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Periodically Oscillating Turbulent Flow in a Pipe *

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A fluctuating eddy viscosity model for Reynolds stress is proposed. The validity of this model is verified for all frequencies tested through comparisons between the calculations and measurements. The analysis yields three properly-modified Strouhal numbers: the first of them is adequate to illustrate similarly the velocity distributions in turbulent region over a wide range of frequencies and the time-mean Reynolds numbers, the second is adequate in the vicinity of wall, the last is for the instantaneous friction factor. The following results are obtained: at low frequencies the amplitude of the wall shear stress is proportional to the amplitude of the velocity. At high frequencies it is proportional to both the velocity amplitude and the square root of the frequency. The predicted magnitude of the instantaneous friction factor exceeds that of steady state during accelerating flow and it is smaller than that during decelerating flow.

1. Introduction

The investigations in unsteady-flow problems began to be done intensely during recent years, since in this period there had appeared an urgent need for more accurate calculations of unsteady-state processes in thermo-fluid dynamics.

Oscillating turbulent flow in a pipe, which may be one of the most fundamental flows in engineering, has been studied by F.T. Brown *et al* [1], T. Mizushima *et al* [2], and S. Oomi *et al* [3,4]. Their work [1,3,4] is formulated with the assumption that the eddy viscosity does not vary with time while the flow is oscillating, or that the friction factor of pipe is not different from that of steady flows.

In this paper, a model of fluctuating eddy viscosity is proposed to calculate the velocity distribution and the friction factor of turbulent pipe flow with periodically oscillating components superposed on the main turbulent flow. The validity of this theoretical model is confirmed by experimental measurements. With a properly modified Strouhal number, the amplitude and phase of the velocity and the friction factor can be expressed almost similarly over a wide range of the time-mean Reynolds numbers and the frequencies of the fluctuation.

2. General Statement for Equation

In what follows we shall concern ourselves with a fully developed and incompressible turbulent flow through a straight rigid pipe with a circular cross section. The coordinate along the pipe axis is denoted by x and the normal to it is by r . The equation of motion is

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\nu \frac{\partial U}{\partial r} - \overline{uv} \right) \right] \quad \dots\dots\dots (1)$$

where P and U are pressure and fluid velocity in the direction of the pipe axis, respectively. Total eddy viscosity ν_T , i.e. the sum of kinematic viscosity ν and the eddy viscosity ϵ , and shear stress τ are defined by

$$\nu_T = \nu + \epsilon \quad \dots\dots\dots (2)$$

$$\tau / \rho = -\nu_T \frac{\partial U}{\partial r} \quad \dots\dots\dots (3)$$

Then Eq.(1) reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \nu_T \frac{\partial U}{\partial r} \right) - \frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial U}{\partial t} = 0 \quad \dots\dots\dots (4)$$

On the assumption that $\partial P / \partial r = 0$, the integration of Eq.(4) with respect to r from 0 (pipe axis) to R (pipe wall) yields

$$-\frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{2}{R} \frac{\tau_w}{\rho} + \frac{\partial U_m}{\partial t} \quad \dots\dots\dots (5)$$

where τ_w means the wall shear stress and U_m denotes the mean velocity averaged over the cross section.

In the case of steady flow, where $\partial / \partial t = 0$, Eq.(5) reduces to the simple relation:

$$u_*^2 \equiv \tau_w / \rho = \left(-\frac{1}{\rho} \frac{dP}{dx} \right) \frac{R}{2} = \frac{\lambda}{8} U_m^2 \quad \dots\dots\dots (6)$$

where λ is the friction factor of pipe. And Eq.(4) gives upon similar integration

$$\frac{U}{u_*} = \int_0^{y/R} \frac{1-y/R}{\nu_T / u_* R} d\left(\frac{y}{R}\right) = \frac{1}{R_*} \int_0^{y_*} \frac{R_* - y_*}{\nu_T / \nu} dy_* \quad \dots\dots\dots (7)$$

where $y = R - r$, $y_* = u_* y / \nu$ and $R_* = u_* R / \nu$.

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Consequently in the steady state, U/u_* can be calculated from v_T/u_*R . In the core region near the pipe axis, v_T/u_*R has an approximately constant value⁽⁵⁾, then the velocity distribution becomes parabolic:

$$\frac{U_{CL}-U}{u_*} = \frac{1}{2} \frac{1}{C_c} \left(\frac{r}{R}\right)^2, \quad C_c = \left(\frac{\nu_T}{u_*R}\right)_{\text{core}} \quad \dots\dots\dots(8)$$

where U_{CL} is the velocity at $r = 0$ and C_c denotes the value of ν_T/u_*R in the core region.

3. A Model of Total Kinematic Viscosity ν_T ($= \nu + \epsilon$)

The eddy viscosity ϵ may fluctuate in accordance with flow oscillation. The fluctuation of ϵ will be assumed in the following section on the basis of similarity of ϵ/u_*R in steady flows. In this section an algebraic ν_T model in steady state is developed.

The cross section of a pipe is divided into five flow-regions, namely viscous layer next to the wall, and toward the center of the pipe, buffer I layer, buffer II layer, logarithmic region and core region. An algebraic model for ν_T in each layer is as follows:

at viscous layer

$$\frac{\nu_T}{\nu} = 1 \quad 0 \leq y_* \leq \frac{1}{a} \equiv y_{*VB} \quad \dots\dots\dots(9)$$

at buffer I layer

$$\frac{\nu_T}{\nu} = ay_* \quad \frac{1}{a} \leq y_* \leq \frac{a}{C_B} \quad \dots\dots\dots(10)$$

at buffer II layer

$$\frac{\nu_T}{\nu} = C_B y_*^2 \quad \frac{a}{C_B} \leq y_* \leq \frac{\kappa}{C_B + \kappa^2/4C_m R_*} \equiv y_{*BT} \quad \dots\dots\dots(11)$$

in logarithmic region

$$\left. \begin{aligned} \frac{\nu_T}{\nu} &= \kappa y_* [1 - (\kappa/4C_m)(y_*/R_*)] \\ \kappa/(C_B + \kappa^2/4C_m R_*) &\leq y_* \\ &\leq \frac{1}{\kappa} 2C_m(1 + \sqrt{1 - C_c/C_m})R_* \end{aligned} \right\} \quad \dots\dots\dots(12)$$

in core region

$$\left. \begin{aligned} \frac{\nu_T}{\nu} &= C_c R_* \\ \frac{1}{\kappa} 2C_m(1 + \sqrt{1 - C_c/C_m})R_* &\leq y_* \leq R_* \end{aligned} \right\} \quad \dots\dots\dots(13)$$

where the constant values we used are:

$$\left. \begin{aligned} a &= 0.19, \quad C_B = 0.011, \quad \kappa = 0.41, \quad C_m = 0.077 \\ C_c &= \text{a function of Re (ref. Fig.3)} \end{aligned} \right\} \quad \dots\dots\dots(14)$$

The distributions of ν_T in the present model as well as in models proposed by other authors[1,3] are illustrated in figs. 1 and 2. Measurements[6] are also shown in the same figure. The models [1] and [3] show a jump in ν_T at where y_* is about 30. The velocity distribution in r direction calculated with their ν_T model may have a kink even if the shear stress is continuous at this point. The value of C_c in Eq.(14) is evaluated by use of both Eq.(8) and the velocity distributions[7]. Fig.3 shows the plotted C_c as a function of Reynolds num-

Fig.1 Total eddy viscosity distribution near the wall and its approximations.

Fig.2 Total eddy viscosity distribution over the cross section of a pipe and its approximations.

Fig.3 Variation in C_c with Reynolds number.

ber. A solid curve in Fig.3 is drawn smoothly so that it may fit the data. We use this curve when we determine the value of C_0 at a certain Reynolds number. It will be seen from Figs. 1 to 3 that the actual distribution of v_T can be approximated more accurately by the present author's five-layers model, that is, Eqs.(9) - (14).

The equation of motion is solved in the following section, where it will be seen that the term ϵ/v_T appears in the form of $(1/2r) \cdot d(r^2 \epsilon/v_T)/dr$. Hence, we introduce new variables g and f defined by

$$g = \epsilon/v_T = 1 - v/v_T, \quad f = \frac{1}{2r} \frac{d}{dr}(r^2 g).$$

The variables g and f are functions of only v_T/v . In the present paper, however, we assume a simple form of g given by

$$g = \begin{cases} 0 & 0 \leq y_* \leq y_{*VB} \\ 1 + \left(\frac{y_* - y_{*BT}}{y_{*BT} - y_{*VB}} \right)^3 & y_{*VB} \leq y_* \leq y_{*BT} \\ 1 & y_{*BT} \leq y_* \end{cases} \quad (15)$$

The variable f may be written in the form

$$\frac{f}{R_*} = \frac{1}{R_*} \frac{1}{2r} \frac{d}{dr}(r^2 g) = \frac{g}{R_*} - \frac{1}{2} \left(1 - \frac{y_*}{R_*} \right) \frac{dg}{dy_*} \quad (16)$$

In Fig.4, experimental data show that the equation approximates g within 10 per cent error. The distribution of f versus y_* , which is estimated from the experimental results in Fig.1, is shown in Fig.4 in a shaded region. The value of f near the wall ($y_* < 10$) may have a large error because

Fig.4 Distribution of $(1-v/v_T)$ and f/R_* , and its approximation.

there is a lot of difficulties in experimental measurements of ϵ . The approximations of f with g given by Eq.(15) are illustrated in Fig.4 with the solid and chain lines at Reynolds number, $Re = 10^4$ and $Re = 10^6$, respectively. The circle on these lines indicates the boundary, y_{*BT} , between Buffer II layer and logarithmic region. Each line has a discontinuity at $y_* = y_{*VB}$, i.e. the boundary between the viscous and buffer I layers, because of $g = f = 0$ in the viscous layer in this model.

4. A Model of Fluctuating Eddy Viscosity and the Momentum Equation

Periodically oscillating flow quantities are given, in the most general form, by

$$U_n = U_{n0} + \sum_{m=1}^{\infty} U_{nm} e^{i n \omega t} \quad (17)$$

$$U = U_0 + \sum_{n=1}^{\infty} U_n e^{i n \omega t} \quad (18)$$

$$-\frac{1}{\rho} \frac{\partial P}{\partial x} = K_0 + \sum_{n=1}^{\infty} K_n e^{i n \omega t} \quad (19)$$

$$\epsilon = \epsilon_0 + \sum_{n=1}^{\infty} \epsilon_n e^{i n \omega t} \quad (20)$$

where ω is the frequency of the flow fluctuation; the subscripts 0 and n refer to the time-mean value (real) and the n -th harmonic amplitude (complex), respectively. The quantities ϵ_0 and ϵ_n vanish in the viscous layer. The notations for other flow quantities are expanded in a similar way:

$$\tau = \tau_0 + \sum_{n=1}^{\infty} \tau_n e^{i n \omega t} \quad (21)$$

$$\tau_w/\rho = -\nu \left(\frac{\partial U}{\partial r} \right)_{r=R} = \tau_{w0}/\rho + \sum_{n=1}^{\infty} \frac{\tau_{wn}}{\rho} e^{i n \omega t} \quad (22)$$

$$u_* = u_{*0} + \sum_{n=1}^{\infty} u_{*n} e^{i n \omega t} \quad (23)$$

where $u_* \equiv \sqrt{\tau_w/\rho}$ is the friction velocity.

In these Eqs.(17)-(23), the real part of the right hand side has physical meaning.

If the consideration is restricted to the case of $\tau_w = \tau_{w0} + \tau_{wn} \exp(i n \omega t)$ and $|\tau_{wn}/\tau_{w0}| < 1$ to obtain the relation between u_{*n} and τ_{wn} , a power series expansion of $\sqrt{\tau_w/\rho}$ gives

$$1 + \frac{u_{*n}}{u_{*0}} e^{i n \omega t} = 1 + \frac{1}{2} \frac{\tau_{wn}}{\tau_{w0}} e^{i n \omega t} - \frac{1}{16} \left(\frac{\tau_{wn}}{\tau_{w0}} \right)^2 e^{i 2 n \omega t} - \frac{1}{16} \left(\tau_{wn}/\tau_{w0} \right) \left(\tau_{wn}/\tau_{w0} \right)$$

where an over-bar denotes the complex conjugate. On the right hand side of this equation, if we assume that the absolute magnitude of the second order term ($2n\omega$) should not exceed 5 per cent of that of the first term ($n\omega$), that is,

$$|\tau_{wn}/\tau_{w0}| \leq 0.4, \quad |U_{mn}/U_{m0}| \leq 0.29 \quad (24)$$

where the velocity ratio is deduced from Eq.(48) and $Hn = 1.4$ in Fig.13, then the magnitude of the induced steady term (second order term) becomes less than one per cent of the zero-th order term, 1. On this condition, neglecting the second order terms, we get

$$u_{*n}/u_{*0} = \frac{1}{2}(\tau_{wn}/\tau_{w0}) \dots (25)$$

In steady flows, there exists a similarity in ϵ/u_*R over a wide range of Reynolds numbers as shown in Figs.1-4. The same similarity may exist in a fluctuating flow provided its frequency and the amplitude of the velocity are not so large. Then, we assume

$$\begin{aligned} (\epsilon/u_*R)_{\text{unsteady}} &= (\epsilon_0/u_{*0}R)_{\text{time-mean}} \\ &= (\epsilon/u_*R)_{\text{steady}} \end{aligned}$$

Using Eqs.(20), (23) and (25), we get the following expression for ϵ_n :

$$\frac{\epsilon_n}{\nu\tau_0} = \frac{1}{2} \frac{\tau_{wn}}{\tau_{w0}} g \quad (\nu\tau_0 = \nu + \epsilon_0) \dots (26)$$

where $\nu\tau_0$, τ_{w0} and g are evaluated from Eqs.(9) - (14), (6) and (15) at the time-mean Reynolds number, Re_0 , respectively. The value of τ_{wn} is to be determined when the momentum equation is solved in the next section.

Substituting Eqs.(2), (18), (19) and (20) into Eq.(4), neglecting the higher order terms, and collecting the same harmonic terms, we get

$$\frac{1}{r} \frac{d}{dr} \left[r \nu \tau_0 \frac{dU_0}{dr} \right] + K_0 = 0 \dots (27)$$

$$\frac{1}{r} \frac{d}{dr} \left[r \left(\nu \tau_0 \frac{dU_n}{dr} + \epsilon_n \frac{dU_0}{dr} \right) \right] + K_n - i n \omega U_n = 0 \dots (28)$$

If we put $\epsilon_n = 0$ in Eq.(28), the latter reduces to the equation proposed by F.T.Brown [1] and S.Oomi[3]. Equation (28) is rewritten, by using Eqs.(26), (27) and (6), and by putting $y = R - r$, as:

$$\begin{aligned} \frac{\nu\tau_0}{\nu} \frac{d^2 U_n}{d(y/R)^2} + \frac{1}{1-y/R} \left[\frac{d}{d(y/R)} \right. \\ \times \left. \frac{\nu\tau_0}{\nu} \left(1 - \frac{y}{R} \right) \right] \frac{dU_n}{d(y/R)} - i \frac{R^2 n \omega}{\nu} \\ \times \left(U_n + \frac{i}{n \omega} K_n \right) = \frac{R^2 n \omega}{\nu} \frac{1}{n \omega} \frac{\tau_{wn}}{\rho R} f \end{aligned} \dots (29)$$

The expression for τ_n in Eq.(21) is obtained from the first term of Eq.(28). The volumetric flow rate of the incremental flow Q_n is expressed by U_{mn} . Hence, we have

$$\begin{aligned} U_n &= C_1 I_0(i^{1/2} z) - D_1 K_0(i^{1/2} z) + Y_1 - \frac{i}{n \omega} K_n, \quad z = 2\sqrt{q_1 y} \\ Y_1 &= -\frac{1}{n \omega} \frac{\tau_{wn}}{\rho R} \frac{3}{2(y_{BT} - y_{VB})^2} \left\{ \frac{2}{q_1} (2y - y_{BT}) + i \left(y - y_{BT} + \frac{2}{q_1} \right) \left(y - y_{BT} - \frac{2}{q_1} \right) \right\} \end{aligned} \dots (30)$$

where C_1 and D_1 are integral constants, and I_0 and K_0 are Kelvin functions of order zero.

Buffer II layer: Substitution of Eqs.(32) and (11) into Eq.(29) multiplied by $(1-y)$ leads to

$$y^2(1-y) \frac{d^2 V_n}{dy^2} + y(2-3y) \frac{dV_n}{dy} - i q_1 (1-y) V_n = q_1 \frac{1}{n \omega} \frac{\tau_{wn}}{\rho R} (1-y) f, \quad q_1 = \frac{Q_n}{C_B R_{*0}^2} \dots (31)$$

A solution of the homogeneous equation of Eq.(39) was obtained in terms of an infinite power series:

$$w = y^p \sum_{k=0}^{\infty} C_k y^k \quad C_0 = 1 \quad C_{k+1} = \frac{(2k+1)\rho + k(k+2)}{(k+1)(2\rho + (k+2))} C_k \quad (k=0, 1, 2, \dots)$$

$$\frac{\tau_n}{\rho} = \frac{\nu\tau_0}{R} \frac{dU_n}{d(y/R)} + \frac{\tau_{wn}}{\rho} \frac{1}{2} \left(1 - \frac{y}{R} \right) g \dots (30)$$

$$Q_n/\pi R^2 = U_{mn} = 2 \int_0^1 (r/R) U_n d(r/R) \dots (31)$$

5. Solution of Each Layer

Assuming $dK_n/dr=0$, we here obtain a solution of Eq.(29) in each layer. Notations r and y are used instead of r/R and y/R for simplicity, and Ω_n denotes $R^2 n \omega / \nu$.

Viscous Layer: We use a new variable V_n defined as

$$V_n = U_n + i K_n / n \omega. \quad (32)$$

In this layer, y is small enough to assume $y \ll 1$. We can take there $\epsilon=g=f=0$ and $\nu\tau_0=\nu$. Thus Eq.(29) is rewritten in the following form.

$$\frac{d^2 V_n}{dy^2} - \frac{dV_n}{dy} - i \Omega_n V_n = 0, \quad \Omega_n = \frac{R^2 n \omega}{\nu} \dots (33)$$

This has a solution

$$U_n = A e^{\rho_1 y} + B e^{\rho_2 y} - \frac{i}{n \omega} K_n \dots (34)$$

where ρ_1 and ρ_2 are the roots of the characteristic equation of Eq.(33), and A and B are integral constants (complex). Applying no slip condition at the wall and using the definition for τ_{wn} , i.e. Eq.(22), we have

$$A+B = \frac{i}{n \omega} K_n \dots (35)$$

$$\tau_{wn}/\rho = \frac{\nu}{R} (A \rho_1 + B \rho_2) \dots (36)$$

Buffer I Layer: The assumption $y \ll 1$ is also applied in this layer. Substituting Eqs.(32) and (10) into Eq.(29) and putting $y \ll 1$ there, we get

$$\begin{aligned} y \frac{d^2 V_n}{dy^2} + \frac{dV_n}{dy} - i q_1 V_n &= q_1 \frac{1}{n \omega} \frac{\tau_{wn}}{\rho R} f \\ q_1 &= \Omega_n / a R_{*0}, \quad R_{*0} = R u_{*0} / \nu \end{aligned} \dots (37)$$

Referring to Fig.4 and Eq.(16), g/R_* is seen to be negligibly small compared with f/R_* in this layer. Then f in Eq.(37) may be replaced by

$$f = -1.5 (y - y_{BT})^2 / (y_{BT} - y_{VB})^3.$$

A particular integral of Eq.(37) can be obtained easily and it is denoted Y_1 below. A general solution of Eq.(37) is then given by

$$\left. \begin{aligned} U_n &= C_1 I_0(i^{1/2} z) - D_1 K_0(i^{1/2} z) + Y_1 - \frac{i}{n \omega} K_n, \quad z = 2\sqrt{q_1 y} \\ Y_1 &= -\frac{1}{n \omega} \frac{\tau_{wn}}{\rho R} \frac{3}{2(y_{BT} - y_{VB})^2} \left\{ \frac{2}{q_1} (2y - y_{BT}) + i \left(y - y_{BT} + \frac{2}{q_1} \right) \left(y - y_{BT} - \frac{2}{q_1} \right) \right\} \end{aligned} \right\} \dots (38)$$

where C_1 and D_1 are integral constants, and I_0 and K_0 are Kelvin functions of order zero.

Buffer II layer: Substitution of Eqs.(32) and (11) into Eq.(29) multiplied by $(1-y)$ leads to

$$y^2(1-y) \frac{d^2 V_n}{dy^2} + y(2-3y) \frac{dV_n}{dy} - i q_1 (1-y) V_n = q_1 \frac{1}{n \omega} \frac{\tau_{wn}}{\rho R} (1-y) f, \quad q_1 = \frac{Q_n}{C_B R_{*0}^2} \dots (39)$$

A solution of the homogeneous equation of Eq.(39) was obtained in terms of an infinite power series:

$$w = y^p \sum_{k=0}^{\infty} C_k y^k \quad C_0 = 1 \quad C_{k+1} = \frac{(2k+1)\rho + k(k+2)}{(k+1)(2\rho + (k+2))} C_k \quad (k=0, 1, 2, \dots)$$

where ρ is a root of the characteristic equation of Eq.(39) and C_k is given by the above recurrence formula. It may be difficult to get a particular integral of Eq.(39) exactly. Then an approximate solution is obtained by assuming that $y \ll 1$ on the left hand side of Eq.(39). The result is

$$\left. \begin{aligned} U_n &= Cw_1 + Dw_2 + Y_1 - \frac{i}{n\omega} K_n \\ Y_1 &= \left\{ \frac{60y^2}{144+q_1^2} - \frac{18(4y_{BT}+1)y}{36+q_1^2} + \frac{6y_{BT}(3y_{BT}+2)}{4+q_1^2} \right\} yT + i \left\{ \frac{5y^3}{144+q_1^2} - \frac{3(4y_{BT}+1)y^2}{36+q_1^2} \right. \\ &\quad \left. + \frac{3y_{BT}(3y_{BT}+2)y}{4+q_1^2} + \frac{1}{q_1^2} [-y_{BT}^2(2y_{BT}+3) + 2(y_{BT}-y_{VB})^2] \right\} q_1 T \\ T &= q_1 \frac{1}{n\omega} \frac{\tau_{wn}}{\rho R} \frac{1}{2} \frac{1}{(y_{BT}-y_{VB})^2} \end{aligned} \right\} \dots\dots(40)$$

Logarithmic Region: In this region and the following core region, Eqs.(15) and (16) show that $g = f = 1$. This enables us to change the dependent variable to

$$V_n = U_n + \frac{i}{n\omega} (K_n - \tau_{wn}/\rho R) \dots\dots\dots(41)$$

Substituting Eqs.(41) and (12) into Eq.(29) multiplied by $(1-y)/\kappa R_{*0}$, we obtain

$$\left. \begin{aligned} y(1-py)(1-y) \frac{d^2 V_n}{dy^2} + \frac{d}{dy} [y(1-py) \\ \times (1-y)] \frac{dV_n}{dy} - i q_T (1-y) V_n &= 0 \\ p = \frac{\kappa}{4C_m}, \quad q_T = \frac{Q_n}{\kappa R_{*0}} \end{aligned} \right\} \dots\dots\dots(42)$$

A solution of this equation is given by

$$\left. \begin{aligned} U_n &= Ew_1 + Fw_2 - \frac{i}{n\omega} (K_n - \tau_{wn}/\rho R) \\ w_1 &= \sum_{k=0}^{\infty} C_k y^k, \quad w_2 = w_1 \ln y + \sum_{k=1}^{\infty} a_k y^k \\ k^2 C_k - [(k-1)k(p+1) + i q_T] C_{k-1} \\ &\quad + [(k-2)kp + i q_T] C_{k-2} = 0 \quad (k \geq 2) \\ C_0 &= 1, \quad C_1 = i q_T, \quad a_k = \left[\frac{\partial C_k(\rho)}{\partial \rho} \right]_{\rho=0} \end{aligned} \right\} \dots\dots\dots(43)$$

Core Region: Using Eq.(41) and r instead of y in Eq.(29) which is divided by $C_0 R_{*0}$, we obtain Kelvin's differential equation.

$$\frac{d^2 V_n}{dr^2} + \frac{1}{r} \frac{dV_n}{dr} - i q_c V_n = 0, \quad q_c = \frac{Q_n}{C_0 R_{*0}} \dots\dots\dots(44)$$

On the condition that the velocity should have a finite value at $r=0$, a solution is given by

$$U_n = G I_0(i^{1/2} z) - \frac{i}{n\omega} \left(K_n - \frac{\tau_{wn}}{\rho R} \right), \quad z = r \sqrt{q_c} \dots\dots\dots(45)$$

The solutions for each layer has been obtained with eleven unknown constants, i.e. $K_n, \tau_{wn}, A, B, \dots, G$. These constants are to be determined from the following three conditions: (1) The velocity U_n and its gradient are connected continuously at the boundary between respective layers; (2) the wall conditions of Eqs.(35) and (36) must be fulfilled; and (3) either K_n or U_n at any point $y(\neq 0)$ must be given.

It may be added here that K_n, U_n and τ_{wn} are proportional to each other, because of the linearity of Eq.(29).

6. Amplitude of Flow Rate and Instantaneous Friction Factor

Substituting Eqs.(17), (19) and (22) into Eq.(5), and collecting the same harmonic terms, we obtain the following relation.

$$\frac{i n \omega U_{mn}}{K_n} + \frac{2 \tau_{wn}/\rho R}{K_n} = 1 \dots\dots\dots(46)$$

This is the equation concerning the amplitudes of flow rate and the wall shear stress relative to the amplitude of pressure gradient. On the other hand, we can get a similar relation by integrating Eq.(29). The result shows that the left hand side of Eq.(46) is slightly different from unity, because in the latter case we used the discontinuous distributions of f and dv_r/dr at the boundary between respective layers. If its error exceeds 5% in some cases, we will reject the calculated result.

A part of the pressure gradient K_n , which balances with the wall shear stress τ_{wn} , is represented by the following equation in terms of a new friction factor λ_n and the cross sectional mean velocity U_{mn} .

$$K_n - i n \omega U_{mn} = \frac{2 \tau_{wn}}{\rho R} \equiv \lambda_n \frac{1}{2D} U_{m0} U_{mn} \dots\dots(47)$$

where $D(=2R)$ denotes the diameter of the pipe. Dividing Eq.(47) by Eq.(6) with the subscript 0, which means the time mean values, we obtain

$$\left. \begin{aligned} \frac{\tau_{wn}}{\tau_{w0}} &= H_n \frac{U_{mn}}{U_{m0}} \\ H_n &\equiv \frac{\lambda_n}{\lambda_0} = 8 \frac{2 \tau_{wn}/\rho R}{n \omega U_{mn}} \left(\frac{Q_n}{\lambda_0 R_{*0}} \right) \end{aligned} \right\} \dots\dots\dots(48)$$

where Re_0 denotes DU_{m0}/ν and λ_0 means the friction factor for the steady turbulent pipe flow at Re_0 .

We here define an instantaneous friction factor λ in the same way as in steady flows:

$$\left. \begin{aligned} \frac{2}{\rho R} \left\{ \tau_{w0} + \sum_{n=1}^{\infty} \Re(\tau_{wn} e^{i n \omega t}) \right\} \\ \equiv \lambda \frac{1}{2D} \left\{ U_{m0} + \sum_{n=1}^{\infty} \Re(U_{mn} e^{i n \omega t}) \right\}^2 \\ R_e \equiv \frac{D}{\nu} \left\{ U_{m0} + \sum_{n=1}^{\infty} \Re(U_{mn} e^{i n \omega t}) \right\} \end{aligned} \right\} \dots\dots\dots(49)$$

where the symbol $\Re$ denotes the real part of the complex variable. The instantaneous friction factor λ in Eq.(49) may be rewritten in two ways for convenience. The one way is to represent λ in terms of λ_0 :

$$\lambda = \lambda_0 \left(\frac{R_{e0}}{R_e} \right)^2 \left\{ 1 + \sum_{n=1}^{\infty} \Re \left(H_n \frac{U_{mn}}{U_{m0}} e^{in\omega t} \right) \right\} \dots (50)$$

The other way is to use λ_s , which is obtained experimentally in steady state, for example, $\lambda_s = \alpha Re^{-1/k}$ where in Blasius' formula $\alpha = 0.3164$ and $k = 4$. Then we have

$$\lambda/\lambda_s = \left(\frac{R_{e0}}{R_e} \right)^{2-1/k} \left\{ 1 + \sum_{n=1}^{\infty} \Re \left(H_n \frac{U_{mn}}{U_{m0}} e^{in\omega t} \right) \right\} \dots (51)$$

Provided that the value of H_n is obtained by using the present theory, we can easily calculate the instantaneous friction factor from Eq.(50) or (51).

In a quasi-steady theory, it is assumed that the instantaneous friction factor λ coincides with that obtained in a steady state. Hence, the ratio of λ/λ_s is equal to unity at any instance.

7. Comparisons with other theories

At low frequencies, the quasi-steady theory describes the flow fairly correctly. At high frequencies, F.T.Brown[1] and S.Oomi[3] assumed that there is no fluctuation in the eddy viscosity whereas the flow rate is oscillating. At intermediate frequencies, however, the eddy viscosity is fluctuating according to the flow oscillation. This is the case the present au-

thors treat. Hereafter, the result obtained with either Eq.(26) or $\epsilon_n = 0$ are compared with the results of the quasi-steady and Oomi's theories. These results for $Re_0 = 5.7 \times 10^4$ are shown in Figs.5 and 6. A properly modified Strouhal number $\sqrt{\Omega_n}/R_{*0}$ is used as the abscissa, since the volumetric flow rate through the pipe is predominantly determined from the velocity profiles in the logarithmic and the core regions and there the equation of motion has a common parameter $\sqrt{\Omega_n}/R_{*0}$ except a slight change in C_c with Reynolds number as shown in Fig.3.

It can be seen from Figs.5 and 6 with a help of Eq.(46) that at low frequencies the pressure gradient K_n balances with the wall shear stress τ_{wn} as the quasi-steady theory indicates. At high frequencies, since the magnitudes of K_n and $n\omega U_{mn}$ become extremely large compared with $2\tau_{wn}/\rho R$, the quasi-steady theory can predict almost accurately the relation between K_n and $n\omega U_{mn}$ but it predicts the phase angle between K_n and τ_{wn} to be -90° whereas other theories predict it to be -45° .

Compare the present $\epsilon_n = 0$ model with Oomi's model. Both models give almost the same results in the range of $\sqrt{\Omega_n}/R_{*0} > 0.6$. Oomi's model, however, leads to less accurate results of U_{mn} and $|\tau_{wn}|$ at lower frequencies, i.e. $\sqrt{\Omega_n}/R_{*0} < 0.3$.

In comparison among the three models except Oomi's model, the present $\epsilon_n \neq 0$ model (using Eq.(26)) gives almost the same result as obtained by the quasi-steady model at low frequencies, and gives almost the same result as obtained by the present $\epsilon_n = 0$ model at high frequencies. This is clearly understood from Fig.6 which shows the phase angle of τ_{wn} . Consequently, we may say that the present $\epsilon_n \neq 0$ model gives a reasonably good prediction for all frequencies.

Fig.5 Variation in velocity averaged over the cross section of a pipe with frequency (comparison of the present theory and other theories).

Fig.6 Variation in the wall shear stress with frequency (comparison of the present theory and other theories).

Fig.7 Comparison of the time-mean velocity distribution in oscillating turbulent flow and the velocity distribution in a steady state.

8. Experimental Verification

Experimental apparatus consists of a blower and a polyvinyl chloride pipe, having an inner diameter D of 99.4 mm and a length of 135D. An X-type hot-wire probe is located at 105D downstream from the pipe inlet. The pipe is connected horizontally to the suction side of the blower. A butterfly-type valve, which is rotatable, is placed between the blower and the pipe to generate an oscillation of the flow.

In every one cycle of flow oscillation, 360 deg, a lamp and photo-transistor system generates 24 clock pulses at an interval of 15 deg. Each clock pulse has a duration of 3 deg during which data are accumulated by 6 kHz sampling pulses. The data accumulation is repeated cycle by cycle until the number of data amounts to 3 000 for each corresponding clock pulse to make a short-time average of the data. Repeating the same method at various positions from the wall to the pipe axis, we obtain the instantaneous flow situations over the cross section of the pipe.

The experimental conditions we tested are $Re_0 = 1.4 \times 10^5$, $\sqrt{\Omega_1}/R_{*0} = 0.726$, $|U_{m1}/U_{m0}| = 0.355$, $|U_{m2}/U_{m0}| = 0.106$, and $|U_{m3}/U_{m0}| = 0.023$. The time-mean velocity distribution obtained is shown in Fig.7. The time-mean velocity profile agrees fairly well with the prediction by Eqs.(9) - (14) and also with that obtained in a steady turbulent flow. In consequence, it is interpreted that if the value of Re_0 is equal to that of Re of a steady flow, then the time-mean flow-situation almost coincides with that of the steady flow.

Instantaneous velocity profiles obtained are shown in Fig.8 by using ωt as a parameter. The white circles and solid lines represent the measurements and predictions, respectively, for accelerating flow, and the shaded circles and the broken lines do for decelerating flow. According to Fig.8, the measurements agree with the predictions at any time and position except at $\omega t = 2\pi/3$ near the wall.

We, here, compare the predictions with measurements obtained by S.Oomi[8,9] concerning the cross-sectional mean flow, i.e. Eq.(46). The present predictions at time-mean Reynoldsnumbers, $Re_0 = 1 \times 10^4$ and 4×10^5 , are shown in Figs.9 and 10. It may be seen that the abscissa $\sqrt{\Omega_n}/R_{*0}$ is an appropriate parameter to the description of the cross-sectional mean flow, since the two predicted lines are almost the same over a wide range of Reynolds numbers. The cross-sectional mean flow conditions vary remarkably within the frequency range $\sqrt{\Omega_n}/R_{*0}$ from 0.1 to 1.0.

9. Numerical Calculation

Velocity profiles of oscillating components are shown in Fig.11. The vertical axis is nondimensionalized by the velocity of the pipe axis, U_{CL} . The upper curves show the magnitude

Fig.8 Variation in velocity profile with time.

Fig.9 Variation in velocity averaged over the cross section of a pipe with frequency (comparison of measurements and predictions).

Fig.10 Variation in the wall shear stress with frequency (comparison of measurements and predictions).

of the velocity, while the lower curves do the distribution of phase angles. The velocity profile at $\sqrt{\Omega_n}/R_{*0} = 0.2$ is not shown in the same figure because it is almost the same as that of $\sqrt{\Omega_n}/R_{*0} = 0.5$. In the turbulent region, *i.e.* roughly $y/R > 0.2$, it is clearly seen that if a magnitude of $\sqrt{\Omega_n}/R_{*0}$ is given, then one velocity profile is correspondingly determined without relation to Re_0 . Distributions of absolute values have plug-like profiles in high frequencies, and the phase angle has a maximum at $\sqrt{\Omega_n}/R_{*0} = 1$ and then it decreases with an increase in $\sqrt{\Omega_n}/R_{*0}$.

Unfortunately, in Fig.11, the velocity profile near the wall is not shown systematically by $\sqrt{\Omega_n}/R_{*0}$, although it is a suitable parameter for the turbulent region. The flow near the wall is considerably influenced by the wall shear stress. This effect becomes larger at higher frequencies, since the dynamic boundary layer then becomes thinner as is shown in Fig.11. Thus, $\sqrt{\Omega_n}/R_{*0}$ is replaced by $\sqrt{\Omega_n}/R_{*0}$ obtained in Eq.(39) for buffer II layer. Fig.12 shows the velocity profiles near the wall. It will be seen that the similarity of the velocity profiles exists at high frequencies regardless of Re_0 .

Instantaneous friction factor λ can be easily calculated from H_n and either Eq. (50) or (51) as mentioned before. Thus we must get the value of H_n . H_n can be obtained from the experimental results shown in Figs.9 and 10 by using Eq.(48). Then H_n is shown in Fig.13. The average value $\sqrt{\Omega_n}/\lambda_0 Re_0$ of $\sqrt{\Omega_n}/R_{*0}$ for U_{mn} and $\sqrt{\Omega_n}/R_{*0}$ for τ_{wn} is chosen on the abscissa since Eq. (48) includes U_{mn} and τ_{wn} . The calculated H_n varies universally with $\sqrt{\Omega_n}/\lambda_0 Re_0$ regardless of Re_0 . It agrees fairly well with the measurements except large discrepancies in phase angle at high frequencies as was shown in Fig.10. At low frequencies, the predicted H_n is nearly equal to 1.4 so that we can take $\lambda_n = 1.4\lambda_0$. At high frequencies, however, H_n tends asymptotically to $16(\sqrt{\Omega_n}/\lambda_0 Re_0)\exp(i\pi/4)$ and then Eq.(48) leads to

$$\tau_{wn}/\rho \approx 1.0\sqrt{\omega\nu}U_{mn}e^{i(\pi/4)} \dots (52)$$

It is found from Fig.13 that the threshold value of $\sqrt{\Omega_n}/\lambda_0 Re_0$ separating the low and high frequency regions may be 0.1. The phase angle of H_n there is seen to be nearly 22.5 deg, *i.e.* a half of 45 deg.

The calculated ratios of λ/λ_s , when using Eq.(51) and the theoretical H_n , are shown in Fig.14 under the condition of $|U_{mn}/U_{m0}| = 0.25$ and $k = 4$. In the case of extremely low frequencies, the value of λ should agree with that of a steady flow, as is given by Blasius' formula. Therefore $\lambda/\lambda_s = 1$ at any instance. The present theory, however, indicates that λ moves along the line denoted by $H_n=1.4$ in Fig.14. The discrepancy between two theories would result mainly from a linearization of the momentum equation. At intermediate frequencies, the predicted λ/λ_s moves along a closed curve, like an aerofoil section, in clockwise direction as shown in the same

Fig.11 Variation in velocity profile with frequency (over the cross section of a pipe).

Fig.12 Variation in velocity profile with frequency (near the wall).

figure. And the area of the closed curve increases with frequency. At high frequencies, λ/λ_s is asymptotic to the curve calculated from $H_n = 16(\sqrt{\Omega_n}/\lambda_0 Re_0)\exp(i\pi/4)$ though it is not shown in Fig.14.

Roughly speaking, λ is smaller than that of steady state, λ_s , in a decelerating flow and $\lambda > \lambda_s$ in an accelerating flow. In Fig.14, the direction of movement of λ/λ_s , either clockwise or counterclockwise, is uniquely determined by the sign of $\angle H_n$.

Although a some of measurements of $\angle H_n$ obtained by H.Kusama⁽¹⁰⁾ and by S.Oomi(Fig.13) show $\angle H_n < 0$ at low frequencies, most of measurements shown in Fig.13 and the results of L.C.Thomas⁽¹¹⁾ indicate $\angle H_n > 0$ as the present theory predicts.

In addition, it is noted that λ/λ_s can be almost similarly shown in Fig.14 by the parameter $\sqrt{\Omega_n}/\lambda_0 Re_0$ over a wide range of Re_0 .

10. Summary

A fluctuating eddy viscosity model is proposed to analyze oscillating turbulent flows in a straight pipe. The validity of the present theory is verified through comparisons between predictions and measurements. Concerning the oscillating components of the flows, the conclusions arrived at may be summarized as follows:

Similarity to the cross-sectional mean flow conditions exists in nondimensional frequency regime $\sqrt{\Omega_n}/Re_0$ regardless of time-mean Reynolds number Re_0 . The flow conditions have a transitional frequency band of $\sqrt{\Omega_n}/Re_0$, that is anticipated to be about 0.1 to 1.0.

Velocity profiles can be described in a single curve over a wide range of Re_0 by using a parameter $\sqrt{\Omega_n}/Re_0$ in the turbulent region and by $\sqrt{\Omega_n}/Re_0$ near the wall with only weak dependence on Re_0 at low frequencies.

Changes in the instantaneous friction factor λ are similarly shown by a parameter $\sqrt{\Omega_n}/\lambda_0 Re_0$ over a wide range of Re_0 .

Amplitude of the wall shear stress τ_{wn} is proportional to U_{mn} at low frequencies and to $\sqrt{\nu \omega \nu} U_{mn}$ at high frequencies. The threshold frequency exists at about $\sqrt{\Omega_n}/\lambda_0 Re_0 = 0.1$.

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References

- (1) Brown, F.T., et al., *Trans. ASME*, Ser.D, Vol.91, No.4 (1969-12), p.678.
- (2) Mizushima, T., et al., *J. Chem. Engng. Japan*, Vol.6, No.6 (1973), p.487.
- (3) for example, Oomi, S. and Usui, T., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.43, No.347 (1975-7), p.2030.
- (4) Oomi, S., et al., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.43, No.375 (1977-11), p.4109.
- (5) Reynolds, A. J., *Turbulent Flow in Engineering*, (1974), p.119, John-Willy & Sons.
- (6) Hussain, A.K.M.F. and Reynolds, W.C., *Trans. ASME*, Ser.I, Vol.97, No.4 (1975-12), p.568.
- (7) for example, Laufer, J., NACA Rep., 1174, (1954).

Fig.13 Variation in H_n with frequency.

Fig.14 Friction factor of pipe in oscillating turbulent flows.

- (8) Oomi, S., et al., Paper of Japan Soc. Mech. Engrs. (in Japanese), No.770-8 (1977-7), p.114.
- (9) Oomi, S. and Iguchi, M., Paper of Japan Soc. Mech. Engrs. (in Japanese), No.770-13 (1977-10), p.4.
- (10) Kusama, H., et al., Paper of the 768th Meeting of Japan Soc. Mech. Engrs. (in Japanese), (1967-5), p.151.
- (11) Thomas, L.C. and Shukla, R.K., *Trans. ASME*, Ser.I, Vol.98, No.1 (1976-3), p.27.