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A Study on a Rotor Passing through a Resonance*

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A numerical and physical study is made concerning the non-stationary transitions of motion of a rotor with the condition that it can easily pass through a resonance.

As a result of this study, theoretical design criteria for a rotor with a usable limit are introduced, and these criteria are compared with the results of experiments in which an actual rotor is used.

1. Introduction

For the problems of a rotor passing through resonances, it is technologically important to know the transient behavior of motion at resonance, and many kinds of studies have been reported from various points of view.^{1) 2) 3)}

The author has also studied this problem, and as a result of investigating the transient behavior of motion of the rotor under the critical condition of whether the rotor can just barely pass through the resonance or not, he has introduced a method for estimating the condition of the rotor under which it can pass through a resonance.⁴⁾

This result is available for estimating a design limit of the rotor.

However, it is more useful when designing rotors to be able to predict the margins of energy source or the maximum displacement of motion, when rotors can easily pass through resonances.

In the present paper, by using a relation in which the unbalance constant μ of a rotor is given by 3 constants of the estimable value K for a rotor passing through a resonance, the acceleration coefficient of rotation q and the damping factor ζ of the oscillating system, the transient behavior of motion of a rotor which can easily pass through a resonance is presented clearly and concisely.

Furthermore, the transient behavior of motion of an actual rotor passing through a resonance is investigated experimentally and the results obtained are compared with the theoretical ones.

2. Nomenclature

x, y	:displacements in directions of abscissa and ordinate
m	:rotating unbalance mass
M	:mass of a rotor excepting m
c	:damping coefficient of system
k	:spring constant of system
e	:radius of gyration of unbalanced mass
I	:polar moment of inertia of rotor

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θ	:angular displacement of rotation
$T(\theta)$	:apparent driving torque of energy source acting on rotating shaft, $T(\theta) = k_0 (T_M - T_{SL})$
h	:angular acceleration of rotation corresponding to $T(\theta)$, given by the dimension H_z/s^2
μ	:unbalance constant
q	:acceleration coefficient of rotation, f_n^2/h
K	:estimable value, which is the value used to estimate the conditions of a rotor which can pass through a resonance
Re	:magnification factor of displacement of a rotor
ζ	:damping factor of system
ω_n, f_n	:natural frequency of system, $f_n = \omega_n/2\pi$
τ_n	:non-dimensional time, $\omega_n t$
P	:critical speed, at which maximum displacement of motion of a rotor is generated
P_D	:angular velocity of rotation, at which maximum dynamic load torque due to motion of a rotor is generated
$R.T_{DL}$	:transient dynamic load torque normalized with driving torque of energy source at resonance
k_0	:gear reduction ratio
T_M	:driving torque of energy source
T_{SL}	:static load torque due to frictional resistance, etc.

3. Maximum displacement and maximum dynamic load torque when a rotor easily passes through a resonance

3.1 Equations of motion

Fig.1 shows a schematic diagram of the system treated in this paper.

The differential equations of motion for a rotor excited by an unbalanced mass, which moves with a circular mode of motion in a plane perpendicular to the rotating shaft, are given by considering the influence of the motion of a rotor on an energy source and by neglecting the effect of gravity as follows;

$$\left. \begin{aligned} (M+m)\ddot{x} + c\dot{x} + kx &= me(\ddot{\theta}^2 \cos\theta + \ddot{\theta} \sin\theta) \\ (M+m)\ddot{y} + c\dot{y} + ky &= me(\ddot{\theta}^2 \sin\theta - \ddot{\theta} \cos\theta) \\ (I+me^2)\ddot{\theta} &= T(\theta) + mxe \sin\theta - mye \cos\theta \end{aligned} \right\} (1)$$

where, if the following constants are defined,

Furthermore, the input data for the numerical analysis of the non-stationary transitions of motion of a rotor are chiefly limited in this paper as follows;

$$2 \leq q \leq 32, 0 \leq \zeta \leq 0.15, 0 \leq K \leq 0.8 \quad \dots (9)$$

Next, even if several points within the limits of the parameters given by Eq. (9) are used as input data, many combinations of these three parameters are possible and, therefore, many numerical calculations are required.

Furthermore, in order to determine whether or not a rotor has exceeded the critical speed, it is necessary to know the non-stationary transitions of motion of the rotor until its rotational speed is at least twice the natural frequency.

Therefore, a new time limit τ_{end} and the time mesh $\delta\tau$ must be chosen for every new acceleration coefficient of rotation q in the numerical calculations, because the angular velocity of rotation obtained in the calculations depends on the acceleration coefficient q , the time limit τ_{end} and the time mesh $\delta\tau$.

In the following, a practical method of numerical calculation in which the time limit τ_{end} and the time mesh $\delta\tau$ are automatically given in the numerical calculation program and in which the number of numerical calculations n is given independent of q is presented.

If $\mu=0$ in the third equation in Eq. (3), then that equation becomes as follows;

$$\frac{d^2\theta}{d\tau^2} = \frac{1}{2\pi q} \quad \therefore \quad \frac{d\theta}{d\tau} = \frac{\tau}{2\pi q} \quad \dots (10)$$

where, if

$$\frac{d\theta}{d\tau} = \frac{1}{\omega_n} \frac{d\theta}{dt} = 1.0$$

, the angular velocity of rotation is equal to the natural frequency.

And, if the non-stationary transitions of motion of a rotor are numerically calculated until the rotational speed reaches p times the natural frequency, the time limit τ_{end} is obtained from Eq. (10) as follows;

$$\tau_{\text{end}} = 2\pi pq \quad \dots (11)$$

, where if the time mesh is expressed by δ ($=\delta\tau$) and the number of numerical calculations is expressed by n , we obtain

$$n\delta = 2\pi pq \quad \therefore \quad n = \frac{2\pi pq}{\delta} \quad \dots (12)$$

, where if δ is given as follows;

$$\delta = \alpha q \quad \dots (13)$$

, the number of numerical calculations n is obtained using Eqs. (12) and (13) as follows;

$$n = 2\pi p/\alpha \quad (\text{constant}) \quad \dots (14)$$

As shown in Eq. (14), when the number of numerical calculations n is given as a constant independent of the acceleration coefficient q , every numerical calculation has a constant degree of precision independent

of q in the case where a rotor has an ideal source of energy corresponding to the condition $\mu=0$.

Next, when a rotor has an angular velocity of rotation ω_n , the following equation is valid,

$$\frac{d\theta}{d\tau} = p \quad \therefore \quad \theta = p\tau \quad \dots (15)$$

where if

$$\theta = 2\pi, \quad \tau = m\delta \quad \dots (16)$$

, m denotes the number of numerical calculations in a cycle of rotation at an angular velocity ω_n , and so

$$m = \frac{2\pi}{p\alpha q} \quad \dots (17)$$

is obtained from Eqs. (15) and (16).

In the case where $\mu \neq 0$, the angular velocity of rotation of a rotor cannot reach ω_n , because the increase in the angular velocity of rotation is influenced by the unbalance constant μ at the critical speed.

This is the reason why, instead of p , the automatically estimated time limit τ_{end} is used to finish the numerical calculation.

Furthermore, the number of numerical calculations in a cycle of rotation in the case where $\mu \neq 0$ is a little larger than that in the case where $\mu = 0$ in regions near the time limit τ_{end} , because the number of numerical calculations in a cycle of rotation depends on the angular velocity of rotation.

The limit of rotational speed p in the numerical calculations is given as $p = 2.5$ when the transient values for maximum dynamic load torque, maximum displacement, critical speed and so on are calculated continuously.

These transient values are obtained with a degree of precision that allows them to be used in practical applications.

In the case of studies on the non-stationary transitions of motion in regions distant from a resonance, it is necessary to choose an optimum value for $p\alpha q$ in Eq. (17).

For example, $m \approx 8$ in the case where $p = 5.0$, $\alpha = 0.02$ and $q = 8$

3.4 A consideration of the non-stationary transitions of motion in a rotor which easily passes through a resonance

The magnification factor Re of the displacement of motion of a rotor and the dynamic load torque $R.T_{DL}$, normalized by the driving torque of an energy source at the resonance are obtained as follows;

$$Re = \sqrt{R_x^2 + R_y^2} \quad \dots (18)$$

$$R.T_{DL} = -2\pi pq (R_x \sin\theta - R_y \cos\theta) \quad (19)$$

Fig. 2 shows the non-stationary characteristics of Re and $R.T_{DL}$ in the case where the input data are

$$(A) \quad \zeta = 0.05, \quad q = 8.0, \quad K = 0.5$$

, and Fig. 3 shows the non-stationary phase difference between the centrifugal force

vector, which is produced by rotating an unbalanced mass, and the displacement vector of motion of the rotor.

In these figures, the abscissa shows the angular velocity of rotation $\dot{\theta}/\omega_n$.

It can be seen from Fig.2 that the maximum displacement $(Re)_{\max}$ of the rotor is $(Re)_{\max} = 7.2$, which occurs at an angular velocity of rotation $P = 1.19$ - the critical speed - and that the maximum dynamic load torque $(R.T.)_{\max}$ normalized by the driving torque is $(R.T.)_{\max} = 0.492$, which occurs at an angular velocity of rotation $P_D = 1.14$.

As a result, it is seen that the relation $P_D < P$ is valid even in the case where a rotor can easily pass through a resonance.

However, if P_D is compared with the natural frequency $(\dot{\theta}/\omega_n) = 1.0$, the angular velocity of rotation P_D , with a maximum dynamic load torque, is rather distant from the resonant frequency, in this case.

Furthermore, at the angular velocity of rotation P_D , the phase difference is $\phi_{ns} = 0.64\pi$, and the amplitude of displacement of motion is $Re = 6.43$.

although the phase difference ϕ_{ns} and the angular velocity of rotation P_D , which correspond to the maximum dynamic load torque, are approximately given by $\phi_{ns} = \pi/2$ and $P_D = 1.0$, respectively, when the rotor has the limit of the critical condition under which the rotor just barely passes through the resonance, it can be seen from Figs.2 and 3 that the phenomena of the rotor easily passing through the resonance are rather different from the phenomena of the rotor under the critical condition.

Therefore, it can not be always said that the estimable value K , which is introduced when the rotor has the critical condition, is valid as an approximate value for the maximum dynamic load torque normalized with the driving torque of an energy source in any case that a rotor easily passes through a resonance.

Nevertheless, in an example shown in Fig.2, the estimable value is $K=0.5$, and the

Fig. 2 Non-stationary displacement of motion and dynamic load torque.

maximum dynamic load torque is $(R.T.)_{\max} = 0.492$, and both of them are very close values to each other.

Therefore, it is technologically important for a rotor to study the relation between the estimable value K and the maximum dynamic load torque in every condition that a rotor easily passes through a resonance.

Meanwhile, the dotted line which is shown in Fig.3 is a non-stationary transition of phase difference ϕ_{ns} when the input data are given as follows;

(B) $\zeta = 0.11$, $q = 2.0$, $K = 0.2$

It can be seen from the ϕ_{ns} characteristics with the parameters of (A) and (B) in Fig.3 that, in the angular velocity of rotation after the resonance, the maximum phase difference $(\phi_{ns})_{\max} = 2\pi$ radian which corresponds to a cycle of rotation is generated in case of (A) and $(\phi_{ns})_{\max} = 1.3\pi$ in case of (B).

After the maximum phase difference is generated, both of these phase difference ϕ_{ns} gradually decrease undulating around $\phi_{ns} = \pi$ which is near the phase difference in the steady state of motion.

In Fig.4, the abscissa shows the angular phase of rotation θ when the initial angular phase of rotation is $\theta_0 = 0$, and the ordinate shows the phase of displacement vector of motion.

The dotted linear lines in Fig.4 indicate the phase characteristics when the phase differences are zero, that is, they are the phase characteristics of the centrifugal force vector.

Therefore, the phase difference ϕ_{ns} between the centrifugal force vector and the displacement vector of motion is obtained by measuring a vertical or horizontal length between a dotted linear line and a solid line which shows the phase characteristics of the displacement vector of motion, as shown in Fig.4.

The phase difference characteristics in the region near $(\phi_{ns})_{\max}$ in Fig.3 corresponds to the phase characteristics in the region near M_1 in Fig.4 in case of (A), and to the phase characteristics in the region near M_2 in Fig.4 in case of (B).

In the region near M_1 , the phase of

Fig. 3 Non-stationary phase difference.

displacement vector of motion is stagnant in its increase, and it agrees with the phase of rotation θ which is given by a dotted line, that is, the phase difference $(\phi_{ns}) = 0$ or 2π occurs at $\theta = 19.5\pi$.

The phase of displacement vector is stagnant, while the phase of rotation advances rapidly and the phase difference ϕ_{ns} increases rapidly.

In short, the phase of displacement vector of motion is gradually increased and comes to follow the phase of rotation.

It is considered that these phenomena of phase differences are caused by the residual component of the free oscillations of motion which are generated when a rotor passes through a resonance.

If the free oscillations of motion persist after a rotor passes through a resonance, the difference between the phase velocity of rotation and the phase velocity of the free oscillations of motion of the rotor enlarges as the angular velocity of rotation increases, and then there occur some points where the phase of rotation outstrips the phase of the resultant oscillations of motion of the rotor.

In case of (B) in Fig. 3, as the rotor has a large damping factor $\zeta = 0.11$, the oscillations of motion damp rapidly, and then the maximum phase difference is $(\phi_{ns}) = 1.3\pi$, after which the phase difference ϕ_{ns} dwindles with undulation.

If the damping factor of the system is very small, the component of free oscillations of motion persists for a long time, and then there occur the phenomena where the phase of the resultant oscillations of motion is outstripped by the phase of rotation many times.

Furthermore, it seems that the magnitude of residual component of free oscillations of motion is influenced by the angular acceleration of rotation when a rotor passes through the resonance, and by the time elapsed after the resonance, etc.

Fig. 4 Transient phase of displacement vector of motion, which depends on phase of rotation θ .

3.5 Relation between estimable value K and maximum dynamic load torque normalized with driving torque of energy source at resonance

Figs. 5 and 6 show the relations between the estimable value K and the maximum value of the normalized dynamic load torque, $(R.T_{DL})_{max}$, with the parameters of damping factor ζ .

Fig. 5 shows the relation when the acceleration coefficient q is 32, and Fig. 6 shows the relation when $q=2$.

Furthermore, each value of $(R.T_{DL})_{max}$, which is given in Figs. 5 and 6, is a value which is obtained when the estimable value K is given by an integral number times of 0.1.

In Figs. 5 and 6, the linear solid line shows the following relation;

$$(R.T_{DL})_{max} = K \quad \text{..... (20)}$$

In case of an example shown in Fig. 2, the acceleration coefficient $q = 8.0$ which lies between $q = 2$, which is the acceleration coefficient in Fig. 6, and $q = 32$ in Fig. 5, was given as one of the input data in the numerical calculations, and in this case, the relation given by Eq. (20) was ap-

Fig. 5 Maximum dynamic load torque normalized with driving torque of energy source at resonance when $q = 32$.

Fig. 6 Maximum dynamic load torque normalized when $q = 2$.

proximately valid, as previously described.

Moreover, in case of Figs. 5 and 6, the maximum values of the normalized dynamic load torque, $(R.T.)_{DL, max}$, obtained by the numerical calculations are scattered in the region near the linear line given by Eq.(20) with not so much deviation.

These values of $(R.T.)_{DL, max}$ are obtained in the region of $0 \leq K < 1.0$.

Furthermore, when $(R.T.)_{DL, max}$ is approximated by Eq.(20), the maximum numerical error is smaller than +10 % in the region of $0 \leq K < 1.0$.

Such a degree of the maximum numerical error can be practically accepted in the rotor design.

As a result, it is seen that the maximum value of the normalized dynamic load torque, $(R.T.)_{DL, max}$, can be predicted from the estimable value K in every case where a rotor easily passes through the resonance.

As the estimable value K is valid as an approximate value for the maximum dynamic load torque normalized, even in the case where a rotor can easily pass through the resonance, the margin B of the driving torque of energy source of the rotor is given as follows;

$$B = (1 - K) \times 100 \% \quad \text{..... (21)}$$

Fig. 7 Critical speed generating maximum displacement of motion when $q = 2, 8$ and 32 .

Fig. 8 Critical speed when $q = 4$ and 16 .

where, it is necessary for designing a rotor on safety side to estimate the margin of the driving torque of energy source at about 10 % smaller than a value estimated by Eq.(21).

Furthermore, in case of $\zeta = 0.2$ in Fig. 5, ρ_{max} which is used for calculating the estimable value K is approximated by a value obtained by $\rho_{max} = 1/2\zeta$, because ρ_{max} given by Eq.(5) has a large numerical error when the damping factor ζ of the system is $\zeta = 0.2$.

Although the results above obtained are studied in the limit of $q \leq 32$, the relation given by Eq.(20) can be approximately valid over a pretty wide range of q than $q = 32$.

For practical application, the maximum of μ and/or the optimum values of q, ζ, ω_n , etc. which correspond to a margin required can be predicted using Eq.(7), etc.

3.6 Maximum displacement of motion and critical speed

Figs.7 and 8 show the relation between the estimable value K and the critical speed P at which the maximum amplitude of displacement of a rotor is generated.

The signs on the solid lines in Figs. 7 and 8 are the same parameters of damping factor as shown in Figs.5 and 6.

As understood from Figs.7 and 8, the larger the estimable value K is, the less the critical speed P decreases, and it gradually approaches the natural frequency $(\theta/\omega_n) = 1.0$ as K increases.

Furthermore, the decreases in the critical speed is not so steep in the region of the estimable value $K < 0.8$.

Although the critical speed in the region of $K \geq 0.8$ is not so much drawn in the figures, it is conjectured that the critical speed P decreases rapidly as the estimable value K increases in this region, because the condition of the rotor approaches the critical condition as K increases.

Fig.9 shows the relation between the estimable value K and the ratio P_D/P , with the parameters of q and ζ .

Fig. 9 Relation between estimable value K and ratio of angular velocity of rotation P_D generating maximum dynamic load torque versus critical speed P .

As understood from Fig.9, the angular velocity of rotation P_D , where the maximum dynamic load torque occurs, is always smaller than the critical speed P in any condition that a rotor easily passes through a resonance, that is, P_D/P is always smaller than 1.0.

Furthermore, the smaller both of ζ and q are, the smaller the value of P_D/P .

Table 1 shows the magnification factor of the maximum displacement of motion, $(Re)_{\max}$, of a rotor with a circular oscillating system of single-degree-of-freedom, which is numerically calculated by the input data K , q and ζ within the limit of Eq.(9).

Table 1 forms a simple diagram because the estimable value K is substituted for the unbalance constant u of a rotor, and has a clear limit, $0 \leq K < 1.0$.

Fig.10 shows the relation between the estimable value K and the maximum displacement factor R.M.K, where R.M.K is given by a ratio of the value of $(Re)_{\max}$ in the case where the estimable value $K \neq 0$ against the value of $(Re)_{\max}$ in the case where $K = 0$.

In Fig.10, the maximum displacement factor R.M.K when $q = 4, 16$ and $\zeta = 0.03, 0.07$ are compared.

Naturally, in Fig.10, the maximum displacement of motion $(Re)_{\max}$ increases as the estimable value K increases.

Furthermore, it can be seen from Fig. 10 that when the damping factor ζ is constant, the smaller the acceleration coefficient q is, that is, the larger the angular acceleration of rotation h is, the more the maximum amplitude of displacement of motion is influenced by the estimable value K ; and further that when the acceleration coefficient q is constant, the smaller the damping factor is, the more the maximum amplitude of displacement of motion is influenced by the estimable value K , too.

The cause of these phenomena is considered as follows; the smaller both of the acceleration coefficient q and the damping

Table 1 Magnification factor of maximum displacement of motion.

q	K	0.0	0.03	0.05	0.07	0.09	0.11	0.15
2.0	0.0	6.02	5.14	4.66	4.25	3.89	3.59	3.08
	0.2	6.20	5.28	4.78	4.34	3.97	3.64	3.11
	0.4	6.43	5.46	4.92	4.45	4.05	3.71	3.14
	0.6	6.87	5.80	5.18	4.63	4.18	3.79	3.17
	0.8	7.42	6.31	5.56	4.91	4.37	3.91	3.21
4.0	0.0	8.15	6.48	5.65	4.99	4.45	4.0	3.30
	0.2	8.44	6.69	5.81	5.10	4.52	4.05	3.31
	0.4	8.84	6.95	5.99	5.23	4.61	4.10	3.33
	0.6	9.40	7.34	6.25	5.40	4.72	4.17	3.35
	0.8	10.3	8.0	6.68	5.67	4.88	4.26	3.36
8.0	0.0	11.2	8.10	6.74	5.72	4.94	4.32	3.42
	0.2	11.6	8.34	6.89	5.82	5.0	4.36	3.43
	0.4	12.2	8.66	7.09	5.94	5.07	4.40	3.43
	0.6	13.0	9.11	7.36	6.10	5.16	4.44	3.43
	0.8	14.3	9.88	7.77	6.32	5.27	4.50	3.43
16.0	0.0	15.5	9.89	7.79	6.34	5.30	4.53	3.47
	0.2	16.1	10.2	7.94	6.43	5.35	4.55	3.47
	0.4	16.9	10.5	8.13	6.52	5.39	4.57	3.46
	0.6	18.0	11.0	8.37	6.64	5.45	4.59	3.45
	0.8	19.9	11.8	8.72	6.79	5.51	4.60	3.43
32.0	0.0	21.1	11.6	8.61	6.74	5.49	4.60	3.45
	0.2	21.9	11.9	8.73	6.80	5.51	4.61	3.44
	0.4	23.0	12.2	8.88	6.86	5.53	4.61	3.43
	0.6	24.5	12.7	9.07	6.93	5.56	4.61	3.41
	0.8	26.9	13.4	9.31	7.01	5.57	4.60	3.38
1/(2 ζ)	∞	16.7	10.0	7.14	5.56	4.55	3.33	

factor ζ of the system are, the larger the difference between the resonant displacement of a rotor in the steady state of motion and the maximum displacement of motion of a rotor regarded as having an ideal source of energy.

Furthermore, the increase in the maximum displacement of motion of a rotor, which is due to the increase in the estimable value K and to the slow passage of a rotor through the resonance, depend on the difference between the resonant displacement of motion of the rotor in the steady state of motion and the maximum displacement of motion of the rotor having an ideal source of energy.

4. An experiment for a rotor passing through a resonance and comparison of experimental result with theoretical one

4.1 Outline of experimental apparatus and method

An impeller type full automatic washing machine which is usually sold in the market is used for the experiment.

The experimental device consists of the outer frame and the main body of the machine, and the main body is suspended through the medium of 4 pieces for vibration isolation from the four top corners of the outer frame which is specially made of an iron-frame structure for the experiment.

In this experimental device, the main body is almost symmetrically suspended at the four points in a horizontal plane with the same height as the center of gravity of the main body.

Consequently, the motion of the main body which is suspended by the vibration isolation pieces is nearly identified as two uncoupled modes of vibration, which are independent of each other.

One of these modes is a translation mode of motion in a plane including the vertical axis of rotation, and the other is a rolling mode of motion around the horizontal axis which is perpendicular to the plane for the translation mode of motion, and which passes the center of gravity of the main body.

In the experiment, an already-known unbalanced weight is fixed on the inner wall of the rotating basket for washing and dehydrating, and the non-stationary transitions of motion of the main body and the

Fig.10 Maximum displacement of motion normalized by maximum displacement of a rotor regarded as having an ideal energy source.

rotational speed characteristics of the rotating basket are measured when the rotating basket is started.

Furthermore, the already-known unbalanced weight is fixed on the basket wall with the same height as the center of gravity, and furthermore, the horizontal displacement of the main body is measured in the centroid plane, too, because an experiment on the transition mode of motion is one of the objects of the study.

Furthermore, the measurement of the non-stationary transitions of motion of the main body of the machine is continued until the rotational speed of the basket sufficiently exceeds the first order natural frequency of the machine, at which a rotor has the translation mode of motion.

4.2 Experimental results of an actual rotor passing through a resonance

Fig. 11 shows the relation between the unbalanced weight fixed on the inner wall of the rotating basket and the critical speed which is the angular velocity of rotation where the maximum displacement of motion of the experimental machine is generated.

The experimental values of the critical speeds are marked by the sign • in Fig. 11.

In the experiment, the magnitude of

Fig. 11 Comparison of experimental value for critical speed of a practical rotor with theoretical value.

Fig. 12 Experimental examples of non-stationary transitions of motion in an actual rotor.

unbalance is expressed in units of weight because the radius of gyration of the unbalanced weight fixed on the basket wall is approximated as a constant.

As the critical speed which is measured in the experiment disperses a little in each experiment, the experimental values make a little width in Fig. 11.

Naturally, it can be seen from Fig. 11 that the critical speed which is measured in the experiment decreases as the unbalanced weight increases.

However, the critical speed does not change so much with the unbalanced weight in the region where the unbalanced weight is small, and the critical speed remarkably changes, decreasing as the unbalanced weight increases in the region where the unbalanced weight is larger than $mg = 1.8 \text{ kg}$.

When the unbalanced weight $mg \geq 2.55 \text{ kg}$, the angular velocity of rotation is stagnant near the first order resonant frequency, and the machine is reduced to a critical condition where it can not pass through the resonance, and so the critical speed does hardly change with the unbalanced weight in this region of unbalanced weight.

Fig. 12 shows examples of the non-stationary transitions of motion of the experimental machine, which were measured in the region of the angular velocities of rotation near the critical speed.

The upside of the figure shows an example when the rotor can pass through the resonance, and the downside shows an example of the machine vibrating with a large and constant amplitude of displacement of motion because it cannot pass through the first order critical speed.

Table 2 shows the relation between the unbalanced weight mg and the maximum amplitude of displacement of motion, $X_{\max}$, of the main body of the machine, measured in the experiment.

Although the maximum displacement of motion increases as the unbalanced weight increases, the relation between the unbalanced weight and the maximum displacement of motion can not be perfectly approximated by only a linear line, and there is a tendency that the maximum displacement of motion is a value a little larger than the

Table 2 Maximum displacement of motion, estimable value, critical speed, etc. which depend on unbalanced weight mg in experimental machine.

U.B (kg)	$X_{\max}$ (mm)	μ ($\times 10^{-3}$)	$\rho_{\max}$	K	P	$f_n \times P$ (Hz)
$mg=0.0$	0.0	0.0	—	0.0	1.45	1.70
1.0	17.0	2.348	2.86	0.111	1.43	1.67
2.0	41.0	8.504	3.62	0.509	1.31	1.53
2.2	49.0	10.09	4.0	0.667	1.24	1.45
2.3	54.0	10.72	4.13	0.731	1.21	1.41
2.4	59.5	11.67	4.24	0.817	1.15	1.35
2.6	67.5	13.70	4.46	1.01	1.0	1.17

($q = 2.63 \text{ c}$, $Mg = 27.5 \text{ kg}$, $f_n = 1.17 \text{ Hz}$
 $I = 4.61 \text{ kg.cm.s}^2$)

value which is proportional to the unbalanced weight.

4.3 An estimation of the condition for the experimental machine to be able to pass through the first order natural frequency

The machine constants in the experimental machine are

$$Mg = 27.5 \text{ kg}, e = 17 \text{ cm}, f_n = 1.17 \text{ Hz}, \\ I = 4.61 \text{ kg.cm.s}^2, k_0 = 1.37$$

where the experimental machine has a rotational speed reduction mechanism and k_0 is a constant rotational speed reduction ratio in the machine.

As the driving torque of energy source $T(\dot{\theta})$ given by Eq.(1) is regarded as an apparent torque including a static load torque $S(\dot{\theta})$ which is due to a frictional and pneumatic resistance acting on the rotating shaft of basket, the driving torque of energy source $T(\dot{\theta})$ given by Eq.(1) is expressed as follows;

$$T(\dot{\theta}) = k_0 (T_M - T_{SL}) \quad \dots\dots (22)$$

where $S(\dot{\theta}) = k_0 T_{SL}$, and T_M is the driving torque of energy source and T_{SL} the static load torque, which act on the rotating shaft of motor.

As the experimental machine has a rotational speed reduction mechanism, the driving torque of energy source and the static load torque, which act on the rotating shaft of basket, are given by k_0 times those torques which act on the rotating shaft of motor.

Substituting Eq.(22) into the third equation in Eq.(1), and considering a rotor without any unbalanced weight, the following equation is obtained by substituting $m = 0$ into the third equation in Eq.(1),

$$I \ddot{\theta} = k_0 (T_M - T_{SL}) \quad \dots\dots (23)$$

where, putting the driving torque of motor as $T_M = 0$, we obtain

$$S(\dot{\theta}) = k_0 T_{SL} = -I \ddot{\theta} \quad \dots\dots (24)$$

Eq.(24) means the rotational speed reduction characteristics of a rotor without any unbalanced weight.

Therefore, substituting the angular deceleration of rotation $\ddot{\theta}$, which is averaged by measuring a slope of the rotational speed reduction characteristics which is approximated by a linear line at an angular velocity of rotation of a rotor, into Eq.(24), the static load torque $S(\dot{\theta})$ of the machine, which depends on the angular velocity of rotation, is obtained.

In case of the experimental machine, the angular deceleration of rotation h_D of the rotating basket without any unbalanced weight in the region of the angular velocities of rotation near the first order natural frequency is obtained as,

$$h_D = \ddot{\theta}/2\pi = -0.0507 \text{ Hz/s}$$

Substituting h_D into Eq.(24), we obtain

$$T_{SL} = 2\pi \times 4.61 \times 0.0507 / 1.37 = 1.07 \text{ kg.cm}$$

Furthermore, as the rotational speed reduction characteristics is approximated

by a linear line in the region near the first order natural frequency, T_{SL} can be considered a constant in the region near the resonance.

Next, the experimental machine has an induction motor with 4 poles, which is started by an electric condenser, and the driving torque of motor T_M is given in the region near the first order natural frequency as follows;

$$T_M = 12 \text{ kg.cm}$$

where, T_M is approximately regarded as a constant in the region near the first order natural frequency because T_M has a gentle, flat characteristics in this region, although it depends on the angular velocity of rotation $\dot{\theta}$.

Thus, the driving acceleration of rotation h which is due to the apparent driving torque of energy source $T(\dot{\theta})$ is given as

$$h = \frac{k_0 (T_M - T_{SL})}{2\pi (I + me^2)} = 0.52 \text{ Hz/s}$$

Furthermore, as the first order natural frequency f_n where the experimental machine has a translation mode of motion is

$$f_n = 1.17 \text{ Hz}$$

, the acceleration coefficient of rotation q is obtained as

$$q = 2.63 \text{ (c)}$$

because $q = f_n^2/h$.

Next, although the damping factor of the system rather depends on the displacement of motion of the main body in the experimental machine, the damping factor is approximately $\zeta = 0.15$ on the average.

In the case where a rotor has a damping factor ζ of this degree, the maximum displacement of motion normalized by the eccentricity of the rotor is scarcely influenced by the estimable value K , and as understood from Table 1, it can be approximated by a value of $(1/2\zeta)$ with a numerical error of less than 10 %.

Furthermore, as above described, the damping factor of the system rather depends on the displacement of motion of the main body of machine.

So, for simplicity of calculation, the following equation is used for estimating the value of $\rho_{\max}$ in the experimental machine,

$$\rho_{\max} = \frac{(M + m)}{me} X_{\max} \quad \dots\dots (25)$$

where, the experimental value which is shown in Table 2 is substituted for $X_{\max}$.

The maximum dynamic load torque $(T_{D\max})_M$ on the rotating shaft of rotor, which is generated in the experimental machine, is obtained by using Eq.(6) as follows;

$$(T_{D\max})_M = 10.67 \text{ kg.cm} \quad \text{when } mg = 2.5 \text{ kg}$$

$$(T_{D\max})_M = 11.53 \text{ kg.cm} \quad \text{when } mg = 2.55 \text{ kg}$$

where $(T_{D\max})_M = T_{D\max}/k_0$.

As the apparent driving torque of energy source on the rotating shaft of motor is given as

$T(\dot{\theta})/k_0 = T_M - T_{SL} = (12.0 - 1.07) = 10.93 \text{ kg.cm}$
 , the apparent driving torque of energy source is larger than the maximum dynamic load torque which is generated when the unbalanced weight mg fixed on the wall of rotating basket is 2.5 kg, that is,

$T(\dot{\theta})/k_0 = 10.93 \text{ kg.cm} > (T_{Dmax})_M = 10.67 \text{ kg.cm}$
 , and so in this case, the experimental rotor can pass through the first order natural frequency.

Furthermore, in the case where $mg = 2.55 \text{ kg}$, the experimental rotor cannot pass through the resonance because

$(T_{Dmax})_M = 11.53 \text{ kg.cm} > T(\dot{\theta})/k_0 = 10.93 \text{ kg.cm}$
 The theoretical result above obtained agrees well with the experimental result which is shown in Fig.11.

4.4 margin of driving torque of energy source and critical speed, in the experimental machine

The estimable value K and the margin B of the driving torque of energy source, which correspond to the unbalanced weight fixed on the rotating basket in the experimental machine, are shown in Table 2.

They are calculated by Eqs.(4) and (21) when the acceleration coefficient of rotation is $q = 2.63 \text{ c}$.

For example, if $mg = 2.0 \text{ kg}$, then the estimable value is $K = 0.5$, that is, the margin B of the driving torque of the motor is about 50 % in the experimental machine.

The maximum unbalanced weight mg which is generated in the rotating basket of an impeller type full automatic washer which is sold in the market is considered to be smaller than $mg = 1.8 \text{ kg}$, that is, it can be predicted that the machine has a more than 60 % margin of the driving torque of motor.

The critical speed which corresponds to a free estimable value K , in the case where the experimental machine has the acceleration coefficient of rotation $q = 2.63 \text{ c}$ and the damping factor $\zeta = 0.15$, is approximately calculated using the two critical speeds when $q = 2$ and 4, with the parameter of $\zeta = 0.15$, which are read from Figs. 7 and 8.

Namely, the critical speed P at $q = 2.63 \text{ c}$ is approximately calculated by regarding it as a point on a linear line which is obtained by connecting two points which are the critical speeds at $q = 2$ and 4.

The critical speed P in the experimental machine, which is obtained by the method above described, is shown in Table 2, corresponding to the estimable value K .

Furthermore, the actual critical speed in the experimental machine, which is given by f_n times the critical speed P , is shown in Table 2, too, where the first order natural frequency f_n of the experimental machine is $f_n = 1.17 \text{ Hz}$.

The sign $\blacktriangle$ on the solid line which is shown in Fig.11 represents the actual critical speed which is theoretically obtained by the method above described, and it

agrees rather well with the experimental value of the critical speed which is given by the sign $\bullet$ which is shown in the same figure.

5. Conclusion

(1) Non-stationary transitions of motion when a rotor easily passes through a resonance are studied and clearly understood, because the limit of the estimable value K for a rotor passing through a resonance, $0.0 \leq K \leq 1.0$, is substituted for the limit of the unbalance constant μ , which is generated in many kinds of rotors.

The results of these studies are used to develop the design criteria for a rotor with a usable limit.

(2) Using an actual rotor, an experiment is made to determine whether it can pass through a resonance or not and it is shown that the experimental results agree well with the theoretical ones.

Furthermore, it is shown that, in an actual rotor, the critical speed which is obtained theoretically is in rather good agreement with the critical speed that is obtained experimentally.

(3) A clear physical explanation is made of an outstripping of the phase of the displacement vector of motion by the phase of the centrifugal force vector by more than a cycle of rotation in the region of angular velocities of rotation away from the resonance.

(4) A simple, precise method of calculation with which the non-stationary transitions of motion in an accelerated, unbalanced rotor with a single-degree-of-freedom can be numerically analyzed is introduced.

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