539.3.082.53

Paper No. 180-10

Characterization of Mechanical and Optical Properties of Rheo-optically Simple Materials*

By Kunio Hayashi**, Katsuhisa Gozu***,
Yuzo Abe⁺, and Takashi Tokita⁺⁺

Classical photoelasticity can be extended to analyse the stress-strain fields in viscoelastic materials. In this photoviscoelastic analysis, it is necessary to record and approximate with mathematical formulae a complete time history of the mechanical and optical characteristic properties of the viscoelastic materials. A new procedure, where the nonlinear minimization technique "SUMT" was utilized, was proposed to approximate those properties with Prony series representations. This procedure has been proved to be very rigorous and effective by applying it to two polymers, polyurethane elastomer and epoxy resin. Furthermore, it was considered appropriate to regard these two polymers as rheo-optically simple materials. Thus, their mechanical and optical behaviours were linear and the principle of reduced time (or time-temperature superposition principle) could be applied well.

1. Introduction

High polymers have been widely put to practical use as machine parts and structural elements during recent years. Therefore the characterization of their mechanical behaviour has received great attention. As theoretical analysis of stresses and strains in viscoelastic materials is limited in application, the establishment of some powerful experimental methods is eagerly waited for. The mechanical behaviour of viscoelastic materials especially is highly dependent on their temperature and there might be thermal stresses due to temperature distribution. In such a situation the mathematical treatment of the problem is very difficult. As an effective experimental method for analysing stresses and strains in elastic materials, photoelasticity has been applied successfully. It is a great advantage of this method that the overall stress field can be analysed simultaneously. This classical photoelasticity can be extended to encompass the study of mechanical behaviour of viscoelastic materials, and Williams⁽¹⁾, (2), Kunio and Miyano⁽³⁾, (4) showed the usefulness of this new optical method, viz. photoviscoelasticity. The proceedings of the symposium on "The Photoelastic Effect and its Applications" contains two important papers on photoviscoelasticity by Theocaris⁽⁵⁾ and

Coleman⁽⁶⁾.

In order to analyse stresses and strains in viscoelastic materials by photoviscoelasticity, it is necessary to record a complete time history of their mechanical and optical characteristic properties. The mechanical properties can be expressed by the creep compliances and relaxation moduli, and the optical properties by the stress-optical and the strain-optical coefficients in creep or relaxation. These properties are highly dependent on time and temperature and so they are usually determined through the basic uniaxial tension tests (constant strain rate tests and creep tests) under several temperature levels. From the master curves, obtained by basic tests, the material characteristic properties are determined through the Laplace transformation, and inverting based on the theory of linear viscoelasticity. Therefore, it is quite important to express the master curves by convenient formulae. Some procedures of this curve fitting were developed and applied to some polymers⁽³⁾, (7) but they are not sufficient in their applicability and accuracy. As Miyano pointed out in his paper⁽³⁾, the coefficients of the approximate Prony series representation must be in the same sign, when the material is assumed to be a generalized Maxwell or Voigt-model. In this paper a new procedure (where the nonlinear minimization technique "SUMT" was utilized) is proposed and has been proved to be very rigorous and effective by applying it to two polymers, viz. polyurethane elastomer and epoxy resin. As the usual computer library contains the program of SUMT, it is quite reasonable to apply this collocation procedure for characterization of the mechanical and optical properties of viscoelastic materials.

2. Basic equations of linear photoviscoelasticity

The constitutive equations of linear

* Received 30th October, 1978.

** Associate Professor,
Department of Mechanical Engineering
Faculty of Science and Technology,
Sophia University, Tokyo.

*** Research Assistant,
Department of Mechanical Engineering
Faculty of Science and Technology
Sophia University, Tokyo

⁺ Juki Co., Ltd.

⁺⁺ Nissan Diesel Motor Co., Ltd.

photoviscoelastic materials can be written as follows in the Laplace plane,

$$S(p) = 3K(p)\bar{e}(p), \quad \bar{S}_{ij}(p) = 2G(p)\bar{e}_{ij}(p) \quad (1)$$

$$\left. \begin{aligned} S_{ij} &= \sigma_{ij} - \frac{1}{3}\delta_{ij}S, \quad S = \sigma_{kk} \\ e_{ij} &= \varepsilon_{ij} - \frac{1}{3}\delta_{ij}e, \quad e = \varepsilon_{kk} \end{aligned} \right\} \quad (2)$$

Assuming the linearity for optical properties too, the relations between the experimental fringe orders and the principal stress or strain differences are

$$\left. \begin{aligned} \bar{n}(p) &= C\varepsilon(p) \{ \bar{\varepsilon}_1(p) - \bar{\varepsilon}_2(p) \} = C\varepsilon(p) \bar{\gamma}_{\max}(p) \\ \bar{n}(p) &= C\sigma(p) \{ \bar{\sigma}_1(p) - \bar{\sigma}_2(p) \} = C\sigma(p) \bar{\tau}_{\max}(p) \end{aligned} \right\} \quad (3)$$

From equations (1) and (3) is derived the following relation between the birefringence-strain coefficient $C\varepsilon$ and the birefringence-stress coefficient $C\sigma$.

$$C\varepsilon(p) = 2G(p)C\sigma(p) \quad (4)$$

Incompressibility being assumed, the mechanical and optical properties of a linear photoviscoelastic material are connected by the following equations.

$$\left. \begin{aligned} K(p) &= \infty, \quad \nu(p) = 0.5, \quad E(p) = 3G(p) \\ C\varepsilon(p) &= \frac{2}{3}E(p)C\sigma(p) \end{aligned} \right\} \quad (5)$$

When the material properties are to be determined by uniaxial tensile tests of stripped test specimens, there are the relations between axial stress σ , axial strain ε and fringe order n :

$$\left. \begin{aligned} \bar{\sigma}(p) &= 3G(p)\bar{\varepsilon}(p) = E(p)\bar{\varepsilon}(p) \\ \bar{n}(p) &= \frac{3}{2}C\varepsilon(p)\bar{\varepsilon}(p), \quad \bar{n}(p) = C\sigma(p)\bar{\sigma}(p) \end{aligned} \right\} \quad (6)$$

Because of easiness of experiments, constant strain-rate tests and constant load tests under several fixed temperatures are adopted. Measuring the variation of fringe order with time $n(t)$, photoviscoelastic properties are obtained by the following equations.

(1) Constant strain-rate tests (strain-rate R)

$$\left. \begin{aligned} \varepsilon(t) &= Rt \quad (t \geq 0) \\ E_{rel}(t) &= d\sigma(t)/Rdt \\ C\varepsilon_{rel}(t) &= \frac{2}{3}dn(t)/Rdt \end{aligned} \right\} \quad (7)$$

where $E_{rel}(t)$ is the relaxation modulus and $C\varepsilon_{rel}(t)$ is the relaxation birefringence-strain coefficient. Recalling eq. (6), Laplace transformation of eq. (7) yields

$$E(p) = p\bar{E}_{rel}(p), \quad C\varepsilon(p) = p\bar{C}\varepsilon_{rel}(p) \quad (8)$$

(2) Constant load tests (creep tests, constant stress)

$$\left. \begin{aligned} \sigma(t) &= \sigma_0 \quad (t \geq 0) \\ D_{crp}(t) &= \varepsilon(t)/\sigma_0, \quad C\sigma_{crp}(t) = n(t)/\sigma_0 \end{aligned} \right\} \quad (9)$$

Where $D_{crp}(t)$ is the creep compliance and $C\sigma_{crp}(t)$ is the creep birefringence-stress coefficient. Using eq. (6), Laplace transformation of eq. (9) yields

$$E(p) = 1/p\bar{D}_{crp}(p), \quad C\sigma(p) = p\bar{C}\sigma_{crp}(p) \quad (10)$$

While the equations

$$\left. \begin{aligned} p\bar{D}_{crp}(p) &= 1/p\bar{E}_{rel}(p) \\ p\bar{C}\varepsilon_{rel}(p) &= \frac{2}{3}p\bar{E}_{rel}(p) \cdot p\bar{C}\sigma_{crp}(p) \end{aligned} \right\} \quad (11)$$

can be deduced from eqs. (5), (8) and (10), they give the interrelations between the material properties obtained by the two independent tests mentioned above. The characteristic properties of viscoelastic materials are strongly dependent on temperature and it is assumed that the time-temperature shift law holds for the linear photoviscoelastic materials treated in this paper.

That is, the characteristic property $x(t, T)$ at any temperature T is related to that $X(t)$ at the reference temperature T_0 by the equation

$$x(t, T) = X(t/a_{T_0}(T)) = X(\xi), \quad \xi = t/a_{T_0}(T) \quad (12)$$

where t is real time, ξ is the reduced time and $a_{T_0}(T)$ is the time-temperature shift factor which is peculiar to the material. There is a well established formulation for this shift factor, WLF equation, proposed by Williams et al.⁽¹⁾ Using this time-temperature relation, eqs. (7) and (9) can be rewritten

$$\left. \begin{aligned} \frac{T_0}{T}E_{rel}(\xi) &= \left[\frac{\sigma T_0(Ra_{T_0}T)}{\xi} \times \frac{d \log \{ \sigma T_0 / (Ra_{T_0}T) \}}{d \log \xi} \right]_{\xi=t/a_{T_0}} \\ C\varepsilon_{rel}(\xi) &= \frac{2}{3} \left[\frac{n/Ra_{T_0}}{\xi} \cdot \frac{d \log (n/Ra_{T_0})}{d \log \xi} \right]_{\xi=t/a_{T_0}} \end{aligned} \right\} \quad (13)$$

$$\left. \begin{aligned} \frac{T_0}{T}D_{crp}(\xi) &= \left[\frac{\varepsilon T}{\sigma_0 T_0} \right]_{\xi=t/a_{T_0}} \\ \frac{T_0}{T}C\sigma_{crp}(\xi) &= \left[\frac{n T}{\sigma_0 T_0} \right]_{\xi=t/a_{T_0}} \end{aligned} \right\} \quad (14)$$

Applying eqs. (13) and (14) to the data obtained by the basic uniaxial tests under several fixed temperatures, the master curves of characteristic properties over a wide range of time at the reference temperature T_0 are determined as well as the time-temperature shift factor $a_{T_0}(T)$.

3. Mathematical representations of photoviscoelastic properties

When the photoviscoelastic properties such as $E_{rel}(\xi)$, $C\varepsilon_{rel}(\xi)$, $D_{crp}(\xi)$ and $C\sigma_{crp}(\xi)$ under the reference temperature are determined from the experimental data through the basic uniaxial tests, representations of these properties in the Laplace plane are determined by using eqs. (8) and (10). Then the desired photoviscoelastic properties are characterized by inverse transformation. It is desirable therefore to have some method of mathematical representation in both the real and transformed variables which is not only accurate but tractable.

In attacking this problem, Schapery⁽⁷⁾ has proposed an effective collocation scheme using the Prony series expansion,

$$F(\xi) = K + \sum_{i=1}^N \alpha_i \exp(-\xi/\tau_i) \quad (15)$$

Once the measured master curve $F(\xi)$ is approximately represented by eq. (15), its Laplace transform is straight forward

$$p\bar{F}(p) = K + \sum_{i=1}^N \kappa_i / (1 + \frac{1}{p\tau_i}) \quad (16)$$

When a property in the Laplace plane is written in the form of eq. (16) contrarily, eq. (15) gives that in the reduced time through inverse transformation.

In order to determine τ_i and κ_i in eq. (15), Schapery selected N decades of time over the transition range of the experimental data, and set such a range that one fell in each decade over this range. The experimental data were collocated at N points, which led to a large set of linear simultaneous equations and gave not always satisfactory results. Miyano et al. have proposed another collocation method, where setting a limit for the difference between experimental data and mathematical representations, τ_i and κ_i were determined one by one from the end region of the transition range. This procedure gave fairly good results but was a little complicated and so it seemed to require some skill. We propose a new collocation method where nonlinear minimization techniques are applied. Setting the transition range $[\xi_1, \xi_u]$ from experimental data or curves to be formalized ($[P_l, P_u]$ for curves in the Laplace plane), τ_i 's are taken as equally divided $(N-1)$ points over this range. Considering the material properties in the rubbery and glassy range,

$$\left. \begin{aligned} f(\infty) &= \lim_{\xi \rightarrow \infty} F(\xi) = K \\ f(0) &= \lim_{\xi \rightarrow 0} F(\xi) = K + \sum_{i=1}^N \kappa_i \end{aligned} \right\} \quad (17)$$

The overall error of the approximate representation (15) is estimated as the sum total of those at M equally divided points ξ_j ($j=1, \dots, M$),

$$H = \sum_{j=1}^M \{F(\xi_j) - f(\xi_j)\}^2 \quad (18)$$

$$H = \sum_{j=1}^M \{g_j + \sum_{i=1}^N \kappa_i \exp(-\xi_j/\tau_i)\}^2 \quad (19)$$

$$g_j = K - f(\xi_j) = f(\infty) - f(\xi_j)$$

Furthermore, when the viscoelastic material can be regarded to behave as a generalized Maxwell or Voigt model, the sign of the coefficients κ_i 's of the approximate representation are to be the same, as Miyano et al. has pointed out. (3)

Then the problem of the mathematical representation of master curves for the material properties can be written as follows.

$$\left. \begin{aligned} \min H &= \sum_{j=1}^M \{g_j + \sum_{i=1}^N \kappa_i \exp(-\xi_j/\tau_i)\}^2 \\ \text{sub. to } &\left(\begin{aligned} &\kappa_i \geq 0 \text{ (or } \kappa_i \leq 0) \text{ (} i=1, 2, \dots, N) \\ &\sum_{i=1}^N \kappa_i = -g_0 \text{ where } g_0 = K - f(\xi_0) \end{aligned} \right) \end{aligned} \right\} \quad (20)$$

$$\xi_0 = 0$$

This is a nonlinear minimization problem with equality and inequality constraints. The procedure of curve fitting in the Laplace plane can be formulated quite in the same way. SUMT (Sequential Unconstrained Minimization Technique) is utilized as the minimization technique, because this technique is used widely and found in a usual computer library.

4. Experiments

Two photoviscoelastic materials, polyurethane rubber and epoxy resin, produced by Riken Keiki Co., Ltd., Japan, were examined. For constant strain-rate tests, tensile specimens (Fig. 1) were pulled at constant crosshead velocities and temperatures in an Instron-type testing machine. The standard velocity was 1 mm/min, and a few other speeds up to 50 mm/min were employed. A constant dead load was applied suddenly to the specimens in constant stress tests. The time lag error in the application of the load was assumed to be negligibly small. Ne-He laser light source, 6328Å, was used in the optical experiments. The variation of the intensity of light was measured by means of a solar cell.

Fig. 1 Tensile specimen

Fig. 2 Schematic diagram of tensile test machine and temperature control chamber

4.1 Polyurethane rubber

As the known glass temperature of this material was near -50°C , the tests were carried out under 13 constant temperatures, ranging from 0°C to -52°C , in a cooling bath shown in Fig. 2. Coolant, i.e., alcohol, was kept at a fixed temperature by the use of dry ice. The maximum variation of temperature during the test was less than $\pm 0.5^\circ\text{C}$. Applying the time-temperature shift law to the constant strain-rate data, the master curves of tensile stress and of fringe orders were obtained for the reference temperature $T_0 = 273^\circ\text{K}$ as functions of the reduced time (Fig. 3 and 4). The relaxation modulus $E_{rel}(\xi)$ shown in Fig. 5 was determined from the master curve in Fig. 3 by using eq. (13).

The constant load tests data were also shifted to get a master curve of the creep compliance $D_{crp}(\xi)$ (Fig. 6) and one of the creep birefringence-stress coefficient $C\sigma_{crp}(\xi)$ (Fig. 15) using eq. (14). The time temperature shift factor $a_{T_0}(T)$ which was used to get several master curves was shown in Fig. 7, and WLF equation fitted quite well.

4.2 Epoxy resin

As the glass temperature was near 110°C , the constant temperature chamber belonging to the Instron-type testing machine and the

Fig.3 Master curve of tensile stress

Fig.4 Master curve of fringes

Fig.5 Master curve of relaxation modulus

optical equipment shown in Fig.8 were used to get mechanical and optical basic data under 8 constant temperatures, ranging from 90°C to 127°C. The variation of temperature during loading was less than $\pm 0.5^\circ\text{C}$. Taking 400°K as the reference temperature, the basic data were dealt with in the same way as for polyurethane to provide the master curves of tensile stress (Fig. 9) and of

fringe orders (Fig.10). One of the relaxation modulus $E_{rel}(\xi)$ is shown in Fig. 11.

Fig.6 Master curve of creep compliance

Fig.7 Time-temperature shift factor

Fig.8 Schematic diagram of temperature control chamber and photoelastic apparatus

Fig.9 Master curve of tensile stress

Fig.12 Master curve of creep compliance

Fig.10 Master curve of fringes

Fig.13 Master curve of the creep birefringence-stress coefficient

Fig.11 Master curve of relaxation modulus

From the constant load tests, the master curves of $D_{crp}(\xi)$ and $C_{\sigma,crp}(\xi)$ were determined as shown in Figs. 12 and 13 respectively. In this case, too, the time-temperature shift factor $a_{T_0}(T)$, shown in Fig. 14, was expressed well by WLF equation.

Thus, it was found that the experimental data yielded a smooth continuous curve with use of the shift factor and therefore the two materials examined could be considered to be linear photoviscoelastic and to behave according to the time-temperature shift law.

5. Examples of the mathematical representation

As the first example, the master curve of the creep birefringence-stress coefficient $C_{\sigma,crp}(\xi)$ for polyurethane rubber obtained from constant load tests was formulated using the procedure proposed in section 3. Judging from the experimental data, the transition region was assumed to be $[10^{-12}, 10^{-4}]$ and the error evaluation points ξ_i were set at the same points as τ_i . Curve fittings were carried out for 3 cases, that is $N = M = 3, 9$ and 17 . Solving the minimization problem, the coefficients β_i of the approximate formula (21) were determined.

$$\frac{T}{T_0} C_{\sigma, \text{crp}}(\xi) = B - \sum_{i=1}^N \beta_i \exp(-\xi/\tau_i), \beta_i \geq 0 \quad (21)$$

Fig.14 Time-temperature shift factor

The approximate curves obtained are shown in Fig.15.

Fig.15 Master curve of the creep birefringence-stress coefficient

Three terms approximation ($N=3$) was not sufficient because of the shortage of terms but the difference between the original curve and the approximate one became very little in the case of $N=9$. When taking 17 terms ($N=17$), the fitting was perfect. The other master curves were also formulated with the Prony series by the minimization procedure and shown in each figure by dotted lines (Fig.5,6,11,12 and 13). The fit-

ting of the Prony series representations to the original master curves was sufficiently perfect.

It was found that the determination of the relaxation birefringence-strain coefficient $C_{\epsilon, \text{rel}}(\xi)$ from the master curves of fringe order (Fig.4 and 10) obtained through the constant strain-rate tests was very difficult because it was necessary to get the gradient of the curves accurately. However, according to the theory of linear photoviscoelasticity, $C_{\epsilon, \text{rel}}(\xi)$ was able to be determined by eq. (11) using the relaxation modulus $E_{\text{rel}}(\xi)$ and the creep birefringence-stress coefficient $C_{\sigma, \text{crp}}(\xi)$. The procedure for this determination was taken as the second example. When $E_{\text{rel}}(\xi)$ and $C_{\sigma, \text{crp}}(\xi)$ had been approximated by the Prony series representations,

$$\left. \begin{aligned} \frac{T_0}{T} E_{\text{rel}}(\xi) &= A + \sum_{i=1}^N \alpha_i \exp(-\xi/\tau_i), \alpha_i \geq 0 \\ \frac{T}{T_0} C_{\sigma, \text{crp}}(\xi) &= B - \sum_{i=1}^N \beta_i \exp(-\xi/\tau_i), \beta_i \geq 0 \end{aligned} \right\} \quad (22)$$

the Laplace transforms of eqs. (22) were

$$\left. \begin{aligned} p \frac{T_0}{T} \bar{E}_{\text{rel}}(p) &= A + \sum_{i=1}^N \alpha_i / (1 + \frac{1}{p\tau_i}) \\ p \frac{T}{T_0} \bar{C}_{\sigma, \text{crp}}(p) &= B - \sum_{i=1}^N \beta_i / (1 + \frac{1}{p\tau_i}) \end{aligned} \right\} \quad (23)$$

Substituting eqs. (23) in eq. (11),

$$p \bar{C}_{\epsilon, \text{rel}}(p) = \frac{2}{3} \left\{ A + \sum_{i=1}^N \alpha_i / (1 + \frac{1}{p\tau_i}) \right\} \times \left\{ B - \sum_{i=1}^N \beta_i / (1 + \frac{1}{p\tau_i}) \right\} \quad (24)$$

The right side of eq.(24) was approximated by the series $C + \sum_{i=1}^N \theta_i / (1 + \frac{1}{p\tau_i})$ and the non-linear minimization technique was applied to determine the coefficients θ_i . Then transforming back of this formula provided the relaxation birefringence-strain coefficient $C_{\epsilon, \text{rel}}(\xi)$ in a reduced time. The result for epoxy resin is shown in Fig.16.

Fig.16 Master curve of relaxation birefringence-strain coefficient

The curve obtained had a little undulation, which was due to the relatively small variation of $C_{\epsilon, \text{rel}}(\xi)$ compared with other properties. Small errors in the processes to determine $E_{\text{rel}}(\xi)$ and $C_{\sigma, \text{crp}}(\xi)$ might deteriorate the accuracy of the approximation of $C_{\epsilon, \text{rel}}(\xi)$. The coefficients of the Prony series which represented the photoviscoelastic properties of epoxy resin were determined through the minimization technique and shown in Table 1.

Table 1 Prony-series coefficients for the optical and mechanical characterization for epoxy

Properties		E_{rel}	D_{crp}	$C_{\sigma,crp}$	$C_{\varepsilon,rel}$
K		1.202	1.122	102.329	81.289
i	$-\log \tau_i$	α_i	γ_i	β_i	θ_i
1	0.0	0.0121	0.00001	0.0463	0.1239
2	0.5	0.0133	0.00002	1.3745	0.3804
3	1.0	0.0161	0.00007	1.3144	0.0370
4	1.5	0.0135	0.01368	0.0372	0.0362
5	2.0	0.0167	0.01202	0.0452	4.4138
6	2.5	0.0523	0.15939	60.6279	3.7230
7	3.0	0.0688	0.38546	28.8909	10.1107
8	3.5	0.0886	0.40022	0.5957	2.4445
9	4.0	0.5266	0.12109	1.8051	2.1636
10	4.5	2.1920	0.00879	2.4357	0.1559
11	5.0	8.2142	0.00678	0.9494	2.7603
12	5.5	14.9162	0.00168	0.3555	6.9196
13	6.0	16.5986	0.00065	0.5214	18.6956
14	6.5	24.7366	0.00004	0.4866	14.1736
15	7.0	11.7543	0.00001	0.0445	21.7044
16	7.5	13.7203	0.00001	0.0498	29.4490
17	8.0	20.9022	0.00005	0.0702	20.1939
18	8.5	4.3545	0.00001	0.0573	0.0003
19	9.0	0.0123	0.00000	0.0361	0.0005
20	9.5	0.0074	0.00001	0.0490	0.0068
21	10.0	0.0098	0.00003	0.0146	0.0000

The third example was the application of this minimization technique to determine the inverse relaxation birefringence-stress coefficient $\bar{C}_{\sigma,rel}^{-1}(\xi)$ of polyurethane rubber, which, as well as the inverse creep birefringence-strain coefficient $\bar{C}_{\varepsilon,crp}^{-1}(\xi)$, was necessary to know the principal stress difference or the principal strain difference from fringe orders obtained through photo-viscoelastic experiments. These properties were related to $\bar{C}_{\sigma,crp}(p)$ and $\bar{C}_{\varepsilon,rel}(p)$ in the Laplace plane as follows. (1)

$$\bar{C}_{\sigma,rel}^{-1}(p) = 1 / \{ p^2 \bar{C}_{\sigma,crp}(p) \} \quad (25)$$

$$\bar{C}_{\varepsilon,crp}^{-1}(p) = 1 / \{ p^2 \bar{C}_{\varepsilon,rel}(p) \} \quad (26)$$

Approximating the creep birefringence-stress coefficient $C_{\sigma,crp}(\xi)$ with the Prony series representation (21), its inverse transform was substituted into eq. (25).

$$p \frac{T_0}{T} \bar{C}_{\sigma,rel}^{-1}(p) = 1 / \{ B - \sum_{i=1}^N \beta_i / (1 + \frac{1}{p \tau_i}) \} \quad (27)$$

The right side of this equation was approximated again by the series representation

$$B^{-1} + \sum_{i=1}^N \gamma_i / (1 + \frac{1}{p \tau_i}), \quad \gamma_i \geq 0 \quad (28)$$

Then, transforming back of eq. (28) provided the inverse relaxation birefringence-stress coefficient $\bar{C}_{\sigma,rel}^{-1}(\xi)$

$$\frac{T_0}{T} \bar{C}_{\sigma,rel}^{-1}(\xi) = B^{-1} + \sum_{i=1}^N \gamma_i \exp(-\xi / \tau_i) \quad (29)$$

The coefficients γ_i in eq. (28) were determined by the use of the nonlinear minimization procedure and the master curve of $\bar{C}_{\sigma,rel}^{-1}(p)$ obtained is shown in Fig. 17. And the numerical values of coefficients β_i and γ_i in the Prony series representation of $C_{\sigma,crp}(\xi)$ and $\bar{C}_{\sigma,rel}^{-1}(\xi)$ are shown in Table 2.

Fig. 17 Master curve of inverse relaxation birefringence-stress coefficient

Table 2 Prony-series coefficients for the $C_{\sigma,crp}(\xi)$ and $\bar{C}_{\sigma,rel}^{-1}(\xi)$ for polyurethane

Properties		$C_{\sigma,crp}(\xi)$	$\bar{C}_{\sigma,rel}^{-1}(\xi)$
K		93.325	0.010715
i	$-\log_{10} \tau_i$	β_i	γ_i
1	4.0	4.083	6.629×10^{-4}
2	4.5	2.383×10^{-3}	8.019×10^{-4}
3	5.0	1.848	6.684×10^{-4}
4	5.5	1.498×10^{-1}	9.484×10^{-4}
5	6.0	1.791×10^{-1}	1.121×10^{-3}
6	6.5	3.029×10^{-1}	1.318×10^{-3}
7	7.0	1.112×10^{-1}	1.570×10^{-2}
8	7.5	5.257	3.731×10^{-2}
9	8.0	3.727	2.481×10^{-2}
10	8.5	1.443	3.812×10^{-2}
11	9.0	1.066	1.519×10^{-1}
12	9.5	7.792×10^{-2}	3.632×10^{-1}
13	10.0	4.063×10^{-2}	4.324×10^{-1}
14	10.5	6.576×10^{-2}	3.561×10^{-1}
15	11.0	6.467×10^{-2}	1.834×10^{-2}
16	11.5	2.013×10^{-4}	1.405×10^{-2}
17	12.0	1.039×10^{-4}	2.514×10^{-2}

Judging from the shape of the master curve obtained, the mathematical representation of the inverse relaxation birefringence-stress coefficient $\bar{C}_{\sigma,rel}^{-1}(\xi)$ seems to have been completed with sufficient accuracy.

6. Conclusions

In the characterization of mechanical and optical properties of photoviscoelastic materials, a new collocation method to represent these properties in the Prony series was proposed. This mathematical representation of the properties is absolutely necessary to analyse the stress-strain fields in viscoelastic materials by means of photoviscoelasticity.

The proposed procedure was applied to two photoviscoelastic materials, polyurethane rubber and epoxy resin. The following

conclusions were drawn.

- (1) Employing the nonlinear optimization procedure (SUMT), the mathematical representations of the photoviscoelastic properties were made logically and with high accuracy.
- (2) Polyurethane rubber and epoxy resin, which were examined in this paper, might be considered to be linear photoviscoelastic materials and the time-temperature superposition principle holds well for these two materials. The computer of Tokyo University, HITAC 8800/8700, and its library program 389C7/TC/SUMT were used in numerical calculations.

References

- (1) Williams, M.L., et al., GALCIT SM 63-31, CIT (1963-10).
- (2) Arenz, R.J., et al., *Exper. Mechanics*, Vol.7, No.4 (1967-4), p.183.
- (3) Miyano, Y., et al., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.34, No.262 (1968-6), p.1060.
- (4) Kunio, T., et al., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), Vol.34, No.263 (1968-7), p.1198.
- (5) Theocaris, P.S., IUTAM Symposium, Ottignies/Belgium (1973-9), P.146.
- (6) Coleman, B.D. and Dill, E.H., IUTAM Symposium, Ottignies/Belgium (1973-9), p.455.
- (7) Schapery, R.A., GALCIT 119, CIT (1962-4).