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Buckling of an Elastically Restrained Elliptical Plate under Uniform Compression*

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It is the object of this paper to study the stability of a complete elliptical plate restrained elastically against rotation along the boundary ellipse. On the basis of the ordinary thin plate theory, the buckling condition for the elliptical plate subjected to uniform compression in its middle plane is derived with use of trigonometric functions, hyperbolic functions, Mathieu functions and modified Mathieu functions which are the solutions of the equilibrium equation of a buckled plate. Theory is developed by paying attention to the orthogonal property of Mathieu functions. The lowest eigenvalues for the buckling mode symmetrical about both axes are presented in tables for a variety of aspect ratios, degrees of edge restraint and Poisson's ratios. Discussion of the degenerate case to a circular plate is also included.

1. Introduction

Elliptical plates are not only used as lids of elliptical pressure vessels but also applied widely in various structures. Hence, the analysis of static and dynamic behaviours of those elliptical plates is of great importance in engineering design for the prevention of failure due to static and dynamic loads.

The stability problems of complete elliptical plates subjected to a uniform compressive force at the elliptical boundary in the middle plane of the plate have been discussed by several investigators. In 1937, Woinowsky-Krieger¹ solved the buckling problem of an elliptical plate with clamped edge by means of the Rayleigh-Ritz method. Afterwards, in 1956-7, Shibaoka^{2,3} developed the exact buckling analysis of an elliptical plate with clamped edge using trigonometric functions, hyperbolic functions, Mathieu functions and modified Mathieu functions satisfying the equilibrium equation of a buckled plate expressed in terms of elliptical coordinates. In 1972, Mazumdar⁴ analyzed the buckling of elastic plates of arbitrary shape on the basis of his concept of 'Lines of Equal Deflection'. As an illustration of the method, he dealt with the case of elliptical plates with clamped and simply supported edges. They carried out the numerical calculation and presented the calculated values of buckling load in tables and figures. In the previous paper, the author⁵ solved the buckling problem of an elliptical plate with simply supported edge using the solutions of the equilibrium equation for the buckled plate like Shibaoka. In the mathematical procedure leading to the buckling condition, although Shibaoka^{2,3} ex-

pressed the Mathieu functions in terms of trigonometric series and applied the orthogonal property of the trigonometric functions, the author⁵ expressed the trigonometric functions in terms of Mathieu functions and applied the orthogonal property of the Mathieu functions. The first five eigenvalues for the buckling mode symmetrical about both axes of a simply supported elliptical plate are presented in a table and shown in a figure with the mode shapes of the buckling. All these exact and approximate solutions of the buckling problems of elliptical plates have dealt with the plates under either clamped or simply supported boundary conditions. However, in practical application, there are numerous cases where the plate edge must be considered to be elastically restrained.

The present paper is concerned with the buckling of a complete elliptical plate whose edge is rigidly supported against deflection but is elastically restrained against rotation in a manner providing a restoring moment proportional to the inclination of the middle plane. It is assumed that the plate is thin and subjected to a uniform distribution of compressive forces acting in its middle plane. The analysis is exactly developed with use of trigonometric functions, hyperbolic functions, Mathieu functions and modified Mathieu functions satisfying the equilibrium equation of a buckled plate expressed in terms of elliptical coordinates as in the references (2) and (3). In making use of the orthogonal property of Mathieu functions as in the reference (5), the buckling condition for the symmetrical mode about both axes of the elliptical plate is derived from the solutions of the equilibrium equation and the boundary conditions of the elastic support. The lowest eigenvalues for three kinds of Poisson's ratios are calculated numerically for a variety of aspect ratios and degrees of edge restraint, and are presented in tables. It is noted that these values give the critical loads of buckling of the system under consideration. In the special cases of clamped and simply

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supported plates the present results are compared with the critical values given by the other works. It is also shown that when the fundamental ellipse tends to a circle, the buckling condition derived in the paper degenerates to that for an elastically restrained circular plate against rotation.

2. Mathematical formulation

It is assumed that a thin elastic plate is subjected to compressive force P per unit length of the edge in the middle plane of the plate. According to the classical small deflection theory, the equilibrium equation of the buckled plate is, using elliptical coordinates $(\xi, \eta)^{2,3,5}$,

$$\left\{ \frac{2}{c^2 (\cosh 2\xi - \cos 2\eta)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) \right\} \left\{ \frac{2}{c^2 (\cosh 2\xi - \cos 2\eta)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) + k_1^2 \right\} w = 0, \quad (1)$$

where

$$k_1^2 = P/D. \quad (2)$$

In Eqs. (1) and (2), the symbols w and c represent the deflection of the plate and the semifocal length of the ellipse respectively, and D is the flexural rigidity given in terms of Young's modulus E , the plate thickness h , and Poisson's ratio ν by the formula $D = Eh^3/12(1-\nu^2)$.

The bending moment M per unit length of a section of a plate perpendicular to the direction of the normal to the edge-line is expressed in terms of the elliptical coordinates as

$$M = -\frac{Dc^2}{2l_1} \left\{ \left(\frac{\partial^2 w}{\partial \xi^2} + \nu \frac{\partial^2 w}{\partial \eta^2} \right) (\cosh 2\xi - \cos 2\eta) - (1-\nu) \frac{\partial w}{\partial \xi} \sinh 2\xi + (1-\nu) \frac{\partial w}{\partial \eta} \sin 2\eta \right\}, \quad (3)$$

where

$$2l_1^2 = c^2 (\cosh 2\xi - \cos 2\eta). \quad (4)$$

In the present problem, the boundary of the plate is assumed to be rigidly restrained against deflection and to be elastically restrained against rotation in a manner providing a restoring edge moment proportional to angle of edge rotation about the axis perpendicular to the direction of the normal to the edge-line. Then, we have, along the elliptical boundary ξ_0 ,

$$w|_{\xi=\xi_0} = 0 \quad (5)$$

and

$$M|_{\xi=\xi_0} = \beta \frac{\partial w}{\partial \xi} \Big|_{\xi=\xi_0}, \quad (6)$$

where β is the rotational spring stiffness depending on the degree of restraint.

The solution of Eq. (1) corresponding to the buckling mode symmetrical about both axes is⁵, using the unknown constants C_{2n} and $\bar{C}_{2n}$,

$$w = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} C_{2m} \cosh 2m\xi A_{2m}^{(2n)} \delta_m + \bar{C}_{2n} Ce_{2n}(\xi, q) \right\} ce_{2n}(\eta, q), \quad \delta_m = 2(m=0), 1(m \neq 0), \quad (7)$$

where

$$q = k_1^2 c^2 / 4 (= P c^2 / 4D). \quad (8)$$

In Eq. (7), ce_{2n} and Ce_{2n} denote a Mathieu function and a modified Mathieu function of order $2n$ respectively, and $A_{2m}^{(2n)}$ denotes the coefficient in the cosine series for the Mathieu function ce_{2n} . Substituting Eq. (7) into Eq. (5) and applying the orthogonal property of Mathieu functions, we get

$$\sum_{m=0}^{\infty} C_{2m} \cosh 2m\xi_0 A_{2m}^{(2n)} \delta_m + \bar{C}_{2n} Ce_{2n}(\xi_0, q) = 0. \quad (9)$$

Next, substitution of Eq. (7) into Eq. (6) yields

$$\begin{aligned} l_1^{-3} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} C_{2m} 4m^2 \cosh 2m\xi_0 A_{2m}^{(2n)} \delta_m + \bar{C}_{2n} (a_{2n} - 2q \cosh 2\xi_0) Ce_{2n}(\xi_0, q) \right] \cosh 2\xi_0 \\ - \frac{1}{2l_1} \sum_{n=0}^{\infty} \left[\sum_{m=0}^{\infty} C_{2m} 4m^2 \cosh 2m\xi_0 A_{2m}^{(2n)} \delta_m + \bar{C}_{2n} (a_{2n} - 2q \cosh 2\xi_0) Ce_{2n}(\xi_0, q) \right] \phi_{2n, 2n}^{(1)} \end{aligned}$$

$$\begin{aligned}
& -(1-\nu) \left\{ \sum_{m=0}^{\infty} C_{2m}^{2m} \sinh 2m\xi_0 A_{2m}^{(2n)} \delta_m + \bar{C}_{2n} Ce'_{2n}(\xi_0, q) \sinh 2\xi_0 \right\} ce_{2n}(n, q) \\
& = -(2\beta/Dc^2) \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty} C_{2m}^{2m} \sinh 2m\xi_0 A_{2m}^{(2n)} \delta_m + \bar{C}_{2n} Ce'_{2n}(\xi_0, q) \right\} ce_{2n}(n, q),
\end{aligned} \quad (10)$$

where

$$\phi_{2n, 2l}^{(j)} = \sum_{i=0}^{\infty} A_{2i}^{(2n)} \{ A_{2i+2j}^{(2l)} \delta_{i+j} + A_{2i-2j}^{(2l)} \delta_{i-j} \}, \quad (11)$$

and the prime denotes the ξ -derivative. The symbol α_{2n} denotes the characteristic value of the Mathieu function of order $2n$. Here, we introduce the formula⁶

$$\frac{1}{(\cosh a - \cos x)^\alpha} = 2^\alpha e^{-\alpha a} \{ F(\alpha, \alpha, 1; e^{-2a}) + 2 \sum_{n=1}^{\infty} \frac{e^{-n\alpha} \Gamma(n+\alpha)}{n! \Gamma(\alpha)} F(\alpha, n+\alpha, n+1; e^{-2a}) \cos nx \},$$

$$-\pi \leq x \leq \pi, \quad \alpha, a > 0, \quad (12)$$

where Γ and F denote the gamma function and the hypergeometrical function respectively. Re-writing Eq.(10) with use of the above formula and the relation between circular functions and Mathieu functions⁷, and moreover applying the orthogonal property of Mathieu functions, we get, after some calculations,

$$\begin{aligned}
& -\frac{\beta c}{\sqrt{2D}} \left\{ \sum_{m=0}^{\infty} C_{2m}^{2m} A_{2m}^{(2n)} \delta_m \sinh 2m\xi_0 + \bar{C}_{2n} Ce'_{2n}(\xi_0, q) \right\} \\
& = \sum_{s=0}^{\infty} \left[\sqrt{2} e^{-3\xi_0} \{ F(\frac{3}{2}, \frac{3}{2}, 1; e^{-4\xi_0}) \phi_{2s, 2n}^{(0)} + 2 \sum_{j=1}^{\infty} (e^{-2j\xi_0} \Gamma(j+\frac{3}{2})/j!/\Gamma(\frac{3}{2})) F(\frac{3}{2}, j+\frac{3}{2}, j+1; e^{-4\xi_0}) \phi_{2s, 2n}^{(j)} \} \right] \\
& \quad \times \left\{ \left[\sum_{m=0}^{\infty} C_{2m}^{4m} A_{2m}^{(2s)} \delta_m \cosh 2m\xi_0 + \bar{C}_{2s} (\alpha_{2s} - 2q \cosh 2\xi_0) Ce_{2s}(\xi_0, q) \right] \cosh 2\xi_0 \right. \\
& \quad \left. - \frac{1}{2} \sum_{l=0}^{\infty} \left[\sum_{m=0}^{\infty} C_{2m}^{4m} A_{2m}^{(2l)} \delta_m \cosh 2m\xi_0 + \bar{C}_{2l} (\alpha_{2l} - 2q \cosh 2\xi_0) Ce_{2l}(\xi_0, q) \right] \phi_{2l, 2s}^{(1)} \right. \\
& \quad \left. - (1-\nu) \left\{ \sum_{m=0}^{\infty} C_{2m}^{2m} A_{2m}^{(2s)} \delta_m \sinh 2m\xi_0 + \bar{C}_{2s} Ce'_{2s}(\xi_0, q) \sinh 2\xi_0 \right\} \right], \quad n = 0, 1, 2, \dots.
\end{aligned} \quad (13)$$

Elimination of the unknown constant $\bar{C}_{2n}$ from Eqs.(9) and (13) yields

$$\sum_{m=0}^{\infty} C_{2m} e_{mn} = 0, \quad n = 0, 1, 2, \dots, \quad (14)$$

where

$$\begin{aligned}
e_{mn} & = \frac{\beta c}{\sqrt{2D}} A_{2m}^{(2n)} \delta_m \{ 2m \sinh 2m\xi_0 - \cosh 2m\xi_0 Ce'_{2n}(\xi_0, q) / Ce_{2n}(\xi_0, q) \} + \sum_{s=0}^{\infty} \left[\sqrt{2} e^{-3\xi_0} \{ F(\frac{3}{2}, \frac{3}{2}, 1; e^{-4\xi_0}) \phi_{2s, 2n}^{(0)} \right. \\
& \quad \left. + 2 \sum_{j=1}^{\infty} (e^{-2j\xi_0} \Gamma(j+\frac{3}{2})/j!/\Gamma(\frac{3}{2})) F(\frac{3}{2}, j+\frac{3}{2}, j+1; e^{-4\xi_0}) \phi_{2s, 2n}^{(j)} \} \right] [A_{2m}^{(2s)} \delta_m (\cosh 2m\xi_0 \cosh 2\xi_0 \\
& \quad \times (4m^2 - \alpha_{2s} + 2q \cosh 2\xi_0) - (1-\nu) \sinh 2\xi_0 \{ 2m \sinh 2m\xi_0 - \cosh 2m\xi_0 Ce'_{2s}(\xi_0, q) / Ce_{2s}(\xi_0, q) \} \} \\
& \quad - \frac{1}{2} \sum_{j=0}^{\infty} \phi_{2j, 2s}^{(1)} A_{2m}^{(2j)} \delta_m \cosh 2m\xi_0 (4m^2 - \alpha_{2j} + 2q \cosh 2\xi_0)], \quad m, n = 0, 1, 2, \dots.
\end{aligned} \quad (15)$$

Non-vanishing solutions for C_{2m} exist only if the determinant formed of the coefficients vanishes. This gives the buckling condition for the elastically restrained elliptical plate against rotation in the form of an infinite determinantal equation(16). The foregoing equations can be particularized to the usual limiting cases of a clamped elliptical plate ($\beta \rightarrow \infty$) and a simply supported elliptical plate ($\beta \rightarrow 0$). When $\beta \rightarrow \infty$, we can write the elements of the determinantal equation

$$\begin{vmatrix}
 e_{00} & e_{10} & e_{20} & \cdot & \cdot \\
 e_{01} & e_{11} & e_{21} & \cdot & \cdot \\
 e_{02} & e_{12} & e_{22} & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot \\
 \cdot & \cdot & \cdot & \cdot & \cdot
 \end{vmatrix} = 0 \quad (16)$$

from the expression(15) in the form of

$$e_{mn} = A_{2m}^{(2n)} \{2m \sinh 2m\xi_0 - \cosh 2m\xi_0 Ce'_{2n}(\xi_0, q) / Ce_{2n}(\xi_0, q)\}, \quad m, n = 0, 1, 2, \dots \quad (17)$$

The infinite determinantal equation having the elements given by the expression(17) is a buckling condition for the clamped elliptical plate. It is noted that an exchange of rows and columns of Eq.(16) gives the form of buckling condition derived by Shibaoka^{2,3}. When $\beta \rightarrow 0$, Eq.(16) is expressed in the product of two determinants having the elements

$$\begin{aligned}
 e_{pn}^{(1)} &= \sqrt{2} e^{-3\xi_0} \{F(\frac{3}{2}, \frac{3}{2}, 1; e^{-4\xi_0}) \phi_{2p, 2n}^{(0)} \\
 &+ 2 \sum_{j=1}^{\infty} (e^{-2j\xi_0} \Gamma(j+\frac{3}{2})/j!/\Gamma(\frac{3}{2})) F(\frac{3}{2}, j+\frac{3}{2}, j+1; e^{-4\xi_0}) \phi_{2p, 2n}^{(j)}\}, \quad p, n = 0, 1, 2, \dots, \quad (18)
 \end{aligned}$$

and

$$\begin{aligned}
 e_{pm}^{(2)} &= A_{2m}^{(2p)} \delta_m [\cosh 2m\xi_0 \cosh 2\xi_0 (4m^2 - a_{2p}^2 + 2qc \cosh 2\xi_0) \\
 &- (1-\nu) \sinh 2\xi_0 \{2m \sinh 2m\xi_0 - \cosh 2m\xi_0 Ce'_{2p}(\xi_0, q) / Ce_{2p}(\xi_0, q)\}] \\
 &- \frac{1}{2} \sum_{j=0}^{\infty} \phi_{2j, 2p}^{(1)} A_{2m}^{(2j)} \delta_m \cosh 2m\xi_0 (4m^2 - a_{2j}^2 + 2qc \cosh 2\xi_0), \quad p, m = 0, 1, 2, \dots \quad (19)
 \end{aligned}$$

The infinite determinantal equation having the elements given by the expression(19) is a buckling condition for the simply supported elliptical plate⁵.

Let us consider a limiting case where the fundamental ellipse tends to a circle. Then we have⁷

$$\begin{aligned}
 \xi_0 &\rightarrow \infty, \quad q \rightarrow 0, \quad 2qc \cosh 2\xi_0 \rightarrow (k_1 b)^2, \\
 A_{2m}^{(2n)} &\rightarrow 1/\sqrt{2} (m=n=0), \quad 1 (m=n \neq 0), \quad 0 (m \neq n), \\
 a_{2n} &= 4n^2, \\
 Ce_{2n}(\xi_0, q) &\rightarrow p'_{2n} J_{2n}(k_1 b), \\
 dCe_{2n}(\xi, q)/d\xi|_{\xi=\xi_0} &\rightarrow p'_{2n} r dJ_{2n}(k_1 r)/dr|_{r=b},
 \end{aligned}$$

where J_{2n} and b denote the Bessel function of order $2n$ and the radius of a circle respectively, and p'_{2n} is the constant multiplier. With aid of these degenerate forms, we find that the infinite determinantal equation(16) having the elements given by the expression(15) degenerates to an infinite diagonal determinantal equation. Then the diagonal element $e_{nn}=0$, that is,

$$k_1 b J_{2n}(k_1 b) + (\beta b/D - 1 + \nu) J_{2n+1}(k_1 b) = 0 \quad (20)$$

gives the buckling condition for an elastically restrained circular plate with a number $2n$ of nodal diameters.

3. Numerical results

Numerical calculations of the lowest eigenvalues $k_1 b = (P/D)^{1/2} b$ of the buckling problem were carried out for various aspect ratios a/b , edge-restraint parameters $\beta b/D$ and Poisson's ratios ν of an elliptical plate from the infinite determinantal equation (16). In order to determine the actual values of the roots of the infinite determinantal equation, the convergence must be examined. This was done by truncating the infinite determinant at the finite N by N determinant which made the roots converge enough. The rate of convergence decreased with an increasing aspect ratio. Throughout the present numerical calculations, the convergent roots could be found by adopting the 5 by 5 determinant. It is noted here that in respect to the analytical method to take a N by N determinant means to approximate the series expression (7) with the first N terms with respect to m and n . For the modified Mathieu function we used the Bessel function product series⁷ which converges rapidly for a wide range of values of ξ . The lowest eigenvalues calculated numerically in this manner are given in Tables 1 to 3.

In the tables, the limiting cases of $\beta b/D=0$ and $\beta b/D=\infty$ are corresponding to those for a simply supported plate and a clamped plate, respectively. It is seen that in each table the eigenvalue for each aspect ratio is smallest for the case of a simply supported plate and increases with an increasing edge-restraint parameter. As the

Table 1 Lowest eigenvalues $k_1 b$ as functions of a/b and $\beta b/D$ with $\nu=0.0$

a/b	$\beta b/D$					
	0.0	0.1	1.0	10.0	100.0	∞
1.0	1.841	1.951	2.405	3.480	3.794	3.882
1.1	1.767	1.841	2.323	3.345	3.634	3.669
1.2	1.720	1.795	2.270	3.249	3.518	3.551
1.4	1.675	1.751	2.212	3.127	3.371	3.400
1.7	1.662	1.736	2.184	3.038	3.259	3.287
2.0	1.668	1.737	2.179	2.995	3.204	3.230
2.5	1.678	1.740	2.178	2.959	3.158	3.183
3.0	1.678	1.738	2.176	2.939	3.133	3.157
4.0	1.660	1.728	2.170	2.919	3.107	3.131

Table 4 Comparison between the author's results and the others

(a) Dimensionless critical values Pb^2/D of buckling of a simply supported elliptical plate ($\nu=0.3$)

a/b	1.0	1.1	1.2	1.4	2.0	3.0	4.0
Author	4.20	3.85	3.61	3.33	3.05	2.91	2.82
Mazumdar	4.20	3.85	3.62	3.34	3.10	3.08	3.10

(b) Dimensionless critical values Pb^2/D of buckling of a clamped elliptical plate

a/b	1.0	1.1	1.2	1.4	2.0	3.0	4.0
Author	14.68	13.46	12.61	11.56	10.43	9.97	9.80
Woinowsky-Krieger	14.79	13.57	12.76	11.81	11.02	11.01	11.15
Mazumdar	14.67	13.46	12.63	11.66	10.82	10.76	10.83

Table 2 Lowest eigenvalues $k_1 b$ as functions of a/b and $\beta b/D$ with $\nu=0.3$

a/b	$\beta b/D$					
	0.0	0.1	1.0	10.0	100.0	∞
1.0	2.049	2.109	2.521	3.489	3.794	3.832
1.1	1.962	2.023	2.430	3.353	3.634	3.669
1.2	1.901	1.962	2.366	3.255	3.518	3.551
1.4	1.825	1.887	2.288	3.132	3.371	3.400
1.7	1.772	1.833	2.237	3.041	3.260	3.287
2.0	1.747	1.806	2.215	2.997	3.204	3.230
2.5	1.723	1.782	2.199	2.960	3.158	3.183
3.0	1.706	1.762	2.189	2.940	3.133	3.157
4.0	1.679	1.743	2.177	2.919	3.107	3.131

Table 3 Lowest eigenvalues $k_1 b$ as functions of a/b and $\beta b/D$ with $\nu=0.5$

a/b	$\beta b/D$					
	0.0	0.1	1.0	10.0	100.0	∞
1.0	2.166	2.219	2.589	3.495	3.794	3.832
1.1	2.073	2.126	2.492	3.358	3.634	3.669
1.2	2.003	2.057	2.423	3.260	3.518	3.551
1.4	1.911	1.965	2.334	3.135	3.371	3.400
1.7	1.834	1.890	2.269	3.043	3.260	3.287
2.0	1.792	1.848	2.238	2.998	3.204	3.230
2.5	1.750	1.808	2.212	2.960	3.158	3.183
3.0	1.723	1.783	2.198	2.940	3.133	3.157
4.0	1.688	1.753	2.182	2.919	3.107	3.131

edge-restraint parameter becomes large, the eigenvalues approach asymptotically the values for the case of a clamped plate. Also from the tables, we see that the eigenvalues increase with an increasing Poisson's ratio, and the influence of Poisson's ratio on the eigenvalues decreases with an increasing edge-restraint parameter. In the case of $\nu=0$ and $\beta b/D=0, 0.1$, it is explicitly seen that although at the beginning the eigenvalues decrease with an increasing aspect ratio, there are the minimum and maximum values in the range $1.4 < a/b < 3$.

Mazumdar⁴ gave the lowest critical values of buckling of a simply supported elliptical plate of $\nu=0.3$ by using his 'Lines of Equal Deflection' theory. In Table 4(a), his results are given in comparison with the author's tabulated results squared. The critical values of buckling of a clamped elliptical plate were obtained by Woinowsky-Krieger¹ by making two-term approximation of the series by the Rayleigh-Ritz method and

by Mazumdar⁴ by his 'Lines of Equal Deflection' theory. In Table 4(b), their results are given in comparison with the author's tabulated results squared. It is seen from Table 4 that their results are higher than those of the present work, and the discrepancy increases with an increasing aspect ratio. It is a matter of course that the critical values of Shibaoka³ for the buckling of the clamped elliptical plates having the aspect ratios $a/b=2$ and 3 agree very well with the calculated values of the present work in the special case of $\beta b/D=\infty$.

4. Concluding remark

The buckling of a thin elliptical plate with the edge restrained elastically against rotation has been discussed in an exact manner with use of the solutions of the equilibrium equation for a buckled plate subjected to uniform compressive force in its middle plane. The buckling condition for the symmetrical mode about both axes of the ellipse was derived by applying the orthogonal property of Mathieu functions. For all Poisson's ratios and aspect ratios, it is seen from the numerical results that the eigenvalues for the elastically restrained plates under consideration lie between the two limiting values for the clamped and simply supported plates. The results obtained

here are of practical significance since neither clamped nor simply supported boundary conditions can be realized fully in actual mechanical structures.

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