\theta^p \Delta M_{\xi\theta}^p\} \quad (23)$$

2.3 Fourier series expansion and governing equations

Let us assume that the distributed loads can be expanded into the following Fourier series. That is,

$$\begin{aligned} \{\Delta P\} &= \{\Delta P_\xi \Delta P_\theta \Delta P_{\xi\theta}\} \\ &= \sigma_0 \lambda \sum_{k=0}^{\infty} ([A_k] \{p\}_k + [B_k] \{\bar{p}\}_k) \end{aligned} \quad (24)$$

where $\lambda = h/a$, and σ_0 denotes the reference stress which is taken in this study as the initial yield stress of the shell material. We also use the following notations:

$$\begin{aligned} [A_k] &= \begin{bmatrix} \cos k\theta & \cos k\theta & \sin k\theta \end{bmatrix} \\ [B_k] &= \begin{bmatrix} \sin k\theta & \sin k\theta & \cos k\theta \end{bmatrix} \end{aligned} \quad (25)$$

where $\begin{bmatrix} \end{bmatrix}$ denotes a diagonal matrix. The Fourier coefficients of Eq. (24) are as follows:

$$\{p\}_k = \{p_\xi p_\theta p_{\xi\theta}\}_k, \quad \{\bar{p}\}_k = \{\bar{p}_\xi \bar{p}_\theta \bar{p}_{\xi\theta}\} \quad (26)$$

The suffix k denotes the coefficients corresponding to a particular Fourier index k . In what follows, this suffix will be dropped unless it is necessary. For the distributed loads which are symmetric with respect to the

plane of $\theta=0^\circ$, we can take into account only the first terms of the right hand side of Eq.(24). In the following, we call the terms corresponding to $[A_k]$ A-Series and those concerning $[B_k]$ B-Series.

In accordance with Eq.(24) a set of increments are expanded into Fourier series as follows:

$$\left. \begin{aligned} \{\Delta U\} &\equiv \{\Delta U_{\xi} \Delta W \Delta U_{\theta}\} = (\alpha \sigma_o / E) \sum_{k=0}^{\infty} ([A_k] \{u\}_k + [B_k] \{\bar{u}\}_k) \\ \{\Delta \Phi\} &\equiv \{\Delta \Phi_{\xi} \Delta \Phi_{\theta}\} = (\sigma_o / E) \sum_{k=0}^{\infty} (\cos k\theta \sin k\theta) \{\phi\}_k + [\sin k\theta \cos k\theta] \{\bar{\phi}\}_k \\ \{\Delta N\} &\equiv \{\Delta N_{\xi} \Delta N_{\theta} \Delta N_{\xi\theta}\} = \sigma_o h \sum_{k=0}^{\infty} ([A_k] \{n\}_k + [B_k] \{\bar{n}\}_k) \\ \{\Delta M\} &\equiv \{\Delta M_{\xi} \Delta M_{\theta} \Delta M_{\xi\theta}\} = (\sigma_o h^3 / \alpha) \sum_{k=0}^{\infty} ([A_k] \{m\}_k + [B_k] \{\bar{m}\}_k) \\ \{\Delta \epsilon_m\} &\equiv \{\Delta \epsilon_{\xi m} \Delta \epsilon_{\theta m} \Delta \epsilon_{\xi\theta m}\} = (\sigma_o / E) \sum_{k=0}^{\infty} ([A_k] \{e_m\}_k + [B_k] \{\bar{e}_m\}_k) \\ \{\Delta \kappa\} &\equiv \{\Delta \kappa_{\xi} \Delta \kappa_{\theta} \Delta \kappa_{\xi\theta}\} = (\sigma_o / \alpha E) \sum_{k=0}^{\infty} ([A_k] \{c\}_k + [B_k] \{\bar{c}\}_k) \end{aligned} \right\} \quad (27)$$

where

$$\left. \begin{aligned} \{u\}_k &\equiv \{u_{\xi} w u_{\theta}\}_k, \quad \{\bar{u}\}_k \equiv \{\bar{u}_{\xi} \bar{w} \bar{u}_{\theta}\}_k \\ \{\phi\}_k &\equiv \{\phi_{\xi} \phi_{\theta}\}_k, \quad \{\bar{\phi}\}_k \equiv \{\bar{\phi}_{\xi} \bar{\phi}_{\theta}\}_k \\ \{n\}_k &\equiv \{n_{\xi} n_{\theta} n_{\xi\theta}\}_k, \quad \{\bar{n}\}_k \equiv \{\bar{n}_{\xi} \bar{n}_{\theta} \bar{n}_{\xi\theta}\}_k \\ \{m\}_k &\equiv \{m_{\xi} m_{\theta} m_{\xi\theta}\}_k, \quad \{\bar{m}\}_k \equiv \{\bar{m}_{\xi} \bar{m}_{\theta} \bar{m}_{\xi\theta}\}_k \\ \{e_m\}_k &\equiv \{e_{\xi m} e_{\theta m} e_{\xi\theta m}\}_k, \quad \{\bar{e}_m\}_k \equiv \{\bar{e}_{\xi m} \bar{e}_{\theta m} \bar{e}_{\xi\theta m}\}_k \\ \{c\}_k &\equiv \{c_{\xi} c_{\theta} c_{\xi\theta}\}_k, \quad \{\bar{c}\}_k \equiv \{\bar{c}_{\xi} \bar{c}_{\theta} \bar{c}_{\xi\theta}\}_k \end{aligned} \right\} \quad (28)$$

All the quantities of Eq.(28) are dimensionless.

For Eqs.(6), (8)-(10), (20) and (22), in totality, 23 basic equations, we use Eqs.(27) and then obtain the expressions which correspond to a particular Fourier index k . after appropriate manipulations to eliminate the unknowns, we can derive the following governing differential equations. That is, for $k=0, 1, 2, 3, \dots$, we have A-Series:

$$[A]\{Z\}' + [B]\{Z\}' + [C]\{Z\} = \{D\} \quad (29)$$

B-Series:

$$[\bar{A}]\{\bar{Z}\}' + [\bar{B}]\{\bar{Z}\}' + [\bar{C}]\{\bar{Z}\} = \{\bar{D}\} \quad (30)$$

where

$$\{Z\} = \{u_{\xi} u_{\theta} w m_{\xi}\}, \quad \{\bar{Z}\} = \{\bar{u}_{\xi} \bar{u}_{\theta} \bar{w} \bar{m}_{\xi}\} \quad (31)$$

The matrices having the order 4×4 , $[A]$, $[B]$ and $[C]$, are determined by the shell geometry, Poisson's ratio and the Fourier index

k , which coincide with those derived by Budiansky-Radkowski [9] for linear elastic problems. The column vector $\{D\}$ is given by

$$\{D\} = \begin{Bmatrix} p_{\xi} \\ p_{\theta} \\ p_{\xi\theta} \\ 0 \end{Bmatrix} + \begin{Bmatrix} [C1]\{X^p\}' + [C2]\{X^p\} \\ m_{\xi}^p \end{Bmatrix} \quad (32)$$

where

$$\{X^p\} = \{n_{\xi}^p, n_{\theta}^p, n_{\xi\theta}^p, m_{\xi}^p - v m_{\theta}^p, m_{\xi\theta}^p\} \quad (33)$$

$$[C1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \lambda^2(3\omega_{\theta} - \omega_{\xi})/2 \\ 0 & 0 & 0 & -\lambda^2\gamma & 2\lambda^2 k/\rho \end{bmatrix} \quad (34)$$

$$[C2] = \begin{bmatrix} \gamma & -\gamma & k/\rho & -\lambda^2\gamma\omega_{\xi} \\ 0 & -k/\rho & 2\gamma & -\lambda^2 k\omega_{\theta}/\rho \\ -\omega_{\xi} & -\omega_{\theta} & 0 & \lambda^2(\omega_{\xi}\omega_{\theta} - k^2/\rho^2) \end{bmatrix} \quad (35)$$

$$\begin{aligned} &\lambda^2 k(3\omega_{\xi} - \omega_{\theta})/(2\rho) \\ &\lambda^2 \gamma(3\omega_{\theta} + \omega_{\xi})/2 - \lambda^2 \omega_{\xi}'/2 \\ &2\lambda^2 \gamma k/\rho \end{aligned}$$

The matrices of the B-Series, $[\bar{A}]$, $[\bar{B}]$, $[\bar{C}]$ and $\{\bar{D}\}$ are obtainable by replacing k by $(-k)$ in the corresponding matrices of the A-Series and using the Fourier series coefficients superimposed by the bar.

Equation (22) can be expressed as follows:

For the A-Series;

$$\sum_{k=0}^{\infty} [A_k][A_k] \begin{Bmatrix} n^p \\ m^p \end{Bmatrix}_k = [B^p] \begin{Bmatrix} e_m \\ c \end{Bmatrix}_k \quad (36)$$

and for the B-Series;

$$\sum_{k=0}^{\infty} [B_k][B_k] \begin{Bmatrix} \bar{n}^p \\ \bar{m}^p \end{Bmatrix}_k = [\bar{B}^p] \begin{Bmatrix} \bar{e}_m \\ \bar{c} \end{Bmatrix}_k \quad (37)$$

where

$$[B^p] = \int_{-1/2}^{1/2} \rho^* \begin{bmatrix} [D^p] & \lambda \xi [D^p] \\ (\xi/\lambda) [D^p] & \xi^2 [D^p] \end{bmatrix} d\xi \quad (38)$$

and

$$\xi = \zeta/h, \quad [D^p] = (1/E)[D^p] \quad (39)$$

2.4 Boundary conditions and continuity conditions

When the values of $\{U\}$ and M_{ξ} are prescribed at the boundary points, there is no difficulty for solving Eq.(29) or (30). If the other values are prescribed at the boundary points, we have to express the boundary conditions in terms of $\{Z\}$ or $\{\bar{Z}\}$.

In Sanders' shell theory [8] the virtual work at the boundary

$$\pm (U_{\xi} N_{\xi} + U_{\theta} T_{\xi\theta} + W F_{\xi} + M_{\xi} \Phi_{\xi})$$

F_{ξ} and $T_{\xi\theta}$ are the effective transverse shearing force and the effective in-plane shearing force, respectively. They are defined in the incremental form as follows:

$$\Delta F_{\xi} = (1/\alpha\rho) [\partial(\rho\Delta M_{\xi})/\partial\xi + 2\partial\Delta M_{\xi\theta}/\partial\theta - \rho'\Delta M_{\theta}] \quad (40)$$

$$\Delta T_{\xi\theta} = \Delta N_{\xi\theta} + (1/2a)(3\omega_{\theta} - \omega_{\xi})\Delta M_{\xi\theta} \quad (41)$$

ΔF_{ξ} and $\Delta T_{\xi\theta}$ are now expanded into the following Fourier series:

$$\begin{aligned} \begin{Bmatrix} \Delta F_{\xi} \\ \Delta T_{\xi\theta} \end{Bmatrix} = \sigma_0 h \sum_{k=0}^{\infty} \left(\begin{Bmatrix} f_{\xi} \\ t_{\xi\theta} \end{Bmatrix}_k \right. \\ \left. + \begin{Bmatrix} \bar{f}_{\xi} \\ \bar{t}_{\xi\theta} \end{Bmatrix}_k \right) \quad (42) \end{aligned}$$

Making use of Eq.(42), we can express the boundary conditions in which, for example, the values of $\{\Delta N_{\xi} \Delta T_{\xi\theta} \Delta F_{\xi} \Delta \phi_{\xi}\}$ are prescribed. They can be given for the A-Series as follows:

$$[H]\{Z\}' + [J]\{Z\} = \{y\} + \{f^p\} \quad (43)$$

where

$$\begin{aligned} \{y\} &= \{n_{\xi}^p t_{\xi\theta}^p f_{\xi}^p \phi_{\xi}^p\} \\ \{f^p\} &= \{n_{\xi}^p, n_{\xi\theta}^p, m_{\xi\theta}^p, \lambda^2(3\omega_{\theta} - \omega_{\xi})/2, \lambda^2\gamma(m_{\xi}^p - m_{\theta}^p) \\ &\quad + 2m_{\xi\theta}^p k\lambda^2/\rho, 0\} \end{aligned} \quad (44)$$

The matrices $[H]$ and $[J]$, being of the order 4×4 , are identical with those of linear elastic problems which are determined by elastic constants, the Fourier index and the shell geometry.

At a point in which the shell geometry changes discontinuously as shown in Fig.4, we must use the continuity conditions which are derivable from the equilibrium and also kinematic conditions. That is, we must use the following relationships:

$$\{y^+\} = [\Psi]\{y^-\}, \quad \{Z^+\} = [\Psi]\{Z^-\} \quad (45)$$

where

$$[\Psi] = \begin{bmatrix} \cos\psi & 0 & -\sin\psi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\psi & 0 & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (46)$$

Fig.4 Continuity conditions

The suffixes - and + are used to distinguish the quantities before and after the discontinuity, respectively.

3. Numerical Method of Solution

If we solve the governing differential equation (29) or (30) under appropriate boundary conditions, we can analyze the elastoplastic problems of asymmetrical bending. In this study we apply the finite difference method to this solution procedure. Namely, for an arbitrary mesh point of the finite difference approximation we use the following relations:

$$\begin{aligned} \{Z\}'_j &= (\{Z\}_{j+1} - \{Z\}_{j-1}) / (2\Delta\xi) \\ \{Z\}''_j &= (\{Z\}_{j+1} - 2\{Z\}_j + \{Z\}_{j-1}) / (\Delta\xi)^2 \end{aligned} \quad (47)$$

The first-order derivatives appearing in the boundary conditions and/or the continuity conditions are approximated by the forward and the backward finite difference formulas as follows:

$$\begin{aligned} \{Z\}'_1 &= (-3\{Z\}_1 + 4\{Z\}_2 - \{Z\}_3) / (2\Delta\xi) \\ \{Z\}'_N &= (3\{Z\}_N - 4\{Z\}_{N-1} + \{Z\}_{N-2}) / (2\Delta\xi) \end{aligned} \quad (48)$$

Here we assume that the mesh points of the finite difference approximation are arranged as $j=1, 2, 3, \dots, N$ with the distance $\Delta\xi$. $\{Z\}'_j$, for example, denotes the value of $\{Z\}'$ at the mesh point j . We can thus obtain a set of simultaneous equations with respect to $\{Z\}_j$ from the governing differential equations, the boundary conditions and/or the continuity conditions. These equations are quite similar to the so-called tridiagonal equations in which only the three components adjacent to the diagonal elements are non-vanishing. They can be solved with efficiency and good accuracy by means of the solution procedures which were previously developed by the authors [3-6].

The right hand side of Eq.(29) or (30) includes the terms given by the external load increments and the terms of pseudo load increments occurring due to plastic deformation. The latter terms can not be calculated unless the corrected values of $\{Z\}$ or $\{\bar{Z}\}$ are determined. We will solve such a nonlinear problem by means of the following iterative computation. The solution procedure for the A-Series involves the following steps:

- (1) Obtain a linear elastic solution and then set the shell at the elastic limit state.
- (2) Give an external load increment and set the counter of loading steps equal to zero.
- (3) Assume the values of $\{n_j^p\}$ and $\{m_j^p\}$.
- (4) Start iterative computation.
- (5) Obtain the incremental solutions for all the Fourier indices under consideration.
- (6) Compute the improved values $\{n_j^{p*}\}$

and $\{m_j^p\}$ of $\{n_j^p\}$ and $\{m_j^p\}$ from the Fourier analysis of Eq. (36).

(7) If the solutions do not converge, return to the step (4), replacing $\{n_j^p\}$ and $\{m_j^p\}$ by $\{n_j^{p*}\}$ and $\{m_j^{p*}\}$. If they do, proceed to the next step.

(8) Integrate the increments by the Euler method.

(9) Return to the step (3), advancing the loading step counter from l to $l+1$, if further loading steps are to be analyzed. Otherwise, finish the computation.

For the criterion of the convergence concerning the above iteration, we use the following relation:

$$\frac{(|\{n_j^{p*}\}_k - \{n_j^p\}_k|^2 + |\{m_j^{p*}\}_k - \{m_j^p\}_k|^2)}{(|\{n_j^{p*}\}_k|^2 + |\{m_j^{p*}\}_k|^2)} < \delta \quad (49)$$

where δ is an arbitrary small tolerance limit of convergence, which is assumed to be 10^{-3} in this study.

4. Numerical Results and Considerations

4.1 Example of cylindrical shell

Let us consider a circular cylindrical shell which has the mean radius a , the axial length l and the thickness h (assumed to be constant), and is subjected to the bending moment $M = M_0 \cos 2\theta$ at the end of the shell as shown in Fig. 5.

We investigate the case in which $l=a$ and $h=0.02a$. Analysis is carried out only for the portion AB because of symmetry. The quantities concerning the shell geometry in this case are as follows:

$$\rho=1, \rho'=0, \gamma=0, \omega_\xi=0, \omega_\theta=1, \lambda=0.02 \quad (50)$$

In consideration of symmetry we assume that at the point A

$$\Delta U_\xi = \Delta W' = \Delta M'_\xi = \Delta T_{\xi\theta} = 0 \quad (51)$$

The boundary conditions at the point B are given by

$$\begin{aligned} \Delta U_\xi &= \Delta U_\theta = \Delta W = 0 \\ \Delta M'_\xi &= \Delta M_0 \cos 2\theta \end{aligned} \quad (52)$$

The computer program developed in this research can also treat a linear elastic problem, if the terms resulting from plastic deformation are neglected. In order to demonstrate the appropriateness of the present

Fig. 5 Circular cylindrical shell subjected to end moment

computer program, we try to obtain in the above-stated manner the linear elastic solutions when the shell is subjected to the end moment $M = M_0 \cos n_l \theta$ as n_l takes the values of 0, 2 and 5.

In Fig. 6 the present solutions (solid curve) are compared with the Budiansky-Radkowski solutions [9]. Fairly good agreement between them is recognized. We show the results obtained when $N=51$. It was ascertained that sufficiently good accuracy could be obtained when $N=51$, by computing other cases of N -value. Poisson's ratio was assumed to be 0.3, and in the following we shall also use the same value.

Fig. 6 Transverse displacement W (elastic solutions)

We shall next obtain an elastoplastic solution of the case in which the shell is subjected to the end moment such that $m_\xi = m_0 \cos 2\theta$. We assume a linear hardening material with an isotropic strain hardening in which

$$E/\sigma_0 = 600, E_p = 0.02E \quad (53)$$

in the numerical computation we assume the number of representative points located along the circumference of the shell for the Fourier analysis K , the division number along the shell thickness M , the number of the maximum Fourier index considered N_ξ as $K=16$, $M=16$ and $N_\xi=10$, respectively. The external load increment is assumed to be one-tenth of the elastic limit load. It was confirmed that we were able to obtain sufficiently accurate results under these computation conditions by carrying out calculation under various numerical conditions. For example, when we put $K=14$ and $N_\xi=18$ and assume the same values for the other, the maximum values of $WE/(a\sigma_0)$ and $N_\xi/(\sigma_0 h)$ fluctuate only within 0.04% and 1.5%, respectively, when the external load is 2.2 times the elastic limit load.

Figure 7 shows deformed configurations of the cross sections perpendicular to the axis of symmetry. The dotted curve in the figure indicates the deformation path of a point. $(m_0)_l$ denotes the value m_0 of $m_0 \times \cos 2\theta$ at the elastic limit state, and $2.2 \times (m_0)_l$ the deformation state subjected to the end moment $2.2(m_0)_l \cos 2\theta$.

Figure 8 illustrates the distributions of n_ξ and n_θ along the ξ -direction in the meridional section of $\theta=0^\circ$. Figure 9 shows

the distributions of m_ξ and m_θ in the same meridional cross section. It might be concluded that effects due to plastic deformation are not remarkable in the present example, while a slight difference between the present solution and the linear elastic one can be recognized in the vicinity of $\xi=0.45$.

Fig.7 Deformed shapes of cross sections

Fig.8 Membrane forces

Fig.9 Bending moments

4.2 Example of circular plate

We shall now analyze the circular plate which Witmer and Kotanchik [7] tried to compute by means of the finite element method. The circular plate is shown in Fig.10. It is considered to be clamped at the edge. The plate is made of 6061-T6 aluminium alloy and subjected to distributed loads which act in two small sectors indicated in the figure. These distributed loads are assumed to be uniform and normal to the plate middle surface. Taking as the reference length a the radius 2.5 in. which is equal to the distance from the center of the plate to the point clamped by the boss, we have for the quantities concerning the shell geometry

$$\rho=\xi, \rho'=1, \gamma=1/\rho, \omega_\theta=\omega_\xi=0, \lambda=0.1 \quad (54)$$

In order to avoid the difficulty occurring at the pole ($\rho=0$), the plate is idealized to have a small circular hole at the center with

Fig.10 Dimensions of circular plate used in experiment [7]

the radius 0.02 in. in the finite element analysis in reference [7]. We also adopt this idealization, and use at the circumference of this small circle the boundary condition of free edge, that is,

$$\Delta N_\xi = \Delta F_\xi = \Delta T_{\xi\theta} = \Delta M_\xi = 0 \quad (55)$$

At the outer circumference of the plate, we apply the boundary conditions of clamped edge at various radii of the plate. The effective radius of the clamped plate is determined such that the numerical results agree well with the experimental results in the elastic region. In general, the effective radius is dependent on θ in case of asymmetrical deformation. For simplicity we neglect in this study this dependency on θ . From such numerical computations it was confirmed that if we used for $\rho=1.06$ the boundary conditions of clamped edge:

$$\Delta U_\xi = \Delta W = \Delta U_\theta = \Delta \phi_\xi = 0 \quad (56)$$

We could obtain the results as shown in Fig. 11 in good agreement with the experimental results. In reference [7] an annular plate element supported at points $\rho=1$ and $\rho=1.136$ is taken into consideration to simulate the edge conditions.

The uniaxial stress-strain relation given in [7] is approximated by the method of least squares as follows:

$$\hat{\eta} = 26.2(\hat{\sigma}_e - 1)^{3.09} \quad (57)$$

where

$$\hat{\eta} = (E/\sigma_0)\eta, \hat{\sigma}_e = \sigma_e/\sigma_0 \quad (58)$$

The material constants were assumed to be

$$\nu=0.3, \sigma_0=22.6 \text{ kg/mm}^2, E/\sigma_0=323 \quad (59)$$

Here η and σ_e are the equivalent plastic strain and the equivalent stress, respectively. The load distributions were approximated by the Fourier series with 9 harmonics as in [7]. Because of symmetry we analyzed only the region of $0^\circ \leq \theta \leq 180^\circ$ and we put $K/2=16$, $M=16$ and $N=51$.

Figures 12 through 14 show the results obtained when the pressure $P/(\lambda\sigma_0) = 19.2$ is applied at the sectors indicated in Fig. 10. In every case, the present solutions

(solid curve) agree well with the finite element solutions, although it could be somewhat difficult to obtain a good quantitative agreement between the present solutions and the measured data.

Fig. 11 Radial strain on plate surface of $\bar{z}=0.5$ (elastic state)

Fig. 12 Transverse displacement

Fig. 13 Radial strain on plate surface of $\bar{z}=0.5$

Fig. 14 Radial strain on plate surface of $\bar{z}=0.5$

5. Conclusions

In the present report, we have treated the elastoplastic problems of shells of revolution subjected to asymmetrical loadings, and proposed a solution procedure using the finite difference technique from the standpoint of the incremental theory of plasticity.

At present, a number of large-scale computer programs using the finite element techniques are available for almost all the kinds of structural analysis. Nevertheless, a solution procedure based on the finite difference technique such as the present one is still one of the most effective solution procedures for at least the elastoplastic problems of axisymmetric shells, because of less input data and also less computation time.

While the present work was carried out, a similar solution procedure was proposed in reference [10] for creep deformation of axisymmetric shells. The authors should note that they learnt much from this solution.

References

- [1] Stricklin, J.A., et al, *Trans. ASME*, Ser.J., 96-2(1974-5), p.87.
- [2] Zienkiewicz, O.C., *The Finite Element Method in Engineering Science*, (1971), McGraw-Hill.
- [3] Hamada, M. and Tanaka, M., *Bull. JSME*, 14-74(1971), p.724.
- [4] Hamada, M. and Tanaka, M., *Bull. JSME*, 15-87(1972), p.1029.
- [5] Hamada, M. and Tanaka, M., *Bull. JSME*, 17-103(1974), p.41.
- [6] Tanaka, M., *Ing.-Arch.*, 41-5(1972), p.342.
- [7] Witmer, F.A. and Kotanchik, J.J., *Proc. 2nd. Matrix Meths. Struct. Mech.*, Wright-Patterson Air Force Base, Ohio, 15-17 Oct., 1968, AFFDL and AFIT, (1969-12), p.13.
- [8] Sanders, J.L., Jr., *NASA Rept.*, No.24 (1959-6).
- [9] Budiansky, B. and Radkowski, P.P., *AIAA Jour.*, 1-8(1963-8), p.1833.
- [10] Takezono, S., et al., *Bull. JSME*, 18-123(1975), p.925.