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Vibration of a Shaft Passing through a Critical Speed*

(4th Report, Effect of Gyroscopic Moment)

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Effects of the gyroscopic moment on the nonstationary vibration, which occurs when a rotor passes through its critical speed at a uniform acceleration rate, are analyzed. The numerical integration results show that the gyroscopic moment makes the maximum amplitude small and makes the rotational speed, where the maximum amplitude occurs, high comparing with the results of a single-degree-of-freedom system. These phenomena may be explained by assuming that the acceleration rate in the gyroscopic system becomes large in appearance. Formulas for the imaginary acceleration rate and the maximum amplitude are derived by considering both the nonstationary vibration of a single-degree-of-freedom system and the change of the natural frequency of the gyroscopic system. The estimated maximum amplitudes are compared with the numerical ones.

1. Introduction

Many studies on the vibration during acceleration through a critical speed have been made since Lewis's work in 1932.⁽¹⁾ Their main objective is to clarify the nonstationary characteristics of the vibration occurring when a rotating shaft with a single disk at the midway of the span (a single-degree-of-freedom system) passes through the critical speed at a uniform acceleration rate.⁽²⁾⁻⁽⁵⁾ The present author has experimentally clarified the nonstationary characteristics⁽⁶⁾ and derived theoretically both the approximate expressions representing the nonstationary motion and a formula for the maximum amplitude of the response.^{(7),(8)}

The effect of the gyroscopic moment on the nonstationary vibration, however, do not seem to have been studied well, even though it is commonly known that the moment makes the rotor critical speed high. Mitropol'skii⁽⁹⁾ shows calculated response curves as well as an analytical method for the nonstationary vibration of a gyroscopic system. But there may be some uncertainty in both the equations of motion and the calculated results. Goloskow and Fillipow⁽¹⁰⁾ refer to Mitropol'skii's method in their book but they do not make any discussions about the effects of the gyroscopic moment. Taniguchi, Sakata and Takenouchi⁽¹¹⁾ obtain the solutions of the nonstationary vibration of an overhung rotor by an analog computer and after comparing with the response curves of two systems, one of which is affected by the gyroscopic effect and the other is not, they state that the gyroscopic moment makes the maximum amplitude small and makes the speed at the maximum amplitude high. Nonami and Miyashita⁽¹²⁾ treat an overhung rotor supported flexibly and obtain the same conclusions as described above and further show

that the beat phenomena remain long if the polar moment of inertia is smaller than the diametral moment of inertia. Aiba⁽¹³⁾ treats the nonstationary vibration of a gyroscopic system with asymmetry in two diametral moments of inertia.

The results obtained by these works partially clarify the effects of the gyroscopic moment qualitatively comparing with two types of nonstationary responses, one taking account of the moment and the other neglecting it. It may be thought that they are not enough to explain the nonstationary characteristics of the gyroscopic system comprehensively, still less estimate the maximum amplitude and its rotational speed quantitatively.

In the present paper both the stationary and the nonstationary vibrations of a rotor with a solid disk attached at the overhung shaft end are considered. Calculated results show the effects of the gyroscopic moment on the nonstationary vibration and that even when the stationary response curves and the acceleration parameters are the same in systems with different ratios of length to diameter of the disk (L/D ratio), the maximum amplitude becomes small as L/D ratio changes small.

In order to explain these calculated results comprehensively and to estimate the maximum amplitude quantitatively, we assume that the acceleration rate in the gyroscopic system would increase in appearance and derive a formula, which makes it possible to calculate the imaginary acceleration rate as a function of the system dimensions and the real acceleration rate, by reconsidering the analytical process of the nonstationary vibration of a single-degree-of-freedom system.

Furthermore, the maximum amplitudes of the nonstationary vibration are estimated by using both the imaginary acceleration rates and the stationary responses. The estimated values are compared with the numerical integration results.

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2. Nomenclature

- a : Overhung length of shaft
 c_R, c_θ : Damping coefficients (see Eq.(1))
 D_R, D_θ : see Eq.(10)
 D_M : Modal damping ratio
 EI : Flexural rigidity of shaft
 e : Eccentricity of disk
 e_M : Modal eccentricity
 I_d, I_p : Diametral and polar moments of inertia of disk (see Fig.1)
 k_{ij} : Nondimensional spring constant
 L/D : Ratio of length to diameter of disk
 m : Mass of disk
 n_R, n_θ : see Eq.(3)
 $\bar{r}_G$: Amplitude of r_G in stationary vibration
 r_G, x_G, y_G : Radius of whirling motion of center of gravity and displacements in x and y directions
 $z_G = x_G + iy_G$
 α : Angular acceleration
 α_{ij} : Spring constant (see Eq.(2))
 β : Angle between eccentricity vector and direction vector of polar moment of inertia projected to x - y plane

- γ : Variational rate of first natural angular frequency of forward precession with the rotational speed
 $\epsilon_r, \epsilon_\theta$: Phase angles between ψ and ψ_r , and ψ and ϕ_θ respectively
 $\bar{\theta}_G$: Amplitude of θ_G in stationary vibration
 $\theta_G, \theta_{Gx}, \theta_{Gy}$: Angle between direction vector of polar moment of inertia and z axis, angles projected to x - z and y - z planes of θ_G
 $\Theta = \theta_{Gx} + i\theta_{Gy}$
 κ : Nondimensional angular speed of rotation at acceleration start
 κ_{ij} : see Eq.(3)
 λ : see Fig.1
 τ : Angle of θ_G at rest state
 $u (=I_p/I_d)$: Moment of inertia ratio
 ϕ_r, ϕ_θ : Whirling angles of center of gravity and direction vector of polar moment of inertia
 ψ : Rotational angle
 ω : Angular speed of rotation
 ω_f : Natural angular frequency
 ω_n : Critical speed (see Eq.(8))
 $\Omega (= \omega_n/\sqrt{\alpha})$: Acceleration parameter
 $\Omega^* (= \omega_n/\sqrt{(1+2\gamma)\alpha})$: Imaginary acceleration parameter
 $i (= \sqrt{-1})$: Imaginary unit

3. Equations of motion

3.1 Equations of motion

A system considered is shown in Fig.1. It consists of a uniform rotating shaft and a solid disk with length L and diameter D attached at overhung shaft end. Here we take three coordinate axes as shown in the figure and assume that the gravity force and the mass of shaft may be neglected and that a viscous damping force acts on the system. The equations of motion for the center of gravity of the disk may be written as follows using the constant acceleration condition.

$$\left. \begin{aligned}
 m\ddot{x}_G + c_R\dot{x}_G + \alpha_{11}x_G + \alpha_{12}\theta_G &= \alpha_{11}e\cos\psi + \alpha_{12}\tau\cos(\psi + \beta) \\
 m\ddot{y}_G + c_R\dot{y}_G + \alpha_{11}y_G + \alpha_{12}\theta_G &= \alpha_{11}e\sin\psi + \alpha_{12}\tau\sin(\psi + \beta) \\
 I_d\ddot{\theta}_G + I_p\ddot{\psi} + I_p\dot{\psi}\dot{\theta}_G + c_\theta\dot{\theta}_G + \alpha_{21}x_G + \alpha_{22}\theta_G &= \alpha_{21}e\cos\psi + \alpha_{22}\tau\cos(\psi + \beta) \\
 I_d\ddot{\theta}_G - I_p\ddot{\psi} + I_p\dot{\psi}\dot{\theta}_G + c_\theta\dot{\theta}_G + \alpha_{21}y_G + \alpha_{22}\theta_G &= \alpha_{21}e\sin\psi + \alpha_{22}\tau\sin(\psi + \beta) \\
 \dot{\psi} &= \alpha
 \end{aligned} \right\} \dots\dots\dots (1)$$

$$[a_{ij}] = \frac{36EI}{a^3} \frac{1}{3+4\lambda} \begin{bmatrix} 1 + \frac{\lambda}{3} & -\frac{a}{6}(3+2\lambda) \\ -\frac{a}{6}(3+2\lambda) & \frac{a^2}{3}(1+\lambda) \end{bmatrix} \dots\dots\dots (2)$$

$$\begin{aligned}
 \lambda &= l/a \\
 I_p &= mD^2/8 \\
 I_d &= m(L^2/3 + D^2/4)/4 \\
 EI &: \text{Flexural rigidity}
 \end{aligned}$$

3.2 Natural angular frequencies and critical speeds

Fig.1 Overhung rotor

In order to obtain natural angular frequencies and solutions of a stationary vibration of the system we assume $\psi=0$, $\dot{\psi}=\omega$, $\psi=\omega t$ in Eq.(1). Using notations Z , Θ and Eq.(3)

$$\mu = I_p/I_d, \quad \kappa_{11} = \alpha_{11}/m, \quad \kappa_{12} = \alpha_{12}/m, \quad \kappa_{21} = \alpha_{21}/I_d, \quad \kappa_{22} = \alpha_{22}/I_d, \quad 2n_R = c_R/m, \quad 2n_\theta = c_\theta/I_d \dots\dots\dots (3)$$

Eq.(1) may be written as

$$\left. \begin{aligned}
 \ddot{Z} + 2n_R\dot{Z} + \kappa_{11}Z + \kappa_{12}\Theta &= \kappa_{11}e e^{i\omega t} + \kappa_{12}\tau e^{i(\omega t + \beta)} \\
 \ddot{\Theta} - i\mu\omega\dot{\Theta} + 2n_\theta\dot{\Theta} + \kappa_{21}Z + \kappa_{22}\Theta &= \kappa_{21}e e^{i\omega t} + \kappa_{22}\tau e^{i(\omega t + \beta)}
 \end{aligned} \right\} \dots\dots\dots (4)$$

Neglecting damping force and supposing solutions of the homogeneous equation of Eq.(4) as $Z = Z_0 e^{st}$ and $\Theta = \Theta_0 e^{st}$, then the characteristic equation becomes

$$(s^2 + \kappa_{11})(s^2 + \kappa_{22}) - \kappa_{12}\kappa_{21} = i\mu\omega s(s^2 + \kappa_{11}) \dots\dots\dots (5)$$

Natural angular frequencies ω_f may be calculated from the following equation, which is derived by substituting the relationship $s = i\omega_f$ into Eq.(5).

$$\omega_f^4 - \mu\omega\omega_f^3 - (\kappa_{11} + \kappa_{22})\omega_f^2 + \mu\kappa_{11}\omega\omega_f + \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} = 0 \dots\dots\dots (6)$$

Putting $\omega_f = \omega_n$, Eq.(6) becomes

$$(1-\mu)\omega_n^4 - [(1-\mu)\kappa_{11} + \kappa_{22}]\omega_n^2 + \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} = 0 \quad (7)$$

Solving Eq.(7), the rotor critical speeds may be obtained as

$$\begin{aligned} \omega_{n1}^2 &= [(1-\mu)\kappa_{11} + \kappa_{22} - \sqrt{[(1-\mu)\kappa_{11} + \kappa_{22}]^2 - 4(1-\mu)(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})}] / [2(1-\mu)] \\ \omega_{n2}^2 &= [(1-\mu)\kappa_{11} + \kappa_{22} + \sqrt{[(1-\mu)\kappa_{11} + \kappa_{22}]^2 - 4(1-\mu)(\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21})}] / [2(1-\mu)] \end{aligned} \quad (8)$$

Considering the relationships $\kappa_{11}, \kappa_{22} > 0$, $\kappa_{12}, \kappa_{21} < 0$, $\kappa_{11}\kappa_{22} > \kappa_{12}\kappa_{21}$, we can easily understand that there exist two critical speeds when $1-\mu > 0$, and a single critical speed when $1-\mu < 0$. As the following discussions are concerned with the first critical speed, we use the value of ω_{n1} in Eq.(8) as the critical speed ω_n .

3.3 Solutions of stationary vibration

After substituting $Z = Z_F e^{i\omega t}$, $\Theta = \Theta_F e^{i\omega t}$ into Eq.(4) and dividing both sides of the equation by ω_n^2 , we have the following equations.

$$\begin{bmatrix} -\left(\frac{\omega}{\omega_n}\right)^2 + 2iD_R\left(\frac{\omega}{\omega_n}\right) + k_{11} & k_{12} \\ k_{21} & -(1-\mu)\left(\frac{\omega}{\omega_n}\right)^2 + 2iD_\theta\left(\frac{\omega}{\omega_n}\right) + k_{22} \end{bmatrix} \begin{Bmatrix} Z_F \\ \Theta_F \end{Bmatrix} = \begin{Bmatrix} k_{11}e + k_{12}\tau e^{i\beta} \\ k_{21}e + k_{22}\tau e^{i\beta} \end{Bmatrix} \quad (9)$$

$$k_{11} = \kappa_{11}/\omega_n^2, \quad D_R = n_R/\omega_n, \quad D_\theta = n_\theta/\omega_n \quad (10)$$

Putting $Z_F = r_G e^{-i\epsilon_r}$, $\Theta_F = \theta_G e^{-i\epsilon_\theta}$, the solutions of the stationary vibration of Eq.(1) can be written as

$$Z = \bar{r}_G e^{i(\omega t - \epsilon_r)}, \quad \Theta = \bar{\theta}_G e^{i(\omega t - \epsilon_\theta)} \quad (11)$$

The maximum amplitudes of the stationary vibration may be obtained approximately by putting $\omega/\omega_n = 1$ in Eq.(9). In case of $\tau = D_\theta = 0$, these may be represented as

$$(\bar{r}_G)_{\max} \approx e/(2D_R), \quad (\bar{\theta}_G)_{\max} \approx \frac{e}{2D_R} \sqrt{1 + (2D_R)^2} \left| \frac{k_{21}}{k_{22} - (1-\mu)} \right|$$

3.4 Solutions of nonstationary vibration

After rewriting Eq.(1) by introducing new variables $t' (= \omega_n t)$ and Eq.(12)

$$x_G = r_G \cos \varphi_r, \quad y_G = r_G \sin \varphi_r, \quad \theta_G = \theta_G \cos(\varphi_\theta + \beta), \quad \varphi_{G\tau} = \theta_G \sin(\varphi_\theta + \beta) \quad (12)$$

we may finally obtain the following equations by using Eqs.(3) and (10)

$$\left. \begin{aligned} r_G'' - r_G \varphi_r'^2 + 2D_R r_G' + k_{11} r_G + k_{12} \theta_G \cos(\varphi_\theta - \varphi_r + \beta) &= k_{11} e \cos(\psi - \varphi_r) + k_{12} \tau \cos(\psi - \varphi_r + \beta) \\ r_G \varphi_r'' + 2r_G' \varphi_r' + 2D_R r_G \varphi_r' + k_{12} \theta_G \sin(\varphi_\theta - \varphi_r + \beta) &= k_{11} e \sin(\psi - \varphi_r) + k_{12} \tau \sin(\psi - \varphi_r + \beta) \\ \theta_G'' - \theta_G \varphi_\theta'^2 + \mu \psi' \theta_G \varphi_\theta' + 2D_\theta \theta_G' + k_{21} r_G \cos(\varphi_r - \varphi_\theta - \beta) + k_{22} \theta_G &= k_{21} e \cos(\psi - \varphi_\theta - \beta) + k_{22} \tau \cos(\psi - \varphi_\theta) \\ \theta_G \varphi_\theta'' + 2\theta_G' \varphi_\theta' - \mu \psi'' \theta_G - \mu \psi' \theta_G' + 2D_\theta \theta_G \varphi_\theta' + k_{21} r_G \sin(\varphi_r - \varphi_\theta - \beta) &= k_{21} e \sin(\psi - \varphi_\theta - \beta) + k_{22} \tau \sin(\psi - \varphi_\theta) \\ \psi'' &= \alpha/\omega_n^2 = 1/Q^2 \end{aligned} \right\} \quad (13)$$

where the notations ' and '' denote the first and the second derivatives with respect to t' . The solutions of the nonstationary vibration can be obtained by integrating Eq.(13) numerically. Here the Runge-Kutta method is used. The initial conditions of Eq.(13) have the following relationship with the solutions of the stationary vibration represented in Eq.(11).

$$\left. \begin{aligned} r_G &= \bar{r}_G, \quad \theta_G = \bar{\theta}_G, \quad \varphi_r = -\epsilon_r, \quad \varphi_\theta + \beta = -\epsilon_\theta, \quad \psi = 0, \quad r_G' = 0, \quad \theta_G' = 0 \\ \varphi_r' &= \kappa, \quad \varphi_\theta' = \kappa, \quad \psi' = \kappa, \quad r_G'' = 0, \quad \theta_G'' = 0, \quad \varphi_r'' = 0, \quad \varphi_\theta'' = 0, \quad \psi'' = 0 \end{aligned} \right\} \quad (14)$$

4. Calculated results

In order to study the nonstationary vibration occurring when a rotor passes through a critical speed, it is necessary to understand the characteristics of the critical speed and the stationary vibration of the system. These characteristics of a gyroscopic system will be revealed by calculated results. In the calculations, some of the system parameters are set constant as $l+a=400\text{mm}$, $EI=2.096 \times 10^8 \text{Nmm}^2$ and $m=2\text{kg}$, and the effects of the gyroscopic moment will be discussed by changing two parameters a and L/D ratio.

4.1 Critical speed

Solid lines in Fig.2 represent the

Fig.2 Variation of critical speed against L/D ratio

variations of the first critical speed ω_n against L/D ratio for various overhung lengths a , which are calculated from Eq.(8). Dotted lines in the same figure show the results in case of no gyroscopic effect, which are obtained by putting $\mu=0$ in Eq.(8). The figure shows that the first critical speed of the gyroscopic system becomes high as L/D ratio and a becomes small. In the following discussions, we consider four cases where L/D ratio is equal to 0.02, 0.07, 0.866 and 4.0 and a is always equal to 80 mm.

4.2 First natural angular frequency of forward precession

It is well known that natural angular frequencies of the gyroscopic system ω_f vary with the speed of rotation ω . The characteristic curves of the first natural angular frequencies of forward precession (ω_f/ω_n) against ω/ω_n are shown in Fig.3 for various values of L/D ratio. It should be noted here that ω_n is the first critical speed calculated from Eq.(8) and is not same for each value of L/D . This figure shows that the nondimensional natural angular frequency of forward precession increases proportionally with the nondimensional speed of rotation near the first critical speed and the inclination of the curve becomes steep as L/D ratio decreases.

Fig.3 Variation of the first natural angular frequency of forward precession against L/D ratio ($a=80\text{mm}$)

4.3 Stationary responses

There are many parameters which affect the vibrational characteristics. We further limit ourselves to consideration of the cases where the rotor has a static unbalance only (then $e \neq 0$ and $\tau = \beta = 0$) and the damping force due to D_θ may be neglected (then $D_R \neq 0$ and $D_\theta = 0$). Figure 4(a) represents stationary responses of r_G near the first critical speed for four values of L/D ratio which are calculated from Eq.(11) under the conditions of $D_R=0.01$ and $e=1$. This figure shows that the Q factors (which indicate the sharpness of resonance) are different for each L/D ratio even though the D_R values are same, and that all the maximum amplitudes are same even though the Q factors are different. These two characteristics were observed in the responses of θ_G . Comparing these calculated results with unbalance responses of a single-degree-of-freedom system, it may be deduced that if L/D ratio is changed, the modal damping ratio $D_M (=1/2Q)$ varies even when D_R is unchanged and that if L/D ratio is changed, the modal eccentricity $e_M (=2D_M \times \text{maximum amplitude})$ varies even when e does not change.

On the basis of the Q factors obtained from Fig.4(a), (i) D_M values when D_R is equal to 0.01 and (ii) e_M values when e is equal to unity are given in the (i) and (ii) columns of Table 1, for four cases of L/D ratio. Furthermore in the (iii) and (iv) columns of Table 1, D_R values when $D_M=0.01$ and e values when $e_M=1$ are shown, which are calculated by using the values in (i) and (ii). The values in (iii) and (iv) will be used when nonstationary responses are calculated in order to make it easy to compare the responses with each other. Table 1 shows also the values of μ , $\omega_n/2\pi$ and k_{ij} for reference.

Fig.4 Unbalance response ($a=80\text{mm}$)

Table 1

L/D		0.02	0.07	0.866	4.0
Q		27.2	35.8	50.0	74.8
(i)	D_M value when $D_R=0.01$	0.0184	0.0140	0.0100	0.00668
(ii)	e_M value when $e=1.0$	1.84	1.40	1.00	0.668
(iii)	D_R value when $D_M=0.01$	0.00544	0.00717	0.0100	0.0150
(iv)	e value when $e_M=1.0$	0.544	0.717	1.00	1.50
$\mu = I_p/I_d$		1.999	1.987	1.000	0.0955
$\omega_n/2\pi$		86.79	66.82	55.76	45.95
k_{11}		3.042	5.131	7.368	10.85
k_{12}		-191.2	-322.5	-463.2	-682.0
k_{21}		-0.04776	-0.1846	-0.7137	-0.2610
k_{22}		3.473	13.43	51.90	18.98

$$a=80\text{mm}, \tau=\beta=0, D_\theta=0$$

Recalculated stationary response curves using the values in (iii) and (iv) in Table 1 are shown in Fig.4(b). Only two curves for $L/D=0.02$ and 4 are shown in the figure and the remaining two curves for $L/D=0.07$ and 0.866, which are not plotted, are situated between the former two curves. This figure shows that even though L/D ratios are different, the stationary response curves near the first critical speed keep the same profiles provided that the values of (iii) and (iv) in Table 1 are used.

4.4 Nonstationary responses

4.4.1 When the gyroscopic system is affected by the damping force

As for the four cases with different effects of gyroscopic moment, whose stationary responses are same as shown in Fig.4(b), nonstationary responses are calculated by the method described in 3.4 under the conditions of $\kappa=0.9$ and $\Omega=40$. Calculated results for r_g are shown in Figs.5(a)~(d). It is commonly known that when stationary responses and acceleration parameters Ω are same in different linear systems with one degree of freedom, nonstationary responses become all the same. The responses in Figs.5(a)~(d), however, are different from each other and it may be thought that this is due to the effect of the gyroscopic moment, that is, the difference of L/D .

Comparing Figs.5(a)~(d), the effects of the gyroscopic moment on the nonstationary vibration may be stated as follows.

- ① The maximum amplitude becomes small as L/D ratio becomes small (as the effect of the gyroscopic moment increases).
- ② The nondimensional angular speed of rotation ψ' where the maximum amplitude occurs becomes high as L/D ratio decreases.
- ③ The speed range from the critical speed to a certain speed where the phase angle ϵ_r stops a rapid increase, that is, the second period continues long as L/D ratio decreases.

Comparing these characteristics with the conclusions in reference(12) we notice some discrepancies. They are due to the difference in the way of comparison and in the treating of damping and eccentricity. One more effect of the gyroscopic moment is that ④ the nondimensional whirling speed ϕ_r' changes in a straight line with a positive slope after passing through the critical speed, and the slope becomes steep and the period of the change becomes long as L/D ratio decreases. This tendency can be seen vaguely in Figs.5(a)~(d), but clearly in Figs.6(a),(b), which show the nonstationary responses in case of no damping. Except ① through ④, the nonstationary characteristics of the gyroscopic system are the same as those of a single-degree-of-freedom system.

Let us consider the change of the maximum amplitude against the acceleration parameter Ω quantitatively. In arranging

Fig.5 Variation of nonstationary response against L/D ratio
($a=80\text{mm}$, $D_M=0.01$, $e_M=1$, $\Omega=40$, $\kappa=0.9$)

the results the maximum amplitude ratio (= the maximum amplitude of the nonstationary response / the maximum amplitude of the stationary response) is used. The ratio for Figs.5(a)~(d) can easily be obtained as 0.553, 0.596, 0.635 and 0.635 respectively by dividing each maximum amplitude by 50.0 in Fig.4(b). Adding the cases of $\Omega=80$ and 25, the results are shown in Table 2. The right side of the table shows the results of a single-degree-of-freedom system calculated from Eq.(15).

(3)

$$\text{Maximum amplitude ratio} = 1 - \exp(-2.51D_M\Omega) \quad (15)$$

It is known from the table that the maximum amplitude ratio decreases as L/D ratio decreases and that it has 15% discrepancy in case of $\Omega=25$, $L/D=0.02$ comparing with that of a single-degree-of-freedom system, even though it has only 3% in case of $L/D=0.866$.

4.4.2 When the gyroscopic system is not affected by the damping force

The nonstationary responses of the gyroscopic system without damping are represented in Figs.6(a) and (b). Table 3 shows the variations of the maximum amplitudes against L/D ratio and Ω . The reason why the maximum amplitude is used here instead of the maximum amplitude ratio is that the maximum amplitude of the stationary vibration becomes infinite in this case. Values in the right side column of Table 3 are calculated from Eq.(16), which are derived by the present author for the nonstationary analysis of a single-degree-of-freedom system.

$$\begin{aligned} \frac{\text{maximum amplitude}}{\text{modal eccentricity}} &= 1.25\Omega \left(1 + \frac{0.1838}{1 + 1.0854/\Omega}\right) \quad (7) \\ &= 1.48\Omega \quad \dots\dots\dots (16) \end{aligned}$$

From Figs.6(a),(b) and Table 3, ①, ② and ④ in the preceding section may hold good for the undamped case. These results lead us to think that the nonstationary characteristics of the gyroscopic system may be understood with ease on the basis of the results of a single-degree-of-freedom system provided that the acceleration rate increases in appearance as the natural angular frequency increases with the rotational speed.

5. Estimation of maximum amplitudes

5.1 Estimation of maximum amplitudes

We assume the acceleration effect in a gyroscopic system will appear more remarkably than in a single-degree-of-freedom system and we derive the imaginary acceleration rate to estimate the maximum amplitudes of the gyroscopic systems.

Here we look back upon the analytical process of the nonstationary vibration of a single-degree-of-freedom system without damping. Equation of motion is

$$\ddot{x} + \omega_n^2 x = q \sin \frac{1}{2} \alpha t^2 \quad \dots\dots\dots (17)$$

Fig.6 Variation of nonstationary response against L/D ratio ($a=80\text{mm}$, $D_M=0$, $e_M=1$, $\Omega=40$, $\kappa=0.9$)

Table 2 Variation of maximum amplitude ratio against Ω and L/D ratio (In case of damping)

Ω	L/D				A linear system with one degree of freedom
	0.02	0.07	0.866	4.0	
80	0.771	0.809	0.841	0.841	0.866
40	0.553	0.596	0.635	0.635	0.634
25	0.394	0.438	0.481	0.479	0.466

$a=80\text{mm}$, $D_M=0.01$, $e_M=1$, $\kappa=0.9$

Table 3 Variation of maximum amplitude against Ω and L/D ratio (In case of no damping)

Ω	L/D				A linear system with one degree of freedom
	0.02	0.07	0.866	4.0	
80	86.01	98.50	112.5	114.3	118.1
40	44.17	49.95	56.51	57.25	58.9
25	26.36	30.82	35.61	35.91	36.8
12.5	13.34	14.91	16.47	16.25	18.3
8.0	9.909	10.75	11.30	10.90	11.6

$a=80\text{mm}$, $D_M=0.0$, $e_M=1$, $\kappa=0.9$

If the rest state is selected as the initial condition, then the solution of Eq.(17) may be written as

$$x = \frac{q}{\omega_n} \int_0^t \sin \omega_n(t-\tau) \sin \frac{1}{2} \alpha \tau^2 d\tau \\ = \frac{q}{2\omega_n} \left[\int_0^t \cos \left\{ \frac{1}{2} \alpha \tau^2 - \omega_n(t-\tau) \right\} d\tau - \int_0^t \cos \left\{ \frac{1}{2} \alpha \tau^2 + \omega_n(t-\tau) \right\} d\tau \right] \quad (18)$$

The above integrations can be represented by Fresnel's integrals. Approximating them by series and asymptotic expansions and putting $q = \omega_n^2 e$ (e :eccentricity), we finally obtain a formula for the maximum amplitude Eq.(16). We are going to apply Eq.(18) to the gyroscopic system.

As the first natural angular frequency of forward precession ω_f varies proportionally with the rotational speed near the first critical speed as shown in Fig.3, Eq.(19) may be sustained.

$$\omega_f = \omega_n + \gamma(\omega - \omega_n) = \gamma\omega + (1-\gamma)\omega_n \quad (19)$$

Where γ denotes the slope of $\omega_f/\omega_n - \omega/\omega_n$ curve in Fig.3. We may put $\dot{\psi} = \omega\alpha t$ (provided that $\omega=0$ at $t=0$) under the constant acceleration condition and substituting this equation into Eq.(19), we have

$$\omega_f = \gamma\alpha t + (1-\gamma)\omega_n \quad (20)$$

In the nonstationary vibration of the gyroscopic system, it may be considered that ω_n in Eq.(18) is replaced by ω_f in Eq.(20) with t changed into τ . Equation (18) may be rewritten as

$$x = \frac{q}{2\omega_f} \left[\int_0^t \cos \left\{ \frac{1}{2} \alpha \tau^2 - (\gamma\alpha\tau + (1-\gamma)\omega_n)(t-\tau) \right\} d\tau - \int_0^t \cos \left\{ \frac{1}{2} \alpha \tau^2 + (\gamma\alpha\tau + (1-\gamma)\omega_n)(t-\tau) \right\} d\tau \right] \quad (21)$$

Terms inside of cosine are $(1/2 + \gamma)\alpha\tau^2 + (1-\gamma)\omega_n\tau - \omega_f t$ and $(1/2 - \gamma)\alpha\tau^2 - (1-\gamma)\omega_n\tau + \omega_f t$. Putting γ except the underlined parts equal to zero and $\omega_f = \omega_n$, Eq.(21) becomes

$$x = \frac{q}{2\omega_n} \left[\int_0^t \cos \left\{ \frac{1}{2} (1+2\gamma)\alpha\tau^2 - \omega_n(t-\tau) \right\} d\tau - \int_0^t \cos \left\{ \frac{1}{2} (1-2\gamma)\alpha\tau^2 + \omega_n(t-\tau) \right\} d\tau \right] \quad (22)$$

Comparing Eq.(22) with Eq.(18), it is understood that the acceleration rate in Eq.(22) becomes $(1+2\gamma)\alpha$ and $(1-2\gamma)\alpha$ instead of α in Eq.(18). From the assumption that the acceleration increases as L/D ratio decreases (γ increases), the imaginary acceleration rate in the gyroscopic system may be considered as $(1+2\gamma)\alpha$. Then the imaginary acceleration parameter Ω^* may be represented as follows considering both $\Omega = \omega_n/\sqrt{\alpha}$ and $\Omega^* = \omega_n/\sqrt{(1+2\gamma)\alpha}$.

$$\Omega^*/\Omega = 1/\sqrt{1+2\gamma} \quad (23)$$

In order to obtain γ , we assume $\omega_f = \omega_n + \Delta$ (Δ : small quantity). Substituting the equation into Eq.(6), we have

$$\Delta = \frac{\mu(\omega_n^2 - \kappa_{11}\omega_n)}{4\omega_n^3 - 2\omega_n(\kappa_{11} + \kappa_{22}) - \mu\omega(3\omega_n^2 - \kappa_{11})} (\omega - \omega_n)$$

Comparing this equation with Eq.(19) and putting $\omega = \omega_n$, then γ represented as

$$\gamma = \frac{\mu(\omega_n^2 - \kappa_{11})}{(4-3\mu)\omega_n^2 - (2-\mu)\kappa_{11} + 2\kappa_{22}} = \frac{\mu(1-\kappa_{11})}{4-3\mu - (2-\mu)\kappa_{11} + 2\kappa_{22}} \quad (24)$$

From the above analysis, the maximum amplitude of the nonstationary vibration of the gyroscopic system r_{Gmax} can be estimated as follows.

- (1) Calculate γ from Eq.(24) using the system dimensions.
- (2) Calculate Ω^* from Eq.(23) using γ and Ω .
- (3) When the damping force is considered, r_{Gmax} can be estimated from Eq.(25), which is an extended form of Eq.(15).
- (4) When the damping force is not considered, r_{Gmax} can be estimated from Eq.(26), which is an extended form of Eq.(16).

$$\text{Maximum amplitude ratio} = \frac{r_{Gmax}}{\text{maximum amplitude of stationary vibration}} = 1 - \exp(-2.51D_M\Omega^*) \quad (25)$$

$$\frac{r_{Gmax}}{\text{modal eccentricity}} = 1.25\Omega^* \left(1 + \frac{0.1838}{1+1.0854/\Omega^*} \right) = 1.48\Omega^* \quad (26)$$

5.2 Comparison between estimated values and numerical integration results

5.2.1 When the damping force is neglected

The maximum amplitudes shown in Table 3, which are obtained by the numerical integrations, are compared with values estimated from Eq.(26) and the results are shown in Table 4. Errors of the estimated values from the calculated values (numerical integration results) are also shown in the same table. It is known from Table 4 that in the range

of $\Omega > 25$, errors are below 6% and the estimation by Eq.(26) has a fairly good accuracy. In case of the fast acceleration $\Omega = 8$, the maximum error exceeds 10%. This may arise from the difference in the initial condition between the two.

5.2.2 When the damping force is considered

The maximum amplitude ratios shown in Table 2, which are obtained by the numerical integrations, are compared with the values estimated from Eq.(25) and the results are shown in Table 5. In the range of Table 5,

errors are below 10%, and the estimation by Eq.(25) has an accuracy enough for practical use. In this case, the maximum amplitude of the nonstationary response can be obtained if the maximum amplitude of the stationary response is given.

6. Conclusions

The considered gyroscopic system is a rotating shaft which has a solid disk with L/D ratio attached at the overhung shaft end. The nonstationary vibration occurring when the system passes through the first critical speed was analyzed under the condition of a constant acceleration α . Conclusions are summarized as follows.

- (1) The maximum amplitude of the nonstationary vibration of the gyroscopic system becomes small and the nondimensional rotational speed where the maximum amplitude occurs becomes high as L/D ratio becomes small whereas the stationary response and the acceleration parameter Ω are unchanged.
- (2) The nondimensional whirling speed changes in about a straight line with a positive slope after passage through the critical speed and the slope becomes steep and the period of the change becomes long as L/D ratio decreases.
- (3) The nonstationary characteristics (1) and (2) can be explained by assuming that the acceleration rate in the gyroscopic system increases in appearance.
- (4) The acceleration rate in appearance (the imaginary acceleration rate) α^* is represented as $\alpha^* = (1+2\gamma)\alpha$, where γ denotes the derivative coefficient of the first natural angular frequency of forward precession with respect to the rotational speed.
- (5) The formulas which make it possible to estimate the maximum amplitude of the nonstationary vibration of the gyroscopic system are derived by using the imaginary acceleration rate and the formulas for the maximum amplitude of the nonstationary response of a single-degree-of-freedom system. The estimated results are compared with the numerical integration results and the maximum errors of the estimation are below 10% in the range of $\Omega > 25$.

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Table 4 Comparison between estimated maximum amplitudes and numerical integration results (In case of no damping)

Ω	L/D	0.02	0.07	0.866	4.0
	γ	0.456	0.284	0.0578	0.0160
80	Ω^*/Ω	0.723	0.799	0.947	0.984
	Calculated value	86.01	98.50	112.5	114.3
40	Estimated value	85.3	94.3	111.9	116.2
	Error %	-0.8	-4.2	-0.5	+1.7
25	Calculated value	44.17	49.95	56.51	57.25
	Estimated value	42.6	47.1	55.8	58.0
12.5	Error %	-3.5	-5.7	-1.3	+1.3
	Calculated value	26.36	30.82	35.61	35.91
8	Estimated value	26.5	29.3	34.8	36.2
	Error %	+0.5	-4.9	-2.3	+0.8
4	Calculated value	13.34	14.91	16.47	16.25
	Estimated value	13.2	14.6	17.3	18.0
2	Error %	-1.1	-2.1	+5.1	+10.7
	Calculated value	9.909	10.75	11.30	10.90
1	Estimated value	8.35	9.25	11.0	11.4
	Error %	-15.7	-13.9	-2.7	+4.6

$$a=80\text{mm}, D_M=0, e_M=1, \kappa=0.9$$

Table 5 Comparison between estimated maximum amplitude ratios and numerical integration results (In case of damping)

Ω	L/D	0.02	0.07	0.866	4.0
	Calculated value	0.771	0.809	0.841	0.841
80	Estimated value	0.766	0.799	0.851	0.861
	Error %	-0.6	-1.2	+1.2	+2.4
40	Calculated value	0.553	0.596	0.635	0.635
	Estimated value	0.516	0.552	0.614	0.628
25	Error %	-6.7	-7.4	-3.3	-1.1
	Calculated value	0.394	0.438	0.481	0.479
12.5	Estimated value	0.365	0.394	0.448	0.461
	Error %	-7.4	-10.0	-6.9	-3.8

$$a=80\text{mm} \quad D_M=0.01 \quad e_M=1 \quad \kappa=0.9$$

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