On Synchronization of Rotating Machines with Alternating Flow Hydraulics*

(Part 1. Synchronizing Rotating Motion Under Limited Power Source)

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This paper is concerned with synchronization of rotating machines connected by alternating flow hydraulics.

Taking account of the interaction between driving and driven machines, synchronization phenomena are analyzed by the averaging method and the synchronizing phase angles are derived.

The stability of the steady-state solution is investigated by means of Routh-Hurwitz criterion.

Moreover, the influence of initial conditions on the steady-state rotating motion is considered on the phase plane and critical domain is discussed.

1. Introduction

Recently, alternating flow hydraulics (called AFH hereafter) as a new field of hydraulics has come to attract attention. Unlike the conventional hydraulic driving method which uses continuous pressure and flow source, AFH uses alternating pressure and flow of medium fluid to transmit power.

To achieve transfer of power by means of hydraulics, three methods are available. They are the rotary, the linear and the vibratory methods. Of these methods, the vibratory one has been the least used because the pressure fluctuations which are associated with it are usually considered to be detrimental to the performance of a system.

Medium fluid in the pipings in AFH does not circulate. Therefore, the system is simplified since it can dispense with the hydraulic pump unit and valves which are required otherwise. This merit enables the usage of AFH in the fields where vibrating motion of medium fluid can be advantageously utilized. The merit above mentioned also makes it possible to place free pistons inside the connecting pipings between the driving and the driven machines. Thus, AFH can isolate two different medium fluids in case where one of these two may be too hot or radioactively contaminated.

References (1) through (4) are the research and developments of AFH in such field. K. Foster⁽¹⁾ analysed viscous power losses in a liquid-filled pipe and P.R. Ukrainetz⁽²⁾ manufactured three phase rectification circuits and reported their efficiency obtained by the experiment.

In Japan, Takenaka et al.⁽³⁾ investigated transmission efficiencies of single

pipings and three phase rectification circuits. W.M.J. Schlösser⁽⁴⁾ reported on the application of AFH. The machine they have used, however, was nothing more than an experimental model and any practical application of AFH to direct power transmission has not yet been known. Therefore, the author planned to apply AFH technology for the driving and driven machines (called the two machines hereafter) by directly utilizing three phase AFH without rectification.

In this research, synchronization phenomena which appear on the two machines as in Fig. 5 in case the power of the driving source is limited have been investigated. These synchronization phenomena are similar to the specific self-synchronization phenomena which appear in mechanical vibrating systems. As examples of the mechanical phenomena T.K. Caughey⁽⁵⁾ has explained Hula-Hoop theory, Inouye^{(6),(7)} has described unbalanced rotors and I.I. Blekhman⁽⁸⁾ has tried application of such phenomena to vibrating machines of various types.

In this paper we discuss the synchronization phenomena appearing in two machines connected with three phase AFH pipings. Both machines have a slider crank mechanism instead of the conventional crank shaft. First, taking account of an interaction between the driving and the driven sides which is caused by the limited power source, synchronization phenomena appearing in rotating motion of the two machines are theoretically analyzed by using the averaging method; the solution and the stability of synchronized rotating motion are discussed and the stable synchronizing phase angle is derived. Secondly, the stability of the system influenced by the differences in phase angles and relative angular velocities at the instant of start is studied on the phase plane and the limitation of synchronizing rotating motion is discussed.

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2. Notations

O_i	: Center of the i -th body
(x_i, y_i)	: X,Y co-ordinate that denotes displacement of point O_i
J_i	: Moment of inertia of the i -th body
l_i	: Length of connecting rod
r_i	: Radius of crank
m_i	: Mass of the i -th body
ϕ_{0j}	: Phase angle of crank shaft at the instant of start
θ, ϕ	: Rotating phase angle of crank shaft at arbitrary time
θ_k, ϕ_k	: Rotating phase angle of the k -th eccentric disk
α, ϵ	: Angle of inclination of connecting rod
T^*	: Kinetic energy
V^*	: Potential energy
F^*	: Dissipating function
$T(\theta)$	: Driving torque
$L(\phi)$	: Load torque
β_i	: Coefficient of viscous friction
A_i	: Sectional area of piping
K_i	: Bulk modulus of medium fluid
k_i	: Equivalent spring constant of medium fluid
h_i	: Distance between the driving and the driven pistons
ω	: Synchronizing angular velocity
i	: Number of bodies
j	: $j=1$ denotes driving side and $j=2$ denotes driven side respectively
k	: Number of eccentric disks

3. Constitution of driving and driven machines

In the three phase AFH, the medium fluid is charged previously in three independent pipings and it makes reciprocating motion in different phases. Fig. 1 illustrates the experimental apparatus of the two machines and Fig. 2 shows its hydraulic circuits. In Fig. 1, the driving machine is shown to the left while the driven machine is to the right and both are connected over three hydraulic pipings. The driving machine consists of the electric motor, the hydraulic pump, the hydraulic motor, the control valves and the three phase AFH generator so that required driving torque and velocity for AFH generator can be arbitrarily set. The driven machine consists of the three phase

Fig. 1 Experimental apparatus

AFH motor and the hydraulic pump whose delivery pressure can be controlled to change the load torque of AFH motor.

Fig. 3 illustrates the construction of three AFH generator and motor. Both machines have exactly the same slider crank mechanism which consists of the crank shaft, the eccentric disks, the connecting rods, the pistons and the cylinders. Three connecting rods are hooked up each with three eccentric disks which are placed 120° apart from each other and three pistons are inserted in three cylinders. In this arrangement, three driving pistons make a pseudo-sinusoidal motion. In this model, offset of eccentricity, length of connecting rod, diameters of piston and eccentric disk and length of piping are 10mm, 80mm, 20mm, 75mm and 1200mm respectively. The two machines consisting of the above three phase AFH construction cause a reciprocating motion of the medium fluid in three pipings which are 120° different in phase angle from each other when the crank shaft of driving side is run by a directly coupled hydraulic motor. The medium fluid, therefore, does not circulate but generates three phase AFH. By this medium fluid motion, three pistons of AFH motor are forced to make a reciprocating motion to rotate the crank shaft of driven side. Synchronization of motion between medium fluid and three pistons of the driven machine may result in synchronizing the rotating motion of the driven shaft to that of the driving shaft.

Fig. 2 Hydraulic circuit

Fig. 3 Three phase AFH generator and motor

Fig. 4 shows one of the examples of experimental results when the two machines make a synchronizing rotating motion. In this figure, the pressure of the medium fluid charged in three independent pipings and rotating pulse signals of the driving and the driven shaft are simultaneously recorded when the driving shaft rotates at 25 rps against the load torque of 2kgf·m. The record shows that the pressure waves of medium fluid charged in three pipings make alternating standing waves 120° different in phase angle from each other and the two shafts are synchronously rotating with a certain phase angle difference. By this three phase AFH, the driven shaft is found to make a synchronizing rotating motion to the driving shaft without any external control.

4. Equation of Motion

Fig. 5 represents a model of the two machines where the driving machine is shown to the left and the driven machine to the right. The co-ordinates used in this paragraph is also described in the figure. In this apparatus, three connecting rods are hooked up each with eccentric disks which are placed 120° apart from each other. Fig. 5 shows only one set of slider crank mechanism.

Let θ_i and ϕ_i which are the rotating phase angles of the eccentric disk in driving and driven sides be

$$\theta_i = \theta + \frac{2}{3}\pi(i-1), \quad \phi_i = \phi + \frac{2}{3}\pi(i-1) \quad (i=1,2,3) \quad (1)$$

$$2T^* = \sum_{i=1}^6 \sum_{j=1}^3 m_{ij} (\dot{x}_{ij}^2 + \dot{y}_{ij}^2) + \sum_{i=1}^3 (J_{1j} \dot{\theta}_i^2 + J_{2i} \dot{\phi}_i^2 + J_{5i} \dot{\epsilon}_i^2 + J_{6i} \dot{\phi}_i^2) + J_0 \dot{\theta}^2 + J_7 \dot{\phi}^2 \quad (2)$$

$$2V^* = \sum_{i=1}^3 2(m_{1i} g y_{1i} + m_{2i} g y_{2i} + m_{5i} g y_{5i} + m_{6i} g y_{6i}) + \sum_{i=1}^3 k_i \{x_i(0) - x_i(t)\}^2 \quad (3)$$

$$2F^* = \beta_1 \dot{\theta}^2 + \beta_6 \dot{\phi}^2 + \sum_{i=1}^3 \beta_{3i} (\dot{x}_{3i}^2 + \dot{y}_{3i}^2) + \sum_{i=1}^3 \beta_{4i} (\dot{x}_{4i}^2 + \dot{y}_{4i}^2) \quad (4)$$

Let x_i - y_i co-ordinates be transformed to polar coordinates

$$\begin{aligned} (x_{1j}, y_{1j}) &= (r_1 \cos \theta_i, r_1 \sin \theta_i), \quad (x_{2i}, y_{2i}) = (r_1 \cos \theta_i + b \cos \alpha_i, a \sin \alpha_i) \\ (x_{3i}, y_{3i}) &= (r_1 \cos \theta_i + l_1 \cos \alpha_i + e, 0), \quad (x_{4i}, y_{4i}) = (r_2 \cos \phi_i - l_2 \cos \epsilon_i - f, 0) \\ (x_{5i}, y_{5i}) &= (r_2 \cos \phi_i - d \cos \epsilon_i, c \sin \epsilon_i), \quad (x_{6i}, y_{6i}) = (r_2 \cos \phi_i, r_2 \sin \phi_i) \end{aligned} \quad (5)$$

The slider crank mechanisms of the two machines are considered to be alike, i.e.:

$$\begin{aligned} a=c, \quad b=d, \quad e=f, \quad J_0=J_7, \quad r_1=r_2=r, \quad l_1=l_2=l, \quad J_{1i}=J_{6i}=J_1, \quad J_{2i}=J_{5i}=J_2 \\ \beta_{1i}=\beta_{6i}=\beta_1, \quad \beta_{3i}=\beta_{4i}=\beta_3, \quad m_{1i}=m_{6i}=m_1, \quad m_{2i}=m_{5i}=m_2, \quad m_{3i}=m_{4i}=m_3, \quad (i=1,2,3) \end{aligned}$$

The relationship of Eq. (7) is substituted into Eqs. (2)~(6),

$$\begin{aligned} r_1 \sin \theta_i &= l_1 \sin \alpha_i, \quad r_2 \sin \phi_i = l_2 \sin \epsilon_i, \quad x_i(0) = x_{4i}(0) - x_{3i}(0), \\ x_i(t) &= x_{4i}(t) - x_{3i}(t) \end{aligned} \quad (6)$$

where $x_i(t)$ is relative displacement of x_3 and x_4 .

Applying Lagrange's equation, we obtain the equations of motion:

$$\ddot{\theta} (3J + m_3 r^2 \sum_{i=1}^3 \sin^2 \theta_i) + \frac{\dot{\theta}^2}{2} m_3 r^2 \sum_{i=1}^3 \sin 2\theta_i + \ddot{\phi} (3\beta_1 + \beta_3 r^2 \sum_{i=1}^3 \sin^2 \theta_i)$$

Fig. 4 Synchronizing phenomena of two machines obtained from experiments

Fig. 5 Theoretical model of driving and driven machines

where i shows the number of each eccentric disk. Equivalent spring constant corresponding to compressibility of medium fluid charged in i -th piping may be written as follows:

$$k_i = K_{i-1} \frac{A_i}{h_i} \quad (2)$$

The kinetic energy, the potential energy and the dissipation function of system can be written as:

$$+Mg r \sum_{i=1}^3 \cos \theta_i + r^2 \sum_{i=1}^3 k_i \sin \theta_i (\cos \phi_i - \cos \theta_i) = T(\dot{\theta}) \dots\dots\dots (8)$$

$$\ddot{\phi} (3J + m_3 r^2 \sum_{i=1}^3 \sin^2 \phi_i) + \frac{\dot{\phi}^2}{2} m_3 r^2 \sum_{i=1}^3 \sin 2\phi_i + \dot{\phi} (3\beta_1 + \beta_3 r^2 \sum_{i=1}^3 \sin^2 \phi_i) + Mg r \sum_{i=1}^3 \cos \phi_i - r^2 \sum_{i=1}^3 k_i \sin \phi_i (\cos \phi_i - \cos \theta_i) = -L(\dot{\phi}) \dots\dots\dots (9)$$

where $J = J_0 + J_1 + (m_1 + m_2) r^2$, $M = m_1 + a/\ell$.

5. Analysis of Motion

5.1 Solution of motion

Necessary conditions for the two shafts to make a synchronizing rotating motion are formulated in the case that the driving side is remarkably influenced by the driven side, i.e. in the case of practical limited power. The synchronizing rotating motion herein does not include an oscillating motion but is considered to be a constant rotating motion. Let the phase angles be

$$\theta_i = \tau + \phi_{01} + \theta_1 + \frac{2}{3}\pi(i-1), \quad \phi_i = \tau + \phi_{02} + \theta_2 + \frac{2}{3}\pi(i-1), \quad \tau = \omega t \quad (i=1,2,3) \dots\dots\dots (10)$$

where ϕ_{01} and ϕ_{02} show the initial phase angles of respective crank shafts and θ_1 and θ_2 are phase angles generated by the influence of such motion, which are designated as "Generating Phase Angle". The Generating Phase Angles, θ_1 and θ_2 , are assumed to be slowly varying functions of time. Equation (10) is substituted into Eqs. (8) and (9) and then the averaging method developed by Bogoliubov and Mitropoliski is applied. Since it is assumed that θ_1 and θ_2 are slowly varying, Eqs. (8) and (9) may be averaged over one cycle as follows:

$$I \omega^2 \frac{d^2 \theta_1}{d\tau^2} + c \omega \frac{d\theta_1}{d\tau} + c \omega + \frac{r^2}{2} \sum_{i=1}^3 k_i \sin(\phi_{01} + \theta_1 - \phi_{02} - \theta_2) = T(\omega, \frac{d\theta_1}{d\tau}) \dots\dots\dots (11)$$

$$I \omega^2 \frac{d^2 \theta_2}{d\tau^2} + c \omega \frac{d\theta_2}{d\tau} + c \omega - \frac{r^2}{2} \sum_{i=1}^3 k_i \sin(\phi_{01} + \theta_1 - \phi_{02} - \theta_2) = -L(\omega, \frac{d\theta_2}{d\tau}) \dots\dots\dots (12)$$

where $c = 3(\beta_1 + \beta_3 r^2/2)$, $I = 3(J + m_3 r^2/2)$

The conditions for existence of a stationary motion are

$$\theta_i = \theta_{i0}, \quad \omega = \omega_\ell, \quad \frac{d^2 \theta_i}{d\tau^2} = \frac{d\theta_i}{d\tau} = 0 \quad (i=1,2) \dots\dots (13)$$

Hence Eqs. (11) and (12) become

$$c \omega_\ell + \frac{r^2}{2} \sum_{i=1}^3 k_i \sin(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) = T(\omega_\ell) \dots\dots (14)$$

$$c \omega_\ell - \frac{r^2}{2} \sum_{i=1}^3 k_i \sin(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) = -L(\omega_\ell) \dots\dots (15)$$

the zero subscript denoting the steady-state. For stationary conditions of motion, $\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}$ shows the offset angle between the rotations of the two machines, i.e. the synchronizing phase angle δ . Equations (14) and (15) give the synchronizing phase angle and the synchronizing angular velocity between the rotations of the two machines. Real solutions of Eqs (14) and (15) exist if, and only if,

$$\frac{r^2}{2} \sum_{i=1}^3 k_i \geq L(\omega_\ell) + c \omega_\ell \dots\dots\dots (16)$$

Equation (16) has a simple physical meaning: it is that the fluid torque $r^2 \sum_{i=1}^3 k_i/2$ by three phase AFH charged in the piping must be larger than summation of the load torque $L(\omega_\ell)$ and the viscous torque $c \omega_\ell$. With this restriction, synchronizing angular velocity is obtained as follows:

$$T(\omega_\ell) - L(\omega_\ell) - 2c \omega_\ell = 0 \dots\dots\dots (17).$$

Stationary solution of synchronizing phase angle is also obtained as:

$$\delta_1 = 2n\pi + \gamma, \quad \delta_2 = (2n+1)\pi - \gamma \quad (n: \text{integer}) \dots\dots (18)$$

where

$$\gamma = \sin^{-1} \left\{ \frac{L(\omega_\ell) + c \omega_\ell}{\frac{r^2}{2} \sum_{i=1}^3 k_i} \right\} \dots\dots\dots (19)$$

Hence the phase relationship between the two machines is determined by the driving torque and the load torque. On the other hand, in the case of

$$\frac{r^2}{2} \sum_{i=1}^3 k_i < L(\omega_\ell) + c \omega_\ell \dots\dots\dots (20)$$

stationary solutions of the synchronizing phase angle does not exist, so that synchronizing phenomena do not occur.

The discussion above is for the limited power source. Secondly, the unlimited power source in which the driving side is not at all influenced by the driven side is discussed. In this case θ_i and ϕ_i which are the rotating phase angles of the eccentric disks are assumed as:

$$\theta_i = \tau + \phi_{01} + \frac{2}{3}\pi(i-1), \quad \phi_i = \tau + \phi_{02} + \frac{2}{3}\pi(i-1), \quad \tau = \omega t, \quad \omega = \omega_u \dots\dots\dots (21)$$

Therefore, the synchronizing condition can be written.

$$c \omega_u - \frac{r^2}{2} \sum_{i=1}^3 k_i \sin(\phi_{01} - \phi_{02} - \theta_0) = L(\omega_u) \dots\dots (22)$$

The synchronizing angular velocity ω_u is different from the unlimited power and depends on the driving torque.

5.2 Stability of rotating motions

To study the stability of the sta-

$$I\omega \frac{d^2\eta}{d\tau^2} + c\omega \frac{d\eta}{d\tau} + \frac{r^2}{2} \sum_{i=1}^3 k_i \sin(\phi_{01} + \theta_{10} + \eta - \phi_{02} - \theta_{20} - \xi) = T(\omega, \frac{d\eta}{d\tau}) \quad (24)$$

$$I\omega \frac{d^2\xi}{d\tau^2} + c\omega \frac{d\xi}{d\tau} + \frac{r^2}{2} \sum_{i=1}^3 k_i \sin(\phi_{01} + \theta_{10} + \eta - \phi_{02} - \theta_{20} - \xi) = -L(\omega, \frac{d\xi}{d\tau}) \quad (25)$$

Neglecting the smaller terms of 2nd order about η and ξ , using relationships Eqs. (14) and (15), Eqs. (24) and (25) become

$$I\omega \frac{d^2\eta}{d\tau^2} + (c - T')\omega \frac{d\eta}{d\tau} + \frac{r^2}{2} \sum_{i=1}^3 k_i (\eta - \xi) \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) = 0 \quad (26)$$

$$I\omega \frac{d^2\xi}{d\tau^2} + (c + L')\omega \frac{d\xi}{d\tau} + \frac{r^2}{2} \sum_{i=1}^3 k_i (\eta - \xi) \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) = 0 \quad (27)$$

where

$$T' = \left. \frac{\partial T(\dot{\theta})}{\partial \dot{\theta}} \right|_{\dot{\theta}=\omega_u}, \quad L' = \left. \frac{\partial L(\dot{\phi})}{\partial \dot{\phi}} \right|_{\dot{\phi}=\omega_u}$$

Let the solutions be

$$\eta = Ae^{\lambda\tau}, \quad \xi = Be^{\lambda\tau}$$

Examining the corresponding characteristic equation

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \quad (28)$$

where

$$b_1 = \frac{c - T'}{I\omega}, \quad b_2 = \frac{(c - T')(c + L') + Ir^2 \sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20})}{I^2\omega^2}$$

$$b_3 = \frac{(2c - T' + L')r^2 \sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20})}{2I^2\omega^3}$$

Applying the Rough-Hurwitz criterion to Eq. (28), the stability condition is obtained as follows:

$$(i) \quad \frac{2c - L' + T'}{I\omega} > 0$$

$$(ii) \quad \frac{(c - T')(c + L') + Ir^2 \sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20})}{I^2\omega^2} > 0$$

$$(iii) \quad \frac{(2c - T' + L')r^2 \sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20})}{2I^2\omega^3} > 0$$

$$(iv) \quad \frac{(2c - T' + L')}{I^2\omega^3} \left\{ \frac{(c - T')(c + L')}{I} + \frac{r^2}{2} \sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) \right\} > 0$$

$$(v) \quad \frac{(2c - T' + L')r^2}{2I^4\omega^6} \left\{ \frac{(c - T')(c + L')}{I} + \frac{r^2}{2} \sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) \right\} \sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) > 0$$

Considering $T' < 0$ and $L' > 0$, the stability condition becomes finally:

$$\sum_{i=1}^3 k_i \cos(\phi_{01} + \theta_{10} - \phi_{02} - \theta_{20}) > 0 \quad (29)$$

We examine the stable region of the synchronizing phase angles as in Fig. 6. If $\delta_1 = 2n\pi + \gamma$, Eqs. (26) and (27) define a stable system; in other phase relationship, i.e. $\delta_2 = (2n+1)\pi - \gamma$, Eqs. (26) and (27) define an unstable system. From above result, δ_1 is the stable equilibrium point and δ_2 is the unstable equilibrium point. Therefore, the synchronizing phase angle for the two shafts to make a stable rotating motion has only one solution over

tionary motions, let η and ξ be small perturbations of θ_{10} and θ_{20} , i.e.:

$$\theta_1 = \theta_{10} + \eta, \quad \theta_2 = \theta_{20} + \xi \quad (23)$$

Substituting Eq. (23) into Eqs. (11) and (12), produces

one cycle. The phase relationship between the two crank shafts is shown in Fig. 7, in which (a) is a stable system and (b) is an unstable system. Herein, a stable

Fig. 6 Stationary solutions and stability

system is one for the two shafts to make a constant rotating motion and an unstable system is one to make other motions than the above motion: i.e. oscillating motion, rotation with oscillation, etc.

Let us study the influence of the fluid torque and the load torque on the phase angle. In the case of a constant fluid torque, the relation between the load torque and the phase angle is as shown in Fig. 8. From Fig. 8 when the load torque is equal to zero, the phase angle has a minimum value

Fig. 7 Relationship of synchronizing phase angles

Fig. 8 Effect of load torque

Fig. 9 Effect of hydraulic torque

$2n\pi + \sin^{-1}\{C\omega_L / (r^2 \sum_{i=1}^3 k_i / 2)\}$. If the load torque is increased, the phase angle increases also and if the load torque is equal to $r^2 \sum_{i=1}^3 k_i / 2 - C\omega_L$, the phase angle takes a maximum value $(2n+1/2)\pi$.

In the case of a constant load torque, the relationship between the fluid

torque and the phase angle is as shown in Fig. 9. From Fig. 9, if the fluid torque is equal to $L + C\omega_L$, the phase angle takes the maximum value $(2n+1/2)\pi$. Moreover, if the pressure of medium fluid charged in the piping increases, the phase angle approaches asymptotically $2n\pi$. From the above result, it is found that a stationary synchronizing rotating motion of the two shafts is obtained in connection with the driving torque, the load torque, the fluid torque and the phase angular velocity and that the phase angle must be within $2n\pi$ to $(2n+1/2)\pi$ under the restriction of Eq. (16).

6. Trajectory on phase plane

Let us make a more thorough interpretation by using the phase plane and consider the forms of the trajectory of synchronizing phase angle close to the equilibrium point. Eigenvector of system can be written as:

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \phi_{01} + \theta_1 - \phi_{02} - \theta_2 \\ \dot{\theta}_1 - \dot{\theta}_2 \end{bmatrix} \quad (30)$$

where Z is the eigenvector, z_1 and z_2 are the state variables which represent the synchronizing phase angle and the relative angular velocity respectively. From Eqs. (12) and (13), the state differential equation can be written as:

$$\dot{Z} = F(Z) \quad (31)$$

where F is a function vector, so that

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ T(\dot{\theta}) + L(\phi) - Cz_2 - r \sum_{i=1}^3 k_i \sin z_1 \\ I \end{bmatrix} \quad (32)$$

Therefore, from Eq. (32) two equilibrium states are obtained.

$$Z_1^0 = \begin{bmatrix} 2n\pi + \gamma \\ 0 \end{bmatrix}, \quad Z_2^0 = \begin{bmatrix} (2n+1)\pi - \gamma \\ 0 \end{bmatrix}$$

where Z_1^0 and Z_2^0 are the reference vectors under two equilibrium states. To study characteristics of equilibrium point, let δZ be a small perturbation of Z_0 , i.e.:

$$Z = Z_0 + \delta Z \quad (33)$$

If $F(Z)$ is expanded in a Taylor series about the reference trajectory, we obtain:

$$\delta \dot{Z} = A \delta Z \quad (34)$$

where A is Jacobian matrix, which can be written as:

$$A = \begin{bmatrix} 0 & 1 \\ r^2 \sum_{i=1}^3 k_i \cos z_1^0 & -C \\ -\frac{1}{I} & -\frac{C}{I} \end{bmatrix} \quad (35)$$

The expression for A under each equilibrium state thus becomes:

$$A_1 = \begin{bmatrix} 0 & 1 \\ r^2 \sum_{i=1}^3 k_i \cos(2n\pi + \gamma) & -C \\ -\frac{1}{I} & -\frac{C}{I} \end{bmatrix} \quad (36)$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -\frac{r^2 \sum_{i=1}^3 k_i \cos((2n+1)\pi - \gamma)}{I} & -\frac{C}{I} \end{bmatrix} \dots (37)$$

From Eq. (34) a characteristic equation is now derived in the form:

$$|\lambda I - A| = \lambda^2 + a\lambda + b = 0 \dots (38)$$

where

$$a = \frac{C}{I}, \quad b = \frac{r^2 \sum_{i=1}^3 k_i \cos z_1^0}{I}$$

The form of the trajectory of synchronizing phase angle close to the equilibrium point is determined by eigenvalue of Eq. (38). Classification of the forms of the equilibrium points is shown in Table 1.

To study the influence of initial conditions for the steady-state motion, it is necessary to eliminate the explicit dependence on t from Eq. (32). Therefore, Eq. (32) may now be written as follows:

$$\frac{dz_2}{dz_1} = \frac{T(\dot{\theta}) + L(\dot{\phi}) - Cz_2 - r^2 \sum_{i=1}^3 k_i \sin z_1}{IZ_2} \dots (39)$$

Equation (39) might not be integrated straightforward, therefore trajectory on phase plane would be investigated by analog computer. One of the solutions of Eq. (39) is shown in Fig. 10. In the figure the abscissa is a phase angle z_1 and the ordinate is a relative angular velocity z_2 . There are an infinite number of points that satisfy the stationary solutions, but only two satisfy the solution for one cycle of z_1 . The first, denoted by point A, of them is a stable focal point, and the second, denoted by point B, is an unstable saddle point.

Table 1 Classification of points of equilibrium

Point of equilibrium	Response of Z	Kind of equilibrium
$Z = \begin{bmatrix} 2n\pi + \gamma \\ 0 \end{bmatrix}$	Oscillatory convergence	Stable focal point
$Z = \begin{bmatrix} (2n+1)\pi - \gamma \\ 0 \end{bmatrix}$	Monotonous divergence	Unstable saddle point

Fig. 10 Trajectory on phase angle

These correspond to the point of stable equilibrium and the point of unstable equilibrium described in paragraph 5.2. Notice that as the load torque L is increased under a constant driving torque T , the focus and saddle approach $\pi/2$, and if L is further increased, there are no real equilibrium points.

Thus, the equilibrium points of a nonlinear system are remarkably influenced by the load torque. The trajectory passes through the paths c-d, b-B-e, b-B-A, a-A and g-f as shown in Fig. 10. The path b-B-e shows the stable limit, along which the trajectory starts at the point b, approaches the saddle point B, and comes close to the next saddle point (2π radian apart from B) passing the point e. If the initial conditions lie within the path b-B-e, the path approaches the focal point A, oscillatingly damped. At the focal point A, the two machines make a stable synchronizing rotating motion. If the initial conditions lie above the path b-B-e, the trajectory approaches asymptotically a fixed dynamic path b-B-e or c-d, and the two machines do not make a synchronizing rotating motion. Herein, the path c-d is a limit cycle of the second kind.

7. Conclusion

The results with the rotating machines connected by three phase AFH are summarized below.

(1) Synchronizing rotating motion of the driving and the driven machines ("two machines") connected over three phase AFH pipings is revealed in the experiment.

(2) Taking account of interaction between "two machines" which is caused by the limited power source, synchronization phenomena appearing in the rotating motion of "two machines" are theoretically analyzed by using the averaging method, and the stable synchronizing phase angle is derived.

(3) The phase angle of synchronizing rotating motion of "two machines" is within $2n\pi$ to $(2n+1/2)\pi$.

(4) The influence of initial conditions on the stationary rotating motion is analyzed on the phase plane, critical domain is discussed, and the dynamic behavior of "two machines" is analyzed.

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