539.4.019:539.384.6

Paper No. 180-7

Dynamic Displacements and Stresses in a Circular Cylindrical
Shell of Finite Length with Both Ends Clamped
Which is Subjected to Concentrated Impulsive Loads *

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In this paper, the dynamic displacements and stresses in a circular cylindrical shell of finite length with both ends clamped, which is suddenly subjected to radial concentrated loads on its surface, are analyzed on the basis of Flügge's shell theory with use of the Laplace transformation.

As a result, it is shown that Donnell's shell theory, which is treated in the previous paper, yields good results if the length of the shell is short or the thickness is thin using Flügge's theory as the standard of comparison, and the values of the dynamic maximum stresses are scarcely affected by the axial and circumferential inertia forces of the shell but the values of the lowest frequencies and the transient responses of the stresses are much affected thereby.

1. Introduction

Previously, the authors analyzed the dynamic displacements and stresses in circular cylindrical shells of finite lengths, which are suddenly subjected to radial concentrated loads with use of Donnell type shell theory and proposed a method for analyzing impulsive responses of circular cylindrical shells of finite lengths under arbitrary loading and boundary conditions[1]. But it is somewhat doubtful that Donnell's theory has always the same accuracy as the generally accepted theory such as Flügge's theory. Therefore, the accuracy of Donnell's theory was examined by using various practical methods. Hoff et al. commented using Flügge's theory as the standard of comparison that as the characteristic circumferential wave length increases, so does the error in Donnell's theory[2]~[4].

Concerning dynamic problems, Hamada calculated the natural frequencies of cylindrical shells of finite lengths by a numerical analysis using finite-differences method and indicated similar results to Hoff with respect to the accuracy of Donnell's theory[5].

In a concrete example, Liao investigated the dynamic responses of simply-supported cylindrical shells to a concentrated impulsive load with use of Donnell's and Flügge's theories including the effects of the axial, circumferential and radial inertia forces and showed that Donnell's theory is not recommended if the thickness or the length of the shell increases [6]. Recently, Shirakawa also presented an analytical study of simply-supported cylindrical shells subjected to

concentrated impulsive loads on the basis of Mizoguchi's shell theory and investigated the in-plane, i.e., axial and circumferential, inertia effect[7].

However, in most of the papers published up to date, the displacements are represented by eigenfunctions with trigonometric expression with respect to the axial and circumferential coordinates, so that only the results for the simply-supported circular cylindrical shells are obtained.

In this paper, impulsive responses of a circular cylindrical shell of finite length with both ends clamped, which is suddenly subjected to radial concentrated loads on its surface, are analyzed on the basis of Flügge's shell theory. The purposes of this paper are to demonstrate the differences of the present results from those by Donnell's theory in the authors' previous paper and to examine the in-plane inertia effect of the shell.

2. Fundamental Equations

Stress resultants in a circular cylindrical shell are defined as shown in Fig.1 and (u, v, w) are the displacement components

Fig.1 Stress Resultants in a Cylindrical Shell

* Received 29th March, 1978.

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corresponding to the cylindrical coordinates (x, θ, z) for the midsurface of the shell.

R is the mean radius, h is the thickness, E is Young's modulus, ν is Poisson's ratio and γ is weight per unit volume of the shell and g is the acceleration of gravity and t is time. Then, the fundamental equations for a cylindrical shell employed in this paper are written in the following way.

(I) *For the Analysis of Axisymmetric Motions*

The following uncoupled differential equation for the radial displacement is deduced from two coupled differential equations including the effects of the axial and radial inertia forces.

$$\frac{\partial^6 w}{\partial \xi^6} + J^2 \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{\kappa^2} \frac{\partial^2}{\partial \tau^2} \left(\frac{\partial^2 w}{\partial \xi^2} - w - \kappa^2 \frac{\partial^4 w}{\partial \xi^4} \right) - \frac{1}{\kappa^2} \frac{\partial^4 w}{\partial \tau^4} = 0 \quad (1)$$

Similarly, the differential equation relating the two displacement components (u, w) is given by

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \tau^2} = \nu \frac{\partial w}{\partial \xi} \quad (2)$$

In the case where the effect of the in-plane, i.e., axial, inertia force is neglected, the equation corresponding to Eq.(1) becomes

$$\frac{\partial^5 w}{\partial \xi^5} + \frac{1}{\kappa^2} \frac{\partial^3 w}{\partial \xi \partial \tau^2} + J^2 \frac{\partial w}{\partial \xi} = 0 \quad (3)$$

and the equation corresponding to Eq.(2) becomes

$$\frac{\partial^2 u}{\partial \xi^2} = \nu \frac{\partial w}{\partial \xi} \quad (4)$$

where, $\xi = x/R$, $\tau = ct/R$, $J^2 = (1 - \nu^2)/\kappa^2$
 $\kappa^2 = h^2/12R^2$, $c = \sqrt{gE/\gamma(1 - \nu^2)}$

(II) *For the Analysis of Non-Axisymmetric Motions*

The following uncoupled differential equation for the radial displacement is deduced from three coupled differential equations including the effects of the axial, circumferential and radial inertia forces derived by Flügge[8].

Fig.2 A Circular Cylindrical Shell with Both Ends Clamped Which is Subjected to One Concentrated Impulsive Load

$$\begin{aligned} \Phi^8 w = & \left(\frac{3-\nu}{1-\nu} \nabla^6 - \frac{1}{\kappa^2} \nabla^4 + \frac{1}{\kappa^2} \nabla^2 + 2 \frac{1+\nu}{\kappa^2} \frac{\partial^2}{\partial \xi^2} \right) \\ & \times \frac{\partial^2 w}{\partial \tau^2} - \left(\frac{2}{1-\nu} \nabla^4 - \frac{1}{\kappa^2} \frac{3-\nu}{1-\nu} \nabla^2 \right. \\ & \left. + \frac{1}{\kappa^2} \frac{2}{1-\nu} \right) \frac{\partial^4 w}{\partial \tau^4} - \frac{1}{\kappa^2} \frac{2}{1-\nu} \frac{\partial^6 w}{\partial \tau^6} \dots \dots \dots (5) \end{aligned}$$

Similarly, the differential equations relating the three displacement components (u, v, w) are given by

$$\begin{aligned} \Phi^4 u = & \Psi_1 w - \frac{2\nu}{1-\nu} \frac{\partial^3 w}{\partial \xi \partial \tau^2} \\ & + \kappa^2 \frac{\partial^2}{\partial \tau^2} \left(\frac{2}{1-\nu} \frac{\partial^3 w}{\partial \xi^3} - \frac{\partial^3 w}{\partial \xi \partial \theta^2} \right) \\ \Phi^4 v = & \Psi_2 w + \frac{\partial^2}{\partial \tau^2} \left(\kappa^2 \frac{3-\nu}{1-\nu} \frac{\partial^3 w}{\partial \xi^2 \partial \theta} - \frac{2}{1-\nu} \frac{\partial w}{\partial \theta} \right) \end{aligned} \quad (6)$$

where, $\nabla^2 \equiv \partial^2/\partial \xi^2 + \partial^2/\partial \theta^2$

$$\begin{aligned} \Phi^8 \equiv & \nabla^8 + 2\nu \frac{\partial^6}{\partial \xi^6} + 6 \frac{\partial^6}{\partial \xi^4 \partial \theta^2} + 2(4-\nu) \frac{\partial^6}{\partial \xi^2 \partial \theta^4} \\ & + 2 \frac{\partial^6}{\partial \theta^6} + J^2 \frac{\partial^4}{\partial \xi^4} + \frac{\partial^4}{\partial \theta^4} + 2(2-\nu) \frac{\partial^4}{\partial \xi^2 \partial \theta^2} \end{aligned}$$

$$\Phi^4 \equiv \nabla^4 - \frac{3-\nu}{1-\nu} \frac{\partial^2}{\partial \tau^2} \nabla^2 + \frac{2}{1-\nu} \frac{\partial^4}{\partial \tau^4}$$

$$\begin{aligned} \Psi_1 \equiv & \nu \frac{\partial^3}{\partial \xi^3} - \frac{\partial^3}{\partial \xi \partial \theta^2} \\ & - \kappa^2 \left(\frac{\partial^5}{\partial \xi^5} - 3\kappa^2 \frac{1-\nu}{2} \frac{\partial^5}{\partial \xi^3 \partial \theta^2} - \frac{\partial^5}{\partial \xi \partial \theta^4} \right) \end{aligned}$$

$$\begin{aligned} \Psi_2 \equiv & (2+\nu) \frac{\partial^3}{\partial \xi^2 \partial \theta} + \frac{\partial^3}{\partial \theta^3} \\ & - 2\kappa^2 \left(\frac{\partial^5}{\partial \xi^4 \partial \theta} + \frac{\partial^5}{\partial \xi^2 \partial \theta^3} \right) \end{aligned}$$

In the case where the effect of the in-plane, i.e., axial and circumferential, inertia force is neglected, the equation corresponding to Eq.(5) becomes

$$\Phi^8 w = - \frac{1}{\kappa^2} \frac{\partial^2}{\partial \tau^2} \nabla^4 w \quad (7)$$

and the equations corresponding to Eqs.(6) become

$$\nabla^4 u = \Psi_1 w, \quad \nabla^4 v = \Psi_2 w \quad (8)$$

3. Method of Solution

For a circular cylindrical shell of finite length with both ends clamped as shown in Fig.2, which is suddenly loaded at the center by a concentrated load of magnitude F_0 , the loading function $F(\xi, \theta, t)$ is expressed as

$$F(\xi, \theta, t) = F_0 \delta(\xi) \delta(\theta) H(t) \quad (9)$$

where $\delta(\xi)$ is Dirac's delta function and $H(t)$ is Heaviside's step function. Expanding Eq.(9) into infinite Fourier series with re-

spect to the variable θ , the following equation is derived.

$$F(\xi, \theta, t) = F_0 \delta(\xi) \left\{ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1,2,3,\dots}^{\infty} \cos n\theta \right\} H(t) \quad (10)$$

Because Eq.(10) consists of a constant term and cosine terms, the concentrated load in Fig.2 is represented as a superposition of a uniformly distributed line load and trigonometrically distributed line loads along the circumference as shown in Fig.3.

3.1 Analysis for a Uniformly Distributed Line Load

As shown in Fig.3(a), a radial line load uniformly distributed along the circumference with its magnitude q_0 is suddenly applied at the center of the circular cylindrical shell of finite length $2l$ with both ends clamped.

Because of symmetry of the problem, the boundary conditions may be prescribed at the plane of $x=0$ and $x=l$, such that

(I) At $x=0$

$$u = \frac{\partial w}{\partial x} = 0, \quad \frac{D}{R^3} \frac{\partial^3 w}{\partial \xi^3} (= -Q_x) = \frac{q_0}{2} H(t) \quad (11)$$

(II) At $x=l$

$$u = w = \partial w / \partial x = 0 \quad (12)$$

The shell is at rest and in unstrained state, so that the initial conditions are given by

$$(f)_{t=0} = (\partial f / \partial t)_{t=0} = 0$$

Thus, the Laplace transformation of Eq.(1) becomes

$$\frac{d^6 \bar{w}}{d\xi^6} - \lambda^2 \frac{d^4 \bar{w}}{d\xi^4} + \left(J^2 + \frac{\lambda^2}{\kappa^2} \right) \frac{d^2 \bar{w}}{d\xi^2} - \frac{\lambda^2}{\kappa^2} (1 + \lambda^2) \bar{w} = 0 \quad (13)$$

where,

$$\bar{f} = \int_0^\infty f e^{-pt} dt, \quad \lambda = R_p / c$$

The general solutions for the transformed displacement components $(\bar{u}, \bar{w})$ can be assumed as

$$\bar{u} = \sum_{j=1}^6 f_j c_j e^{\alpha_j \xi}, \quad \bar{w} = \sum_{j=1}^6 c_j e^{\alpha_j \xi} \quad (14)$$

where α_j ($j=1, \dots, 6$) are the roots of the

characteristic equation of sixth degree derived from Eq.(13) and f_j is derived from Eq.(2) by substituting Eqs.(14).

Substituting Eqs.(14) into the Laplace transformations of the boundary conditions, i.e., Eqs.(11) and (12), yields a set of six linear algebraic equations in the six undetermined coefficients c_j ($j=1, \dots, 6$):

$$\begin{bmatrix} a_{11} & \dots & a_{16} \\ \vdots & & \vdots \\ a_{61} & \dots & a_{66} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_6 \end{bmatrix} = \frac{q_0 R^3}{2pD} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (15)$$

where the elements a_{ij} are defined in the following way and the symbol $[A]$ denotes the matrix consisting of these elements a_{ij} .

$$a_{1j} = f_j e^{\alpha_j L}, \quad a_{2j} = e^{\alpha_j L}, \quad a_{3j} = \alpha_j e^{\alpha_j L}$$

$$a_{4j} = f_j, \quad a_{5j} = \alpha_j, \quad a_{6j} = \alpha_j^3$$

$$D = Eh^3 / 12(1 - \nu^2), \quad L = l / R$$

From Eq.(15) the coefficient c_j is obtained as

$$c_j = \frac{q_0 R^3}{2pD} \frac{|A_{6j}|}{|A|} \quad (16)$$

where $|A_{ij}|$ denotes a cofactor of the element a_{ij} in the determinant $|A|$. The solutions of the transformed displacement components $(\bar{u}, \bar{w})$ are thus determined. The Laplace inverse transformations of the solutions $(\bar{u}, \bar{w})$ are determined as the sum of all the residues of simple poles at the origin and on the imaginary axis. Thus, putting the poles as 0 and $\pm i p_m$ ($0 < p_1 < p_2 < \dots$), the inverse transformations of $(\bar{u}, \bar{w})$ are given by the equations

$$\left. \begin{aligned} \frac{Du}{q_0 R^3} &= u_{st} - \kappa^2 \sum_{m=1}^{\infty} \frac{\cos \lambda_m \tau}{\lambda_m^2} \left\{ \frac{\sum_{j=1}^6 f_j |A_{6j}| e^{\alpha_j \xi}}{\sum_{j=1}^6 |B_j|} \right\} \\ \frac{Dw}{q_0 R^3} &= w_{st} - \kappa^2 \sum_{m=1}^{\infty} \frac{\cos \lambda_m \tau}{\lambda_m^2} \left\{ \frac{\sum_{j=1}^6 |A_{6j}| e^{\alpha_j \xi}}{\sum_{j=1}^6 |B_j|} \right\} \end{aligned} \right\} \quad (17)$$

where the elements b_{ij} of the determinant $|B_j|$ are defined in the following way.

(i) In the case where $k \neq j$

$$b_{ij} = a_{ij}$$

Fig.3 Line Loads Acting on a Finite Cylindrical Shell
(a) Uniformly Distributed Line Load
(b), (c) Trigonometrically Distributed Line Load

(ii) In the case where $k = j$

$$b_{1j} = (r_j + Lf_j)e^{\alpha_j L} \beta_j, \quad b_{2j} = Le^{\alpha_j L} \beta_j$$

$$b_{3j} = (1 + L\alpha_j)e^{\alpha_j L} \beta_j, \quad b_{4j} = r_j \beta_j$$

$$b_{5j} = \beta_j, \quad b_{6j} = 3\alpha_j^2 \beta_j$$

where, $f_j = v\alpha_j / (\alpha_j^2 + \lambda_m^2)$

$$r_j = \frac{-v(\alpha_j^2 - \lambda_m^2) + 2v\kappa^2 \alpha_j / \beta_j}{(\alpha_j^2 + \lambda_m^2)^2}$$

$$\beta_j = \frac{\kappa^2 \alpha_j^4 - \alpha_j^2 + 1 - 2\lambda_m^2}{\alpha_j (3\alpha_j^4 + 2\alpha_j^2 \lambda_m^2 + \lambda_m^2 / \kappa^2)}$$

The values of the poles p_m are determined from the roots of the transcendental equation $|A| = 0$ and coincide with the natural frequencies of the circular cylindrical shell with both ends clamped. The roots α_j ($j=1, \dots, 6$) satisfy the following characteristic equation of sixth degree.

$$\alpha^6 + \lambda_m^2 \alpha^4 + (\lambda_m^2 - \lambda_m^2 / \kappa^2) \alpha^2 + \lambda_m^2 (1 - \lambda_m^2) / \kappa^2 = 0 \quad \dots \dots \dots (18)$$

where, $\lambda_m = R p_m / c$.

The first terms of Eqs.(17) are determined from the residues at the origin and coincide with the solutions of the corresponding static problem.

The second terms are derived from the residues at the poles on the imaginary axis. The complete solution is represented as a superposition of the static solution and the solutions for all modes of the natural vibrations of the shell.

For the analysis in the case where the effect of the in-plane inertia force is neglected, Eqs.(3) and (4) may be employed.

The Laplace transformation of Eq.(3) under the same initial conditions becomes

$$\frac{d^5 \bar{w}}{d\xi^5} + \left(J^2 + \frac{\lambda^2}{\kappa^2} \right) \frac{d\bar{w}}{d\xi} = 0 \quad \dots \dots \dots (19)$$

Then the general solutions of the transformed displacement components $(\bar{u}, \bar{w})$ can be assumed as follows.

$$\left. \begin{aligned} \bar{u} &= \sum_{j=1}^4 f_j c_j e^{\alpha_j \xi + \frac{1+\lambda^2}{v} c_5 \xi + c_6} \\ \bar{w} &= \sum_{j=1}^4 c_j e^{\alpha_j \xi + c_5} \end{aligned} \right\} \dots \dots \dots (20)$$

where α_j ($j=1, \dots, 4$) are the roots of the characteristic equation of fourth degree reduced from Eq.(19) and f_j is derived from Eq.(4) by substituting Eqs.(20).

The coefficients c_j ($j=1, \dots, 6$) will be determined with use of the boundary conditions, i.e., Eqs.(11) and (12).

The Laplace inverse transformation may be determined by the same procedure. The solutions for the displacement components are thus given by the equations

$$\left. \begin{aligned} \frac{Du}{q_0 R^3} &= u_{st} - \kappa^2 \sum_{m=1}^{\infty} \frac{\cos \lambda_m \tau}{\lambda_m^2} \frac{1}{\sum_{j=1}^6 |B_j|} \left\{ \sum_{j=1}^4 f_j |A_{6j}| e^{\alpha_j \xi + \frac{1-\lambda_m^2}{v} |A_{65}| \xi + |A_{66}|} \right\} \\ \frac{DW}{q_0 R^3} &= w_{st} - \kappa^2 \sum_{m=1}^{\infty} \frac{\cos \lambda_m \tau}{\lambda_m^2} \frac{1}{\sum_{j=1}^6 |B_j|} \left\{ \sum_{j=1}^4 |A_{6j}| e^{\alpha_j \xi + |A_{65}|} \right\} \end{aligned} \right\} \dots \dots \dots (21)$$

where the elements a_{ij} in the matrix $[A]$ are defined in the following way.

(i) If $j=1, \dots, 4$

$$a_{1j} = f_j e^{\alpha_j L}, \quad a_{2j} = e^{\alpha_j L}, \quad a_{3j} = \alpha_j e^{\alpha_j L}, \quad a_{4j} = f_j, \quad a_{5j} = \alpha_j, \quad a_{6j} = \alpha_j^3$$

(ii) If $j=5, 6$

$$a_{16} = a_{25} = a_{46} = 1, \quad a_{15} = (1 - \lambda_m^2)L/v, \quad a_{26} = a_{35} = a_{36} = a_{45} = a_{55} = a_{56} = a_{65} = a_{66} = 0$$

and the elements b_{ik} in the matrix $[B_j]$ are defined in the following way.

(I) In the case where $k \neq j$

$$b_{ik} = a_{ik}$$

(II) In the case where $k = j$

(i) If $j=1, \dots, 4$

$$b_{1j} = (r_j + Lf_j)e^{\alpha_j L} \beta_j, \quad b_{2j} = Le^{\alpha_j L} \beta_j, \quad b_{3j} = (1 + L\alpha_j)e^{\alpha_j L} \beta_j, \quad b_{4j} = r_j \beta_j, \quad b_{5j} = \beta_j, \quad b_{6j} = 3\alpha_j^2 \beta_j$$

(ii) If $j=5, 6$

$$b_{15} = 2\kappa^2 L/v, \quad b_{16} = b_{25} = b_{26} = b_{35} = b_{36} = b_{45} = b_{46} = b_{55} = b_{56} = b_{65} = b_{66} = 0$$

where, $f_j = \frac{v}{\alpha_j}, \quad r_j = \frac{3}{2\alpha_j^4}, \quad \beta_j = -\frac{1}{2\alpha_j^3}$

The values of λ_m are determined from the roots of the equation $|A| = 0$ and the values of α_j ($j=1, \dots, 4$) satisfy the following equation.

$$\alpha^4 + J^2 - \lambda_m^2 / \kappa^2 = 0 \quad \dots\dots\dots (22)$$

The static solutions corresponding to the first terms of Eqs.(17) and (21) are easily given by the equations

$$\left. \begin{aligned} u_{st} &= \frac{1}{2|A|} \left\{ \sum_{j=1}^4 f_j |A_{6j}| e^{\alpha_j \xi} + \frac{\xi}{V} |A_{65}| + |A_{66}| \right\} \\ w_{st} &= \frac{1}{2|A|} \left\{ \sum_{j=1}^4 |A_{6j}| e^{\alpha_j \xi} + |A_{65}| \right\} \end{aligned} \right\} \quad \dots\dots\dots (23)$$

where α_j ($j=1, \dots, 4$) are the roots of Eq.(22) obtained by equating λ_m to 0.

The coefficient f_j and the elements a_{ij} in the matrix $[A]$ coincide with the forms defined in Eqs.(21), if λ_m equates to 0.

3.2 Analysis for Trigonometrically Distributed Line Loads

As shown in Fig.3(b) and (c), a radial line load distributed in the form of a cosine function along the circumference with its magni-

Substituting the third equation of Eqs.(26), the Laplace transformation of the fundamental equation (5) under the same initial conditions becomes

$$\frac{d^8 \bar{w}_n}{d\xi^8} + a_6 \frac{d^6 \bar{w}_n}{d\xi^6} + a_4 \frac{d^4 \bar{w}_n}{d\xi^4} + a_2 \frac{d^2 \bar{w}_n}{d\xi^2} + a_0 \bar{w}_n = 0 \quad \dots\dots\dots (27)$$

where ($a_6, \dots, a_0$) are the coefficients which depend on the circumferential wave number n and the parameter of Laplace transformation p . Therefore, the general solutions of the transformed displacement components ($\bar{U}_n, \bar{V}_n, \bar{W}_n$) can be assumed as

$$\bar{U}_n = \sum_{j=1}^8 f_j c_j e^{\alpha_j \xi}, \quad \bar{V}_n = \sum_{j=1}^8 g_j c_j e^{\alpha_j \xi}, \quad \bar{W}_n = \sum_{j=1}^8 c_j e^{\alpha_j \xi} \quad \dots\dots\dots (28)$$

where, α_j ($j=1, \dots, 8$) are the roots of the characteristic equation of eighth degree derived from Eq.(27) and (f_j, g_j) are derived from Eq.(6) by substituting Eqs.(28). The coefficients c_j ($j=1, \dots, 8$) will be determined with use of the boundary conditions, i.e., Eqs.(24) and (25).

The Laplace inverse transformation may be determined by the same procedure. The solutions for the displacement components are thus given by the equations

$$\left. \begin{aligned} \frac{DU_n}{q_0 R^3} &= U_n, st - \sum_{m=1}^{\infty} \frac{\cos \lambda_{n,m} \tau}{\lambda_{n,m}^2} \frac{\sum_{j=1}^8 f_j |A_{8j}| e^{\alpha_j \xi}}{\sum_{j=1}^8 |B_j|}, \quad \frac{DV_n}{q_0 R^3} = V_n, st - \sum_{m=1}^{\infty} \frac{\cos \lambda_{n,m} \tau}{\lambda_{n,m}^2} \frac{\sum_{j=1}^8 g_j |A_{8j}| e^{\alpha_j \xi}}{\sum_{j=1}^8 |B_j|} \\ \frac{DW_n}{q_0 R^3} &= W_n, st - \sum_{m=1}^{\infty} \frac{\cos \lambda_{n,m} \tau}{\lambda_{n,m}^2} \frac{\sum_{j=1}^8 |A_{8j}| e^{\alpha_j \xi}}{\sum_{j=1}^8 |B_j|} \end{aligned} \right\} \quad \dots\dots\dots (29)$$

where the elements a_{ij} in the matrix $[A]$ are defined in the following way.

$$a_{1j} = f_j e^{\alpha_j L}, \quad a_{2j} = g_j e^{\alpha_j L}, \quad a_{3j} = e^{\alpha_j L}, \quad a_{4j} = \alpha_j e^{\alpha_j L}, \quad a_{5j} = f_j, \quad a_{6j} = \alpha_j g_j, \quad a_{7j} = \alpha_j, \quad a_{8j} = \alpha_j^3 + f_j \alpha_j^2$$

and the elements b_{ik} in the matrix $[B_j]$ are defined in the following way.

(I) In the case where $k \neq j$

$$b_{ik} = a_{ik}$$

(II) In the case where $k = j$

$$b_{1j} = (r_j + L f_j) e^{\alpha_j L} \beta_j, \quad b_{2j} = (s_j + L g_j) e^{\alpha_j L} \beta_j, \quad b_{3j} = L e^{\alpha_j L} \beta_j, \quad b_{4j} = (1 + L \alpha_j) e^{\alpha_j L} \beta_j$$

tude $q_0 \cos(n\theta)$ is suddenly applied at the center of the circular cylindrical shell of finite length $2L$ with both ends clamped.

Because of symmetry of the problem, the boundary conditions may be prescribed at the plane of $x=0$ and $x=L$, such that

$$\left. \begin{aligned} \text{(I) At } x=0 \\ u = \partial v / \partial x (=S_x) = \partial w / \partial x = 0 \\ \frac{D}{R^3} \left\{ \frac{\partial^3 w}{\partial \xi^3} + \frac{\partial^2 u}{\partial \xi^2} \right\} (= -T_x) = \frac{q_0}{2} \cos n\theta H(t) \end{aligned} \right\} \quad \dots\dots\dots (24)$$

$$\left. \begin{aligned} \text{(II) At } x=L \\ u = v = w = \partial w / \partial x = 0 \end{aligned} \right\} \quad \dots\dots\dots (25)$$

where, $S_x = N_{x\theta} + M_{x\theta} / R$, $T_x = Q_x - \partial M_{x\theta} / R \partial \theta$

The solutions for the displacement components may be sought in the form of separation of variables.

$$\left. \begin{aligned} u &= U_n \cos n\theta, \quad v = V_n \sin n\theta \\ w &= W_n \cos n\theta \end{aligned} \right\} \quad \dots\dots\dots (26)$$

where (U_n, V_n, W_n) are functions of axial coordinate x and time t .

$$b_{5j}=r_j \beta_j, \quad b_{6j}=(g_j+\alpha_j s_j) \beta_j, \quad b_{7j}=\beta_j, \quad b_{8j}=(3\alpha_j+r_j \alpha_j+2f_j) \alpha_j \beta_j$$

$$\begin{aligned} \text{where, } f_j &= \frac{\alpha_j}{d_j} \left\{ -\kappa^2 \alpha_j^4 + (v-3\kappa^4 n^2 \frac{1-v}{2} - \kappa^2 \lambda_{n,m}^2 \frac{2}{1-v}) \alpha_j^2 + n^2 + \kappa^2 n^4 + \frac{2v}{1-v} \lambda_{n,m}^2 - \kappa^2 n^2 \lambda_{n,m}^2 \right\} \\ g_j &= \frac{n}{d_j} \left\{ 2\kappa^2 \alpha_j^4 - (2+v+2\kappa^2 n^2 - \kappa^2 \lambda_{n,m}^2 \frac{3-v}{1-v}) \alpha_j^2 + n^2 - \lambda_{n,m}^2 \frac{2}{1-v} \right\} \\ r_j &= \frac{1}{d_j} \left\{ -5\kappa^2 \alpha_j^4 + \frac{4\kappa^2}{1-v} \frac{\alpha_j^3}{\beta_j} + 3 \left(v-3\kappa^4 n^2 \frac{1-v}{2} - \kappa^2 \lambda_{n,m}^2 \frac{2}{1-v} \right) \alpha_j^2 - \left(\frac{4v}{1-v} - 2\kappa^2 n^2 \right) \frac{\alpha_j}{\beta_j} + n^2 + \kappa^2 n^4 \right. \\ &\quad \left. + \frac{2v}{1-v} \lambda_{n,m}^2 - \kappa^2 n^2 \lambda_{n,m}^2 \right\} - \frac{f_j}{d_j} \left\{ 4\alpha_j^3 - 2 \frac{3-v}{1-v} \frac{\alpha_j^2}{\beta_j} - 2 \left(2n^2 - \frac{3-v}{1-v} \lambda_{n,m}^2 \right) \alpha_j + 2 \frac{3-v}{1-v} \frac{n^2}{\beta_j} - \frac{8}{\beta_j} \frac{\lambda_{n,m}^2}{1-v} \right\} \\ s_j &= \frac{2n}{d_j} \left\{ 4\kappa^2 \alpha_j^3 - \kappa^2 \frac{3-v}{1-v} \frac{\alpha_j^2}{\beta_j} - (2+v+2\kappa^2 n^2 - \kappa^2 \lambda_{n,m}^2 \frac{3-v}{1-v}) \alpha_j + \frac{2}{1-v} \frac{1}{\beta_j} \right\} \\ &\quad - 2 \frac{g_j}{d_j} \left\{ 2\alpha_j^3 - \frac{3-v}{1-v} \frac{\alpha_j^2}{\beta_j} - \left(2n^2 - \frac{3-v}{1-v} \lambda_{n,m}^2 \right) \alpha_j + \frac{3-v}{1-v} \frac{n^2}{\beta_j} - \frac{4}{1-v} \frac{\lambda_{n,m}^2}{\beta_j} \right\} \\ d_j &= \alpha_j^4 - \left(2n^2 - \frac{3-v}{1-v} \lambda_{n,m}^2 \right) \alpha_j^2 + n^4 - \frac{3-v}{1-v} n^2 \lambda_{n,m}^2 + \frac{2}{1-v} \lambda_{n,m}^4 \\ \beta_j &= \left\{ \frac{3-v}{1-v} \alpha_j^6 - \left(3n^2 \frac{3-v}{1-v} + \frac{1}{\kappa^2} - \frac{4}{1-v} \lambda_{n,m}^2 \right) \alpha_j^4 + \left(3n^4 \frac{3-v}{1-v} + \frac{2n^2}{\kappa^2} + \frac{3+2v-8n^2}{\kappa^2} \lambda_{n,m}^2 \right. \right. \\ &\quad \left. - \frac{2}{\kappa^2} \frac{3-v}{1-v} \lambda_{n,m}^2 \right) \alpha_j^2 - \left(n^6 \frac{3-v}{1-v} + \frac{n^4}{\kappa^2} + \frac{n^2}{\kappa^2} - \frac{4n^4}{1-v} \lambda_{n,m}^2 - \frac{2n^2}{\kappa^2} \frac{3-v}{1-v} \lambda_{n,m}^2 \right. \\ &\quad \left. \left. - \frac{1}{\kappa^2} \frac{4}{1-v} \lambda_{n,m}^2 + \frac{1}{\kappa^2} \frac{6}{1-v} \lambda_{n,m}^4 \right) \right\} / \alpha_j (4\alpha_j^6 + 3a_6 \alpha_j^4 + 2a_4 \alpha_j^2 + a_2) \end{aligned}$$

The values of the poles $p_{n,m}$ are determined from the roots of the equation $|A|=0$ and the values of α_j ($j=1, \dots, 8$) satisfy the following equation.

$$\alpha^8 + a_6 \alpha^6 + a_4 \alpha^4 + a_2 \alpha^2 + a_0 = 0 \quad (30)$$

where,

$$\begin{aligned} a_6 &= -4n^2 + 2v + (3-v) \lambda_{n,m}^2 / (1-v) \\ a_4 &= 6n^4 - 6n^2 + v^2 - 3n^2 \lambda_{n,m}^2 (3-v) / (1-v) - \lambda_{n,m}^2 / \kappa^2 + 2\lambda_{n,m}^4 / (1-v) \\ a_2 &= -4n^6 + 2(4-v)n^4 - 2(2-v)n^2 + 3n^4 \lambda_{n,m}^2 (3-v) / (1-v) + 2n^2 \lambda_{n,m}^2 / \kappa^2 \\ &\quad + \lambda_{n,m}^2 (3+2v) / \kappa^2 - 4n^2 \lambda_{n,m}^4 / (1-v) - (3-v) \lambda_{n,m}^4 / (1-v) / \kappa^2 \\ a_0 &= n^8 - 2n^6 + n^4 - \{ (3-v)n^6 / (1-v) + n^4 / \kappa^2 + n^2 / \kappa^2 \} \lambda_{n,m}^2 \\ &\quad + \{ 2n^4 / (1-v) + n^2 (3-v) / (1-v) / \kappa^2 + 2 / (1-v) / \kappa^2 \} \lambda_{n,m}^4 - 2\lambda_{n,m}^6 / (1-v) / \kappa^2 \end{aligned}$$

The first terms of Eqs.(29) coincide with the solutions of the corresponding static problem and the second terms represent the superpositions of all the modes of the natural vibrations of the shell.

In the case where the effect of the in-plane inertia force is neglected, Eqs.(7) and (8) may be employed. By the same procedure, the solutions for the displacement components in this case are easily given and coincide with the forms defined in Eqs.(29).

However, the coefficients ($f_j, g_j, r_j, s_j, \beta_j, d_j$) are given by the equations

$$\left. \begin{aligned} f_j &= \alpha_j \left\{ -\kappa^2 \alpha_j^4 + (v-3\kappa^4 n^2 (1-v)/2) \alpha_j^2 + n^2 + \kappa^2 n^4 \right\} / d_j \\ g_j &= n \left\{ 2\kappa^2 \alpha_j^4 - (2+v+2\kappa^2 n^2) \alpha_j^2 + n^2 \right\} / d_j \\ r_j &= \left\{ -5\kappa^2 \alpha_j^4 + 3(v-3\kappa^4 n^2 (1-v)/2) \alpha_j^2 + n^2 + \kappa^2 n^4 - 4\alpha_j f_j (\alpha_j^2 - n^2) \right\} / d_j \\ s_j &= 2 \left\{ 4\kappa^2 n \alpha_j^3 - n(2+v+2\kappa^2 n^2) \alpha_j - 2\alpha_j g_j (\alpha_j^2 - n^2) \right\} / d_j \\ d_j &= (\alpha_j^2 - n^2)^2, \quad \beta_j = -d_j / \kappa^2 \alpha_j (4\alpha_j^6 + 3a_6 \alpha_j^4 + 2a_4 \alpha_j^2 + a_2) \end{aligned} \right\} \dots (31)$$

and the coefficients ($a_6, \dots, a_0$) in Eq.(30) are given in the following way.

$$\left. \begin{aligned} a_6 &= -4n^2 + 2\nu, & a_4 &= 6n^4 - 6n^2 + J^2 - \lambda_{n,m}^2 / \kappa^2 \\ a_2 &= -4n^6 + 2(4-\nu)n^4 - 2(2-\nu)n^2 + 2n^2 \lambda_{n,m}^2 / \kappa^2, & a_0 &= n^8 - 2n^6 + (1 - \lambda_{n,m}^2 / \kappa^2)n^4 \end{aligned} \right\} \dots (32)$$

The static solutions corresponding to the first terms of Eqs.(29) are easily given by the equations

(I) In the case where $n=1$

$$\left. \begin{aligned} U_{1,st} &= \frac{1}{2|A|} \left\{ \sum_{j=1}^4 f_j |A_{8j}| e^{\alpha_j \xi} + 3(\xi^2 + 4 + 2\nu) |A_{85}| + 2\xi |A_{86}| + |A_{87}| \right\} \\ V_{1,st} &= \frac{1}{2|A|} \left\{ \sum_{j=1}^4 g_j |A_{8j}| e^{\alpha_j \xi} + (\xi^3 - 6\nu\xi) |A_{85}| + (\xi^2 - 2\nu) |A_{86}| + \xi |A_{87}| + |A_{88}| \right\} \\ W_{1,st} &= \frac{1}{2|A|} \left\{ \sum_{j=1}^4 |A_{8j}| e^{\alpha_j \xi} + \xi^3 |A_{85}| + \xi^2 |A_{86}| + \xi |A_{87}| + |A_{88}| \right\} \end{aligned} \right\} \dots (33)$$

where the elements a_{ij} in the matrix $[A]$ are defined in the following way.

(i) If $j=1, \dots, 4$

$$\begin{aligned} a_{1j} &= f_j e^{\alpha_j L}, & a_{2j} &= g_j e^{\alpha_j L}, & a_{3j} &= e^{\alpha_j L} \\ a_{4j} &= \alpha_j e^{\alpha_j L}, & a_{5j} &= f_j, & a_{6j} &= \alpha_j g_j \\ a_{7j} &= \alpha_j, & a_{8j} &= \alpha_j^3 + \alpha_j^2 f_j \end{aligned}$$

(ii) If $j=5, \dots, 8$

$$\begin{aligned} a_{15} &= 3(L^2 + 2\nu + 4), & a_{16} &= a_{46} = 2L, & a_{25} &= L^3 - 6\nu L, \\ a_{26} &= L^2 - 2\nu, & a_{27} &= a_{37} = L, & a_{35} &= L^3, & a_{36} &= L^2, \\ a_{45} &= 3L^2, & a_{55} &= 6(2+\nu), & a_{65} &= -6\nu, & a_{85} &= 12 \\ a_{17} &= a_{28} = a_{38} = a_{47} = a_{57} = a_{67} = a_{77} = 1 \\ a_{18} &= a_{48} = a_{56} = a_{58} = a_{66} = a_{68} = a_{75} = a_{76} = a_{78} \\ &= a_{86} = a_{87} = a_{88} = 0 \end{aligned}$$

$$\text{where, } f_j = \frac{-\kappa^2 \alpha_j^5 + \nu \alpha_j^3 + \alpha_j}{(\alpha_j^2 - 1)^2}$$

$$g_j = \frac{2\kappa^2 \alpha_j^4 - (2+\nu)\alpha_j^2 + 1}{(\alpha_j^2 - 1)^2}$$

and α_j ($j=1, \dots, 4$) are the roots of the following characteristic equation of fourth degree.

$$\alpha^4 - 2(2-\nu)\alpha^2 + J^2 = 0 \dots (34)$$

(II) In the case where $n \geq 2$

$$\left. \begin{aligned} U_{n,st} &= \frac{1}{2|A|} \sum_{j=1}^8 f_j |A_{8j}| e^{\alpha_j \xi} \\ V_{n,st} &= \frac{1}{2|A|} \sum_{j=1}^8 g_j |A_{8j}| e^{\alpha_j \xi} \\ W_{n,st} &= \frac{1}{2|A|} \sum_{j=1}^8 |A_{8j}| e^{\alpha_j \xi} \end{aligned} \right\} \dots (35)$$

where α_j ($j=1, \dots, 8$) are the roots of Eq.(30) obtained by equating $\lambda_{n,m}$ to 0.

The coefficients (f_j, g_j) and the elements a_{ij} in the matrix $[A]$ coincide with the forms defined in Eqs.(29), if $\lambda_{n,m}$ equates to 0.

3.3 Solutions for Concentrated Loads

For the problem of one concentrated load of magnitude F_0 as shown in Fig.2, the solutions for the displacement components are obtained by the superposition of the results given in sections 3.1 and 3.2 equating R_{q0} to F_0 , such that

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} u_0 \\ 0 \\ w_0 \end{bmatrix} + \frac{1}{\pi} \sum_{n=1,2,3,\dots}^{\infty} \begin{bmatrix} U_n \cos n\theta \\ V_n \sin n\theta \\ W_n \cos n\theta \end{bmatrix} \dots (36)$$

For the problem of two equal and opposite concentrated loads as shown in Fig.4, the solutions are obtained in the following way.

Fig.4 A Circular Cylindrical Shell Subjected to Two Equal and Opposite Concentrated Impulsive Loads

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{\pi} \begin{bmatrix} u_0 \\ 0 \\ w_0 \end{bmatrix} + \frac{2}{\pi} \sum_{n=2,4,6,\dots}^{\infty} \begin{bmatrix} U_n \cos n\theta \\ V_n \sin n\theta \\ W_n \cos n\theta \end{bmatrix} \quad (37)$$

where (u_0, w_0) denote the results for the uniformly distributed line load given in section 3.1.

4. Numerical Results

In numerical calculations, $\nu=0.3$ is used and the dimensionless parameters $H(=h/R)$ and $L(=L/R)$ are introduced.

To evaluate the solutions for the concentrated loads, the convergence of the infinite series with respect to n must be examined by Eqs.(36) and (37) and by the equivalent equations for the stress resultants. The radial displacement w is evaluated to an accuracy of three figures by summation of Eqs.(36) and (37) with n ranging from 0 to 30. The bending moments M_x and M_θ diverge to infinity at the loaded point $x=\theta=0$. However, as the value of x increases the infinite series converges more quickly and the displacement components are evaluated to the mentioned accuracy with n ranging from 0 to 6 at $\xi=1$ and the bending moments are evaluated with n ranging from 0 to 30 at $\xi=0.2$ and from 0 to 12 at $\xi=1$. The influences of the wall-thickness and the length of the shell on the convergences are negligible.

4.1 Results for the Static Problem

First, static results are evaluated on the basis of Eqs.(23), (33) and (35), and compared with those from Donnell's theory [1]. In Figs.5 and 6, the effects of the length and the wall-thickness of the shell on the radial displacement at the loaded point are shown by

Fig.5 Effect of the Length of the Shell on the Static Radial Displacement under the Line Load Varying in the Form of Cosine Function ($H=0.05$, $x=\theta=0$)

using n as a parameter in the case of trigonometric line loads and a comparison of Flugge's and Donnell's theories is made.

The error in Donnell's theory decreases as the wall-thickness or the length of the shell decreases. On the other hand, as the wall-thickness or the length of the shell increases the error in Donnell's theory increases unlimitedly in the case where the circumferential wave number n equals to unity, i.e., under the bending deformation like a beam, whereas it is bounded by each limit in the case where the circumferential wave number n is larger than unity. Similarly, in Fig.7, the effect of the length of the shell on the membrane force and the bending moment at the loaded and the clamped points is shown.

Fig.6 Effects of the Wall-Thickness and Length of the Shell on the Static Radial Displacement under the Line Load Varying in the Form of Cosine Function ($x=\theta=0$, $(\cdot)_D$ Means Donnell's Theory, $(\cdot)_F$ Means Flugge's Theory)

Fig.7 Effect of the Length of the Shell on the Static Membrane Force and Bending Moment under the Line Load Varying in the Form of Cosine Function ($H=0.05$, $x=0$ or 1 , $\theta=0$)

Fig. 8 Effects of the Length and Wall-Thickness of the Shell on the Static Radial Displacement under Concentrated Loads ($\alpha = \theta = 0$, Flügge's Theory, Solid Lines Mean One Concentrated Load, Broken Lines Mean Two Concentrated Loads)

Fig. 9 Static Distributions of the Bending Moment Along the Generator under One Concentrated Load ($H = 0.05$, $\theta = 0$, Flügge's Theory)

Fig. 10 Static Distributions of the Membrane Forces Along the Generator under One Concentrated Load ($H = 0.05$, $\theta = 0$, Flügge's Theory)

The errors in the stress resultants show the same trend as those in the radial displacement. In Fig. 8, the relations between the values of the radial displacement at the loaded point and the shape of the shell are shown in the case of the concentrated loads.

The difference in the results between one concentrated load as shown in Fig. 2 and two concentrated loads as shown in Fig. 4 is negligible when the shell is short. As the length increases, the values in the case of two concentrated loads become once larger and later become smaller than those in the case of one concentrated load. Differences between the two theories are recognized in the case of one concentrated load. In Figs. 9 and 10, the static distributions of the bending moments and the membrane forces along the generator through the loaded point are shown in the case of one concentrated load. At the loaded point the values of the stress resultants equal to infinity in all the results and decrease steeply as the distance from the loaded point increases.

4.2 Results for the Dynamic Problem

The influences of the in-plane inertia force of the shell and the differences from the results given by Donnell's theory are in-

Table 1. Values of the Natural Frequencies $\lambda_{n,m}$ ($H = 0.05$, $L = 2$)

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$
1	0.940 006	0.307 658	0.180 525	0.158 751
	0.959 037	0.411 312	0.201 624	0.167 681
	0.954 152	0.411 875	0.205 967	0.178 979
2	0.978 162	0.768 713	0.544 448	0.419 866
	0.983 736	0.818 851	0.528 413	0.440 062
	0.960 154	0.819 799	0.585 408	0.446 954
3	1.001 47	0.926 923	0.803 306	0.698 160
	1.002 33	0.938 709	0.822 344	0.716 464
	0.991 230	0.940 299	0.825 640	0.722 616
4	1.076 08	1.048 96	0.995 848	0.946 523
	1.079 16	1.054 45	1.005 04	0.958 574
	1.077 41	1.056 79	1.008 83	0.964 610
5	1.244 94	1.233 94	1.220 93	1.215 11
	1.248 00	1.238 64	1.226 57	1.223 28
	1.247 26	1.241 66	1.230 76	1.229 23
6	1.509 45	1.235 59	1.519 98	1.542 39
	1.515 79	1.514 57	1.524 22	1.548 52
	1.515 40	1.518 07	1.528 64	1.554 33
7	1.613 12	1.511 49	1.671 68	1.944 11
	1.882 50	1.885 75	1.908 00	1.949 18
	1.882 28	1.889 56	1.912 52	1.954 80
8	1.882 97	1.554 36	1.904 24	2.104 00
	2.342 26	2.348 01	2.377 58	2.429 01
	3.342 12	2.352 00	2.382 14	2.434 44
9	2.341 58	1.880 27	2.249 00	2.424 51
	2.888 82	2.896 02	2.929 96	2.987 69
	2.888 74	2.900 12	2.934 51	2.992 95
10	2.887 34	1.960 28	2.374 72	2.737 83
	3.517 42	3.525 50	3.562 13	3.623 83
	3.517 37	3.529 66	3.566 66	3.628 95

Upper Row: Flügge Including In-Plane Inertia
Middle Row: Flügge Neglecting In-Plane Inertia
Lower Row: Donnell Neglecting In-Plane Inertia

vestigated with respect to natural frequencies and time variations of the displacement components and the stress resultants.

Natural frequencies $\lambda_{n,m}$, which are determined from the root of the transcendental equation $|A|=0$, are numerically found out with use of trial and error procedure or Newton's method. In Table 1, 2 and 3, natural frequencies obtained for three different lengths of the shell are shown within the ranges of n from 0 to 3 and m from 1 to 10.

In the tables, $n=0$ indicates the axisymmetric motions and in the upper row are shown the frequencies based on Eq.(1), in the middle row those based on Eq.(3) neglecting the in-plane inertia effect and in the lower row those neglecting the membrane force N_x and the in-plane inertia effect in the previous paper, and $n \geq 1$ indicates the non-axisymmetric motions and in the upper row are shown the frequencies based on Flügge's equation (5), in the middle row those based on Eq.(7) neglecting the in-plane inertia effect and in the lower row those based on Donnell type theory including the radial inertia effect only in the previous paper. In the case of the axisymmetric motions, regardless of whether or not the membrane force N_x exists, the values of the frequencies scarcely vary.

The in-plane inertia force becomes effective as the length of the shell increases.

In the case of the non-axisymmetric motions, the frequencies from Donnell's theory become

Table 2. Values of the Natural Frequencies $\lambda_{n,m}$ ($H=0.05$, $L=4$)

m	$n=0$	$n=1$	$n=2$	$n=3$
1	0.716 237	0.135 698	0.073 764 3	0.116 228
	0.958 267	0.191 715	0.082 774 6	0.122 742
	0.953 952	0.192 438	0.091 692 6	0.136 817
2	0.933 995	0.435 013	0.247 865	0.192 998
	0.956 162	0.560 090	0.277 739	0.204 376
	0.954 328	0.560 731	0.281 590	0.214 790
3	0.952 824	0.686 593	0.447 692	0.331 101
	0.961 809	0.762 127	0.487 955	0.349 206
	0.956 312	0.762 947	0.491 060	0.356 996
4	0.967 841	0.821 018	0.615 451	0.480 915
	0.974 125	0.853 953	0.649 125	0.501 992
	0.962 120	0.855 035	0.652 171	0.508 741
5	0.988 672	0.890 276	0.741 908	0.620 745
	0.993 240	0.908 299	0.765 163	0.640 488
	0.974 861	0.909 687	0.768 354	0.646 828
6	1.001 51	0.900 539	0.842 001	0.747 709
	1.002 71	0.954 932	0.857 479	0.764 237
	0.998 325	0.956 681	0.860 900	0.770 421
7	1.029 16	0.946 637	0.933 007	0.867 750
	1.037 40	1.007 77	0.943 641	0.881 024
	1.036 61	1.009 91	0.947 318	0.887 151
8	1.051 61	1.002 26	1.027 70	0.988 813
	1.094 00	1.074 73	1.035 42	0.999 455
	1.093 60	1.077 25	1.039 34	1.005 55
9	1.094 89	1.070 60	1.134 36	1.117 76
	1.275 46	1.160 76	1.140 30	1.126 44
	1.172 46	1.163 63	1.144 44	1.132 50
10	1.172 09	1.157 30	1.183 85	1.259 58
	1.043 20	1.268 86	1.262 85	1.266 84
	1.275 29	1.272 04	1.267 16	1.272 86

higher than Flügge's results as the shell becomes long and this inclination becomes prominent in lower modes. The in-plane inertia effect appears notably in the lower modes when the circumferential wave number n is small.

The numerical calculations for the displacement components and stress resultants variations in time are made with use of these natural frequencies. In Figs.11 and 12, the

Fig.11 Radial Displacement Variations at the Loaded Point in Time under One Concentrated Impulsive Load (Flügge's Theory, Including the In-Plane Inertia Effect, $H=0.05$)

Table 3. Values of the Natural Frequencies $\lambda_{n,m}$ ($H=0.05$, $L=8$)

m	$n=0$	$n=1$	$n=2$	$n=3$
1	0.372 497	0.048 136 4	0.042 954 4	0.110 336
	0.954 436	0.068 632 8	0.048 125 8	0.116 434
	0.953 940	0.070 264 4	0.061 664 8	0.128 756
2	0.713 315	0.187 563	0.094 211 9	0.122 072
	0.955 234	0.268 240	0.106 177	0.129 086
	0.953 963	0.268 909	0.113 764	0.142 976
3	0.893 201	0.351 063	0.185 237	0.157 294
	0.956 734	0.474 042	0.208 891	0.166 665
	0.954 087	0.474 665	0.213 575	0.178 669
4	0.931 375	0.506 319	0.289 503	0.214 857
	0.959 258	0.625 891	0.323 762	0.227 728
	0.954 452	0.626 570	0.327 416	0.237 689
5	0.943 402	0.634 147	0.393 502	0.285 449
	0.963 179	0.725 251	0.433 347	0.301 942
	0.955 260	0.726 017	0.436 593	0.310 458
6	0.950 285	0.692 406	0.489 575	0.361 303
	0.968 918	0.790 178	0.529 476	0.380 606
	0.956 773	0.791 051	0.532 563	0.388 212
7	0.956 405	0.728 644	0.574 261	0.437 488
	0.976 919	0.834 681	0.610 505	0.458 323
	0.959 314	0.835 678	0.613 553	0.465 365
8	0.963 366	0.794 325	0.647 201	0.511 234
	0.987 535	0.867 448	0.678 219	0.532 325
	0.963 261	0.868 585	0.681 296	0.539 016
9	0.972 016	0.840 593	0.709 878	0.581 322
	0.997 762	0.893 771	0.735 569	0.601 680
	0.969 041	0.895 065	0.738 717	0.608 150
10	0.982 780	0.875 131	0.764 583	0.647 596
	1.003 50	0.917 018	0.785 556	0.665 599
	0.977 124	0.918 482	0.788 802	0.672 933

Fig. 12 Radial Displacement Variations at the Loaded Point in Time under One Concentrated Impulsive Load (Flügge's Theory, Neglecting the In-Plane Inertia Effect, $H=0.05$)

Fig. 13 Radial Displacement Variations at the Loaded Point in Time under Two Concentrated Impulsive Loads (Flügge's and Donnell's Theories, $H=0.05$, $L=2$)

Fig. 14 Radial Displacement Variations at the Loaded Point in Time under Two Concentrated Impulsive Loads (Flügge's and Donnell's Theories, $H=0.05$, $L=4$)

radial displacement variations in time at the loaded point for the data $L/R=2, 4$ and 8 are shown in the case of one concentrated load.

The effect of the in-plane inertia force becomes important as the length of the shell increases and the periods become long owing to the in-plane inertia force. Similarly, the radial displacement variations in time in the case of two concentrated loads are shown in Figs. 13 and 14. The in-plane inertia force affects the period of oscillations and not the amplitude and these inclinations are independent of the length of the shell.

Fig. 15 Membrane Force Variations at the Clamped End in Time under the Impulsive Line Load Varying in the Form of Cosine Function ($H=0.05$, $L=16$, $n=1$, $\theta=0$, Solid Lines: Including the In-Plane Inertia Effect, Broken Lines: Neglecting the In-Plane Inertia Effect, Flüge's Theory)

Fig. 16 Membrane Force Variations at Five Different Points in Time under the Impulsive Line Load Varying in the Form of Cosine Function (A: Including the In-Plane Inertia Effect, B: Neglecting the In-Plane Inertia Effect, Flüge's Theory, $H=0.05$, $L=16$, $n=1$, $\theta=0$)

Differences between Donnell's and Flügge's theories are not recognized so far as the length of the shell is long. But, as the length increases, the periods given by Donnell's theory become visibly shorter and the amplitudes become smaller than those by Flügge's theory. In Fig. 15, the membrane force variations at the clamped end in time for the data $H=0.05$ and $L=16$ are shown in the case of the trigonometric line load and the magnifications of the initial responses corresponding to Fig. 15 are shown in Fig. 16.

Similarly the initial responses of the bending moment at the clamped end for the same data are shown in Fig. 17. The arrows in Figs. 16 and 17 denote respectively the time when the elastic longitudinal wave emanating from the loaded point due to the load arrives first at each point of $\xi=4, 8, 12$ and 16.

In the long time responses the influence of the in-plane inertia force mainly appears in the period of oscillations. In the initial responses the time when the responses appear becomes different depending on whether the in-plane inertia effect is present or not.

Namely, in the case where the in-plane

Fig. 17 Bending Moment Variations at the Clamped End in Time under the Impulsive Line Load Varying in the Form of Cosine Function (Flügge's Theory, $H=0.05$, $L=16$, $n=1$, $\theta=0$, A : Including the In-Plane Inertia Effect, B: Neglecting the In-Plane Inertia Effect)

inertia effect is neglected, the responses appear before the time when the elastic wave arrives. Contrarily, they appear after the time in the case where the effect is considered.

5. Conclusions

The following conclusions are made as a result of this study.

(1) The differences between Donnell's and Flügge's theories are not recognized so far as the length of the shell is long.

This inclination is independent of the wall-thickness of the shell.

(2) As the length of the shell increases, the periods in Donnell's results become shorter and the amplitudes become smaller than those in Flügge's results.

(3) Concerning the natural frequencies, the in-plane inertia effect appears notably in lower modes when the circumferential wave number is small. This inclination becomes prominent as the length of the shell increases.

(4) In the long time responses the in-plane inertia effect mainly appears in the period of oscillations. In the initial responses the time when the responses appear becomes different depending on whether the in-plane inertia effect is present or not.

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