532.53:532.593

Paper No. 179-12

Effect of the Wind Fluctuation on the Stability of a Liquid Layer Flowing down an Inclined Plane*

by Eiji HASEGAWA** and Nobuyuki TANIYAMA***

The stability problem of a liquid layer flowing down an inclined plane wall is studied theoretically, taking into account the time dependent wind stress on its free surface. The Floquet theory is applied to a differential system with periodic coefficients of this problem. This system is solved by the long wave approximation. The critical Reynolds number for this flow depends on the square of the amplitude of the wind fluctuation. The frequency of the wind fluctuation has the greatest influence on the instability of the liquid layer when the thickness of the liquid layer is of the same order as the thickness of the Stokes layer due to the oscillating motion. In case of low frequency the flow is destabilized by the wind fluctuation, but in a range of high frequencies the flow can be stabilized slightly.

1. Introduction

The flow of a fluid along an inclined flat surface has been used in various engineering processes, for example, wetted-wall towers, forced cooling of a heated body and coating or painting of the mechanical elements. Such film flows make two different types of motions, that is, steady laminar motion and undulatory wavy motion. Either steady or undulatory motion is adopted according to circumstances of the engineering process. The question then should be asked: which type of the motion is more likely to occur under a given flow condition. The problem of the stability of such film flows is important in practical engineering.

The stability of the flow considered depends on the conditions at the entrance region, the flatness of the wall surface, the properties of the free surface and so on. Especially the effect of the conditions of the free surface on the stability of the flow is very important, since the transition from steady flow to undulatory flow is closely related with the existence of a free surface. The free surface is surrounded by the atmosphere with some fluctuations. It is easy to be moved by the air. Therefore the instability of such film flows will be caused by an unsteady wind. Nevertheless, the stability of a liquid film on the wall subject to airflow has scarcely been examined. The only exception is an investigation by Smith into the stability of such a flow on the basis of the theory for the wind-generated waves

on the sea. He represented the wind stress as a constant tangent stress acting on a free surface using the parameters which are related to the airflow perturbation.

A thin liquid flow on an inclined plane encountered in engineering processes is almost exposed to an airflow with a fluctuation due to the turbomachinery. Then the present work investigates how time-dependent airflows are related to the stability of a film flow on an inclined wall. The stress perturbations at the air-liquid surface due to the unsteady wind is represented in a form composed of the steady part and the time-periodic part. The stability is examined by the linear theory. Since the long waves play an important role in instability of this kind of flow, the stability characteristics are treated by making use of the long wave approximations. The effect of the airflow with time-periodic fluctuations on the critical Reynolds number above which wavy motion occurs is mainly investigated.

2. Primary flow

A liquid, having density ρ and kinematic viscosity ν , flows in a uniform film of thickness h upon a plane wall inclined at an angle β to the horizontal under the action of gravity as illustrated in Fig.1. We adopt a coordinate system (x, y, z) in which the undisturbed liquid surface is represented by the plane $z=0$, z increases along the inward normal to the liquid surface, the x direction is chosen in the flow direction for the case of no wind and the y direction is chosen on the liquid surface perpendicular to the x -axis.

We now consider the case when there is a wind blowing in any direction along a free surface with time-periodic fluctuations. It is assumed that the effect of the unsteady wind flow on the stability of the liquid film flow can be represented in the form of a tangential stress acting on the free surface. That is, the tangential

* Received 14th April, 1977.

** Assistant Professor,
Faculty of Engineering,
Keio University, Yokohama.*** Graduate Student,
Faculty of Engineering,
Keio University.

at the free surface. Here, ζ is the displacement of the free surface due to the disturbances which is a function of x, y and t . σ is the coefficient of surface tension. These boundary conditions (12) to (15) were linearized for small disturbances using the same neglect as mentioned in the governing equation. The simultaneous solution of these equations (7) to (15) will provide information from which the instability of the primary flow can be inferred.

The various variables are made dimensionless using the following scales: the thickness h of the uniform film flow for the distance, the surface velocity $U_s = gh^2/2\nu$ in the film flow down a vertical plane in case of no wind for the velocity, h/U_s for the time, U_s/h for the frequency and ρU_s^2 for the wind stress. We shall introduce the following dimensionless parameters: the Reynolds number $R = U_s h / \nu$, the Weber number $T = \sigma / \rho U_s^2 h$ and the dimensionless frequency $\theta = \sqrt{\omega h^3 / 2\nu}$. For simplicity, we use the same letter for the dimensionless variables as that for the dimensional variables.

Since the flow in this problem is homogeneous over the x, y - plane, the perturbation variables $f(t, x, y, z)$ due to the disturbances can be expressed in the form of the Fourier components as

$$f(t, x, y, z) = \tilde{f}(t, z) e^{i(k_x x + k_y y)}, \quad (16)$$

where, k_x and k_y denote the components of the wave number in the x and y directions respectively, and they are related to the magnitude of the wave number k in $k_x^2 + k_y^2 = k^2$. But the amplitude function $\tilde{f}$ of the displacement on the free surface is a function of only t . Representing the perturbation variables u, v, w, p and ζ in the form of Eq.(16), and substituting these variables into our differential system (7) to (15), we have the differential equations for the amplitude functions $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}$ and $\tilde{\zeta}$. If $\tilde{u}, \tilde{v}$ and $\tilde{p}$ are eliminated from the differential system, the equations governing the stability which contain two unknown variables $\tilde{w}$ and $\tilde{\zeta}$ are obtained as

$$\left\{ \frac{\partial}{\partial t} + i(k_x U + k_y V) \right\} \left(\frac{\partial^2}{\partial z^2} - k^2 \right) \tilde{w} - i \left(k_x \frac{\partial^2 U}{\partial z^2} + k_y \frac{\partial^2 V}{\partial z^2} \right) \tilde{w} = \frac{1}{R} \left(\frac{\partial^2}{\partial z^2} - k^2 \right)^2 \tilde{w}, \quad (17)$$

$$z=0: \quad \frac{\partial \tilde{\zeta}}{\partial t} + i(k_x U + k_y V) \tilde{\zeta} = \tilde{w}, \quad (18)$$

$$i \left(k_x \frac{\partial^2 U}{\partial z^2} + k_y \frac{\partial^2 V}{\partial z^2} \right) \tilde{\zeta} - \frac{\partial^2 \tilde{w}}{\partial z^2} - k^2 \tilde{w} = 0, \quad (19)$$

$$\left\{ \frac{dP}{dz} k^2 + T k^4 + \frac{i}{R} (k_x U + k_y V) k^4 \right\} \tilde{\zeta} + \frac{1}{R} \left(\frac{\partial^2}{\partial z^2} - 3k^2 \right) \frac{\partial \tilde{w}}{\partial z} - \frac{\partial^2 \tilde{w}}{\partial t \partial z} - i(k_x U + k_y V) \frac{\partial \tilde{w}}{\partial z} + i \left(k_x \frac{\partial U}{\partial z} + k_y \frac{\partial V}{\partial z} \right) \tilde{w} = 0, \quad (20)$$

$$z=1: \quad \tilde{w} = 0, \quad (21)$$

$$\frac{\partial \tilde{w}}{\partial z} = 0. \quad (22)$$

The differential system for the disturbances obtained above has the time-periodic coefficients through U and V . This fact is the most remarkable feature of our system.

4. Method of solution and the differential system for approximation

Our system (17) to (22) obtained in the above section has the coefficients containing the periodic function $e^{\pm i\theta t}$ in time. We shall now apply the Floquet theorem, which is generalized by Yih to solve the stability of a horizontal liquid layer on an oscillating plane, to the solution of our problem. We shall write

$$\tilde{w}(t, z) = i k e^{\mu t} \chi(t, z), \quad \tilde{\zeta}(t) = e^{\mu t} \eta(t) \quad (23)$$

in which χ and η are periodic in time t and μ is a complex constant. Taking into consideration the equation (10) of continuity, we separate the factor $i k$ from $\tilde{w}$.

We will seek the development of the disturbances in the long run, and call the flow stable if the perturbation quantities remain small for all time, and unstable if they do not. Thus the stability characteristics are determined according to whether the real part of μ is positive or negative. We shall seek a value of μ which satisfies Eqs.(17) to (22).

Since the long waves play an important role in the instability of the flow of this kind, μ, χ and η are expanded in wave number k as

$$\mu = \mu_0 + i k \mu_1 + k^2 \mu_2 + \dots, \quad (24)$$

$$\eta = \eta_0(t) + i k \eta_1(t) + k^2 \eta_2(t) + \dots, \quad (25)$$

$$\chi(t, z) = \chi_0(t, z) + i k \chi_1(t, z) + k^2 \chi_2(t, z) + \dots, \quad (26)$$

in which μ 's are real constants, and η 's and χ 's are all periodic in time t . Substituting Eq.(23) and Eqs.(24) to (26) into Eqs.(17) to (22), the differential equations, which are necessary to obtain the value of μ to order k^2 , are given as follows:

$$\chi_0'''' - R \left(\frac{\partial}{\partial t} + \mu_0 \right) \chi_0'' = 0, \quad (27)$$

$$\chi_1'''' - R \left(\frac{\partial}{\partial t} + \mu_0 \right) \chi_1'' = R \left(\mu_1 \chi_0'' + Q \chi_0'' - Q' \chi_0 \right), \quad (28)$$

The boundary conditions for these equations can be written as

$$z=1: \quad \chi_0 = \chi_0' = 0, \quad (29)$$

$$\chi_1 = \chi_1' = 0, \quad (30)$$

$$z=0: \quad \chi_0'' - Q'' \eta_0 = 0, \quad (31)$$

$$\chi_1'' - Q' \eta_1 = 0, \quad (32)$$

$$\chi_0'''' - R \left(\frac{\partial}{\partial t} + \mu_0 \right) \chi_0' = 0, \quad (33)$$

$$\chi_1'''' - R \left(\frac{\partial}{\partial t} + \mu_0 \right) \chi_1' = R \left\{ \mu_1 \chi_0' + Q \chi_0' - Q' \chi_0 + \left(\frac{dP}{dz} + T k^2 \right) \eta_0 \right\}, \quad (34)$$

$$\left(\frac{d}{dt} + \mu_0 \right) \eta_0 = 0, \quad (35)$$

$$\left(\frac{d}{dt} + \mu_0 \right) \eta_1 = -\mu_1 \eta_0 - Q \eta_0 + \chi_0, \quad (36)$$

$$\left(\frac{d}{dt} + \mu_0 \right) \eta_2 = -\mu_2 \eta_0 + \mu_1 \eta_1 + Q \eta_1 - \chi_1, \quad (37)$$

in which $Q = U \cos \alpha + V \sin \alpha$ and $\alpha = \tan^{-1}(k_y/k_x)$.

Q is a component of the velocity for the primary flow in the direction of the disturbance wave. The primes denote differentiation with respect to z . In the following section, we shall seek the values of μ_0 , μ_1 and μ_2 successively using the above equations (27) to (37).

5. Approximate solutions

The equation (35), which is the kinematic condition on the free surface, for the first approximation is now solved. In the case of no wind $\zeta(t)$ should be written as $e^{-i\omega t}$ in the linear theory, in which c is the phase velocity of the disturbances. Thus the solution of Eq. (35) can be given as

$$\mu_0 = 0, \quad \tau_0 = 1. \quad (38)$$

Solving Eqs. (27), (29), (31) and (33) with the solution (38), χ_0 is given in the form of the sum of the constant part $\bar{\chi}_0(z)$ and the periodic part $\bar{\chi}_0(z)e^{i\omega t}$ in time as follows:

$$\chi_0(z, t) = \bar{\chi}_0(z) + \{\bar{\chi}_0(z)e^{i\omega t} + \text{c.c.}\} \quad (39)$$

in which

$$\bar{\chi}_0(z) = -\sin \beta \cos \alpha (z-1)^2, \quad (40)$$

$$\bar{\chi}_0(z) = \sigma \frac{1-i}{2\theta} \tanh[(1+i)\theta] \times \frac{1 - \cosh[(1+i)\theta(z-1)]}{\cosh[(1+i)\theta]}. \quad (41)$$

Here, $\sigma = \sigma_s \cos \alpha + \sigma_z \sin \alpha$ which is a component of the fluctuating wind stress in the direction of the disturbance wave.

On the basis of the first approximate solutions μ_0 , τ_0 and χ_0 , we now solve the kinematic condition (36) for the second approximation. Since μ_1 is a complex constant and τ_1 is periodic in time, μ_1 is obtained as

$$\mu_1 = -2 \sin \beta \cos \alpha - \Sigma, \quad (42)$$

Here, $\Sigma = \Sigma_s \cos \alpha + \Sigma_z \sin \alpha$ which is a component of the steady wind stress in the direction

$$\begin{aligned} A &= 2 \cos \beta + RTk^2, \\ B &= 2 \cos \beta + RTk^2, \\ &- \frac{4}{3} R \sin \beta \cos \alpha (\sin \beta \cos \alpha - \Sigma) \\ &+ \sigma^2 \frac{2R}{\theta^2} \frac{\sinh 2\theta \sin 2\theta}{(\cosh 2\theta + \cos 2\theta)^2}, \end{aligned}$$

$$C = \sigma^2 \frac{3R}{2\theta^2} \frac{\sinh 2\theta - \sin 2\theta}{(\cosh 2\theta + \cos 2\theta)^2},$$

$$\begin{aligned} \chi_{1s} &= R \left[\frac{\sin \beta \cos \alpha}{6} (\sin \beta \cos \alpha - \Sigma) \left\{ \frac{(z-1)^3}{5} + (z-1)^4 \right\} \right. \\ &+ \frac{\sigma^2}{\theta^2 (\cosh 2\theta + \cos 2\theta)^2} \left\{ \frac{1}{4} [\sin 2\theta \sinh [2\theta(z-1)] - \sinh 2\theta \sin [2\theta(z-1)]] \right. \\ &\left. \left. - \sinh 2\theta \cosh [\theta(z-1)] \sin [\theta(z-1)] + \sin 2\theta \sinh [\theta(z-1)] \cosh [\theta(z-1)] \right\} \right]. \end{aligned}$$

Substituting $\bar{Q}$, Eqs. (43) and (45) into Eq. (44), the solution μ_2 is obtained as

$$\begin{aligned} \mu_2 &= \frac{8R \sin \beta}{15} \left[\cos \alpha (\sin \beta \cos \alpha - \Sigma) - \frac{5 \cot \beta}{4R} - \frac{5Tk^2}{8 \sin \beta} \right. \\ &\left. + \frac{15\sigma^2}{8\theta^2 \sin \beta (\cosh 2\theta + \cos 2\theta)^2} \left\{ \frac{3}{2\theta} (\sinh 2\theta - \sin 2\theta) - \sinh 2\theta \sin 2\theta \right\} \right]. \quad (46) \end{aligned}$$

It is found from Eq. (46) that μ_2 is real.

of the disturbance wave. τ_1 is also given as

$$\tau_1 = \left(-\sigma \frac{(1+i)R}{4\theta^3} \frac{\tanh(1+i)\theta}{\cosh(1+i)\theta} e^{i\omega t} + \text{c.c.} \right). \quad (43)$$

Since μ_1 is real from Eq. (42), the stability characteristics can not be obtained in this order solution.

Before we present the solution χ_1 for the second approximation, we shall consider unknown quantities which are necessary to solve μ_2 making use of the kinematic condition (37) for the third approximation. Since μ_2 is a complex constant and τ_2 is periodic in time, μ_2 can be written in the same form as μ_1 :

$$\mu_2 = -\bar{\chi}_1 + \bar{Q}\bar{\tau}_1 + \bar{Q}^* \bar{\tau}_1^* \quad (44)$$

in which $\bar{Q}$ is the time-periodic amplitude for the velocity component Q of the primary flow in the direction of the disturbance wave, that is, $Q = \bar{Q} + (\bar{Q}e^{i\omega t} + \text{c.c.})$ and $\bar{\tau}_1$ is the coefficient of $e^{i\omega t}$ in Eq. (43). The symbol $*$ denotes a complex conjugate. Then, for prediction of the stability for long waves, none but the time-independent terms in the second approximate solution χ_1 need be calculated. Averaging Eqs. (28), (30), (32) and (34) in time, the equations governing $\bar{\chi}_1(z)$ are given as follows:

$$\bar{\chi}_1''' = R\mu_1 \bar{\chi}_0'' + (\bar{Q}\bar{\chi}_0'' - \bar{Q}'\bar{\chi}_0' + \text{c.c.}), \quad (28)'$$

$$z=1: \quad \bar{\chi}_1 = \bar{\chi}_1' = 0, \quad (30)'$$

$$z=0: \quad \bar{\chi}_1'' + (\bar{Q}''\bar{\tau}_1 + \text{c.c.}) = 0, \quad (32)'$$

$$\begin{aligned} \bar{\chi}_1'' &= R \left\{ \mu_1 \bar{\chi}_0' + (\bar{Q}\bar{\chi}_0' - \bar{Q}'\bar{\chi}_0 + \text{c.c.}) \right. \\ &\left. + \left(\frac{dP}{dz} + Tk^2 \right) \bar{\tau}_0 \right\}. \quad (34)' \end{aligned}$$

The solution $\bar{\chi}_1$ is given from Eqs. (28)', (30)', (32)' and (34)' as

$$\bar{\chi}_1 = \frac{A}{6}(z-1)^3 + \frac{B}{2}(z-1)^2 + C(z-1) + \bar{\chi}_{1s}, \quad (45)$$

in which

6. Discussion

Substituting μ_0 , μ_1 and μ_2 , which are given by Eqs.(38),(42) and (46), into Eq.(24), the solution μ can be obtained to order k^2 . The stability of the flow considered depends on whether the real part of μ is positive or negative. But, since μ_0 is zero and $i k \mu_1$ is purely imaginary, it is found that the flow is unstable or stable according as real number $\mu_2 > \text{or} < 0$. That is, $\mu_2 = 0$ gives a neutral curve. Equating the brackets on the right-hand side of Eq.(46) to zero, the neutral curve in the kR plane is obtained as

$$R = \frac{5 \cot \beta}{4 \sin \beta} \left/ \left\{ \cos \alpha \left(\cos \alpha - \frac{\Sigma}{\sin \beta} \right) - \frac{5 T k^2}{8 \sin^2 \beta} + \sigma^2 \frac{15}{2 \sin^2 \beta} H(\theta) \right\} \right. \quad (47)$$

in which

$$H(\theta) = \left\{ -\frac{\sinh 2\theta}{2\theta} \frac{\sin 2\theta}{2\theta} + \frac{3}{(2\theta)^2} \left(\frac{\sinh 2\theta}{2\theta} - \frac{\sin 2\theta}{2\theta} \right) \right\} / (\cosh 2\theta + \cos 2\theta)^2 \quad (48)$$

Consider that we apply the result Eq.(47) to various special cases. In case that there is no wind and the direction of the disturbance wave agrees with x-axis, we have $\Sigma = \sigma = \alpha = 0$. Therefore the neutral stability curve is given as

$$R \sin \beta = \frac{5}{4} \cot \beta / \left(1 - \frac{5}{8} \frac{T k^2}{\sin^2 \beta} \right) \quad (49)$$

If the reference velocity is taken to be the average velocity of the primary flow instead of the surface velocity, Eq.(49) is transformed as

$$R' = \frac{5}{6} \cot \beta / \left(1 - \frac{5}{18} k^2 T' \right) \quad (49)'$$

where R' and T' are Reynolds number and Weber number on the basis of the average velocity. This formula agrees with the result found by Yih.

In case that there is a constant wind and the effect of surface tension is neglected, Eq.(47) is rewritten as

$$R = \frac{5}{4} \cot \beta / (\cos \alpha (\sin \beta \cos \alpha - \Sigma)) \quad (50)$$

If the direction of z-axis is opposite and the steady wind stress Σ is represented in the dimensional form, Eq.(50) is transformed as

$$\frac{g h^2}{2 \nu^2} = \frac{5}{4} \cos \beta / \left\{ \cos \alpha \left(\sin \beta \cos \alpha + \frac{\tau_s}{\rho g h} \right) \right\} \quad (50)'$$

This result is the same as

$$\frac{2 \tan \beta h^2}{5 \nu^2 \rho} \cos \alpha (\tau_s + \rho g h \cos \alpha \sin \alpha) = 1 \quad (50)''$$

which was obtained by Smith, where $\tau_s = -\Sigma$.

In case that there is a wind blowing with periodic fluctuation in time, the critical Reynolds number is found by equating k on the right-hand side of Eq.(47) to zero. Thus the effect of the time-periodic fluctuation in airflow on the critical Reynolds number R_c is given by the term $\sigma^2 15 H / 2 \sin^2 \beta$ in the denominator on the right-hand side of Eq.(47). Hence its effect is proportional to σ^2 , in which σ is the amplitude of the time-periodic wind. Since that term contains the factor $1/\sin^2 \beta$, the smaller the angle of the inclination, the greater the effect of the wind fluctuation will be. The dependence of the critical Reynolds number on the dimensionless frequency of the time-periodic wind is found from Eq.(48), that is, H .

In the case of small frequency of the unsteady airflow, H is expanded in θ as

$$H(\theta) = \frac{121}{315} \theta^4 - \frac{27323}{51975} \theta^6 + \dots \quad (48)'$$

In this case, it is found from Eq.(48)' that the effect of the fluctuation of the airflow will be small. When the frequency is small, bulk fluids between the solid wall and the free surface are moved with a fluctuation of the airflow. Nevertheless the stability of the flow hardly change. This is an important property.

Fig.2. The effect of the wind frequency on the critical Reynolds number

Fig.2 shows the variation of H with the non-dimensional frequency θ . The flow is destabilized or stabilized according as $H > \text{or} < 0$. H has the maximum value at $\theta = 1.087$ and the flow is most destabilized at that point. When θ is greater than this value, the flow is hardly destabilized. It is interesting that for the range of $3.37 < \theta < 4.54$ the value of H is negative. In this range the motion is stabilized. As θ increases furthermore, the value of H decreases and becomes positive and negative alternately. In that range of θ , H is so small that it is not important in practice.

The influence of the fluctuation of the unsteady wind on the stability of the liquid film flow becomes very strong in case that the dimensionless frequency θ equals to about unity. It is thus found that the flow most destabilizes when the thickness h of the film flow is the same as that of the Stokes layer, that is, the boundary layer caused by the time-periodic wind.

7. Conclusions

The effect of the airflow on the sta-

bility of the liquid film flow down an inclined plane, in particular, the effect of the time-dependent airflow, is examined. As a result, it is found that the flow destabilizes in the range of low frequencies, and it stabilizes in a certain range of high frequencies. When the dimensionless frequency equals almost to unity, the flow is most destabilized.

The authors wish to thank Prof. K. Komotori for his kindness shown in the process of this work.

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