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Dynamic Deformations and Stresses in a Circular
Cylindrical Shell with Both Ends Clamped Subjected
to Translational Excitations at the Base*

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In this paper, an exact analysis for the dynamic stresses and deformations in a circular cylindrical shell with both ends clamped subjected to impulsive motions at the clamped end is presented on the basis of Flügge's dynamic shell theory and with use of Laplace transformation.

The end motions are prescribed by three kinds of loading functions whose velocity histories in time are half sine, rectangular and exponentially decaying pulses, respectively. In the results, it is shown that the dynamic deformations and stresses are much affected mainly by the velocity histories of the end motions and by the relation between the period of the lowest natural frequency of the cylindrical shell and that of the end motions.

1. Introduction

During earthquakes, structures are subjected to dynamic loads caused by the inertia forces of themselves. As the acceleration of earthquake ground motions increases, the magnitudes of the dynamic loads increase, and therefore, those of the dynamic stresses in the structures increase more and more.

In the problems of earthquake engineering, structures are replaced usually by a spring-mass system and the dynamic displacements due to the ground motions are analyzed. On the basis of this analyzed dynamic displacements, the stresses in the structures are statically evaluated. However, there is some doubt that the dynamic stresses in the structures by this approximate method are sufficiently accurate. Therefore, an exact dynamic analysis for the stresses in the structures is necessary. For this purpose, the structures can't be replaced by a spring-mass system but must be treated as a continuous elastic system.

The problems of circular cylindrical shell structures, such as storage tanks or pipe lines, subjected to translational excitations at the base are treated by many investigators in connection with the study of earthquake engineering. Concerning storage tanks, liquid motions in the tanks have been frequently investigated by many authors mainly under the assumption that the walls of the tanks are rigid[1]-[3]. On the other hand, Miles analyzed coupled motions of a cylindrical tank with an incompressible fluid having a free surface subjected to translation, rotation and simple bending and showed the effect of the free surface of fluid on the bending

frequencies of the tank. But, unfortunately, it is assumed that the deflection curves of the tank are approximated by the elementary beam theory[4]. In the analyses of the dynamic interactions between liquid motions and elastic deformations of walls of cylindrical containers, such as in Miles' paper, some approximations have been introduced for the deformations of the containers[5]-[7].

Exact analysis for the dynamic stresses in cylindrical shell structures subjected to translational excitations at the base is not yet made with use of the dynamic shell theory.

In this paper, an exact analysis for dynamic deformations and stresses in a circular cylindrical shell with both ends clamped subjected to impulsive motions at the clamped end is presented on the basis of Flügge's dynamic shell theory and with use of Laplace transformation. The end motions are prescribed by three kinds of loading functions, i.e., cosine-like, ramp-like and exponential functions, whose velocity histories in time are half sine, rectangular and exponentially decaying pulses, respectively. The purposes of this paper are to propose a method for analyzing the dynamic responses of circular cylindrical shells subjected to translational excitations at the base and to demonstrate the effects of the velocity histories of the end motions and the relations between the periods of the natural frequencies of the shell and those of the end motions on the maximum values of the dynamic displacements and stresses in the shell.

2. Formulation of the Problem and Solution

Consider a circular cylindrical shell with both ends clamped subjected to impulsive motions at the clamped end as shown in Fig.1 (b). The end motions are prescribed by the loading function $\delta_0 f(t)$, then the boundary conditions can be written in the following way.

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(i) At $x=0$

$$\left. \begin{aligned} u &= 0, \quad v = \delta_0 \sin \theta f(t), \quad w = \delta_0 \cos \theta f(t) \\ \partial w / \partial x &= 0 \end{aligned} \right\} \dots \dots \dots (1)$$

(ii) At $x=L$

$$u = v = w = \partial w / \partial x = 0 \dots \dots \dots (2)$$

The solutions for the displacement components may be given in the form of separation of variables.

$$u = U \cos \theta, \quad v = V \sin \theta, \quad w = W \cos \theta \dots \dots (3)$$

where (U, V, W) are functions of the axial coordinate x and time t . The shell is initially at rest and in unstrained state, so that the initial conditions are given by

$$(f)_{t=0} = (\partial f / \partial t)_{t=0} = 0$$

Thus, the Laplace transformation of the uncoupled differential equation for the radial displacement, which is deduced from the three coupled differential equations including the effects of the axial, circumferential and radial inertia forces derived by Flügge, is given by the equation [8][9]

$$\frac{d^8 \bar{W}}{d\xi^8} + a_6 \frac{d^6 \bar{W}}{d\xi^6} + a_4 \frac{d^4 \bar{W}}{d\xi^4} + a_2 \frac{d^2 \bar{W}}{d\xi^2} + a_0 \bar{W} = 0 \dots \dots \dots (4)$$

$$\begin{aligned} \text{where } a_6 &= -4 + 2\nu - \frac{3-\nu}{1-\nu} \lambda^2, \quad a_4 = J^2 + \left(\frac{1}{\kappa^2} + 3 \frac{3-\nu}{1-\nu} \right) \lambda^2 + \frac{2}{1-\nu} \lambda^4 \\ a_2 &= -\left(\frac{5+2\nu}{\kappa^2} + 3 \frac{3-\nu}{1-\nu} \right) \lambda^2 - \left(\frac{1}{\kappa^2} \frac{3-\nu}{1-\nu} + \frac{4}{1-\nu} \right) \lambda^4 \\ a_0 &= \left(\frac{2}{\kappa^2} + \frac{3-\nu}{1-\nu} \right) \lambda^2 + \left(\frac{1}{\kappa^2} \frac{5-\nu}{1-\nu} + \frac{2}{1-\nu} \right) \lambda^4 + \frac{1}{\kappa^2} \frac{2}{1-\nu} \lambda^6, \quad \bar{f} = \int_0^\infty f e^{-pt} dt \\ \xi &= x/R, \quad \lambda = R\rho/c, \quad \kappa^2 = h^2/12R^2, \quad J^2 = (1-\nu^2)/\kappa^2, \quad c = \sqrt{gE/\gamma(1-\nu^2)} \end{aligned}$$

The general solutions of the Laplace transformations of the functions (U, V, W) can be assumed as

$$\bar{U} = \sum_{j=1}^8 f_j c_j e^{\alpha_j \xi}, \quad \bar{V} = \sum_{j=1}^8 g_j c_j e^{\alpha_j \xi}, \quad \bar{W} = \sum_{j=1}^8 c_j e^{\alpha_j \xi} \dots \dots \dots (5)$$

where α_j ($j=1, \dots, 8$) are the roots of the characteristic equation of eighth degree derived from Eq.(4) and the coefficients (f_j, g_j) are derived from the differential equations relating the three displacement components. With use of the Laplace transformations of the boundary conditions, i.e., Eqs.(1) and (2), a set of eighth linear algebraic equations in the eight undetermined coefficients c_j ($j=1, \dots, 8$) are obtained as follows :

$$\begin{pmatrix} a_{11} & \dots & a_{18} \\ a_{21} & \dots & a_{28} \\ a_{31} & \dots & a_{38} \\ a_{41} & \dots & a_{48} \\ \vdots & & \vdots \\ a_{81} & \dots & a_{88} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_8 \end{pmatrix} = \delta_0 \bar{f}(p) \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \dots \dots (6)$$

where the elements a_{ij} are defined in the following way and the symbol $[A]$ stands for a matrix consisting of these elements.

$$a_{1j} = f_j, \quad a_{2j} = g_j, \quad a_{3j} = 1, \quad a_{4j} = \alpha_j, \quad a_{5j} = f_j e^{\alpha_j L}, \quad a_{6j} = g_j e^{\alpha_j L}, \quad a_{7j} = e^{\alpha_j L}, \quad a_{8j} = \alpha_j e^{\alpha_j L}, \quad L = l/R$$

Solving Eq. (6), the solutions of the Laplace transformations of the functions (U, V, W) are easily obtained by the equations

(a) Definitions of Stress Resultants in a Circular Cylindrical Shell and Coordinate System

(b) A Circular Cylindrical Shell with Both Ends Clamped Subjected to Translational Excitations at the Base

Fig.1 Illustration of the Problem

$$\left. \begin{aligned} \frac{\bar{U}}{\delta_0} &= \frac{\bar{f}(p)}{|A|} \sum_{j=1}^8 f_j (|A_{2j}| + |A_{3j}|) e^{\alpha_j \xi}, \quad \frac{\bar{V}}{\delta_0} = \frac{\bar{f}(p)}{|A|} \sum_{j=1}^8 g_j (|A_{2j}| + |A_{3j}|) e^{\alpha_j \xi} \\ \frac{\bar{W}}{\delta_0} &= \frac{\bar{f}(p)}{|A|} \sum_{j=1}^8 (|A_{2j}| + |A_{3j}|) e^{\alpha_j \xi} \end{aligned} \right\} \dots\dots\dots (7)$$

where $|A_{ij}|$ denotes the cofactor of the element a_{ij} in the determinant $|A|$. The Laplace inverse transformations of Eqs.(7) are determined as the sum of all the residues of simple poles existing on the imaginary axis. Thus, putting the poles as $\pm i p_m$ ($0 < p_1 < p_2 < p_3 < \dots$), the Laplace inverse transformation of the radial displacement $\bar{W}$ is given by the equation

$$\frac{W}{\delta_0} = 2 \sum_{m=1}^{\infty} \frac{\sum_{j=1}^8 (|A_{2j}| + |A_{3j}|) e^{\alpha_j \xi}}{\lambda_m \sum_{j=1}^8 |B_j|} \int_0^{\tau} f(s) \sin \lambda_m (\tau - s) ds \dots\dots\dots (8)$$

where $\tau = ct/R$, $\lambda_m = R p_m / c$ and the elements b_{ik} in the determinant $|B_j|$ are defined in the following way.

(I) If $k \neq j$

$$b_{ik} = a_{ik}$$

(II) If $k = j$

$$b_{1j} = r_j \beta_j, \quad b_{2j} = s_j \beta_j, \quad b_{3j} = 0, \quad b_{4j} = \beta_j, \quad b_{5j} = (r_j + L f_j) \beta_j e^{\alpha_j L}$$

$$b_{6j} = (s_j + L g_j) \beta_j e^{\alpha_j L}, \quad b_{7j} = L \beta_j e^{\alpha_j L}, \quad b_{8j} = (1 + L \alpha_j) \beta_j e^{\alpha_j L}$$

where $f_j = \frac{\alpha_j}{d_j} \left\{ -\kappa^2 \alpha_j^4 + \left(v - 3\kappa^4 \frac{1-v}{2} - \frac{2\kappa^2}{1-v} \lambda_m^2 \right) \alpha_j^{2+1+\kappa^2} + \left(\frac{2v}{1-v} - \kappa^2 \right) \lambda_m^2 \right\}$

$$g_j = \frac{1}{d_j} \left\{ 2\kappa^2 \alpha_j^4 - \left(2 + v + 2\kappa^2 - \frac{3-v}{1-v} \kappa^2 \lambda_m^2 \right) \alpha_j^{2+1-} - \frac{2}{1-v} \lambda_m^2 \right\}$$

$$r_j = \frac{1}{d_j} \left\{ -5\kappa^2 \alpha_j^4 + 3 \left(v - 3\kappa^4 \frac{1-v}{2} - \frac{2\kappa^2}{1-v} \lambda_m^2 \right) \alpha_j^{2+1+\kappa^2} + \left(\frac{2v}{1-v} - \kappa^2 \right) \lambda_m^2 + \frac{4\kappa^2}{1-v} \frac{\alpha_j^3}{\beta_j} - 2 \left(\frac{2v}{1-v} - \kappa^2 \right) \frac{\alpha_j}{\beta_j} \right\}$$

$$- 2 \frac{f_j}{d_j} \left\{ 2\alpha_j^3 - \left(2 - \frac{3-v}{1-v} \lambda_m^2 \right) \alpha_j - \frac{3-v}{1-v} \frac{\alpha_j^2}{\beta_j} + \frac{3-v}{1-v} \frac{1}{\beta_j} - \frac{4}{1-v} \frac{\lambda_m^2}{\beta_j} \right\}$$

$$s_j = \frac{2}{d_j} \left\{ 4\kappa^2 \alpha_j^3 - \left(2 + v + 2\kappa^2 - \frac{3-v}{1-v} \kappa^2 \lambda_m^2 \right) \alpha_j - \kappa^2 \frac{3-v}{1-v} \frac{\alpha_j^2}{\beta_j} + \frac{2}{1-v} \frac{1}{\beta_j} \right\}$$

$$- 2 \frac{g_j}{d_j} \left\{ 2\alpha_j^3 - \left(2 - \frac{3-v}{1-v} \lambda_m^2 \right) \alpha_j - \frac{3-v}{1-v} \frac{\alpha_j^2}{\beta_j} + \frac{3-v}{1-v} \frac{1}{\beta_j} - \frac{4}{1-v} \frac{\lambda_m^2}{\beta_j} \right\}$$

$$\beta_j = \left[\frac{3-v}{1-v} \alpha_j^6 - \left(\frac{1}{\kappa^2} + 3 \frac{3-v}{1-v} - \frac{4}{1-v} \lambda_m^2 \right) \alpha_j^4 + \left\{ \frac{5+2v}{\kappa^2} + 3 \frac{3-v}{1-v} - 2 \left(\frac{1}{\kappa^2} \frac{3-v}{1-v} + \frac{4}{1-v} \right) \lambda_m^2 \right\} \alpha_j^2 \right. \\ \left. - \left\{ \frac{2}{\kappa^2} + \frac{3-v}{1-v} - 2 \left(\frac{1}{\kappa^2} \frac{5-v}{1-v} + \frac{2}{1-v} \right) \lambda_m^2 + \frac{1}{\kappa^2} \frac{6}{1-v} \lambda_m^4 \right\} \right] / \alpha_j (4\alpha_j^6 + 3a_6 \alpha_j^4 + 2a_4 \alpha_j^2 + a_2)$$

$$d_j = \alpha_j^4 - \left(2 - \frac{3-v}{1-v} \lambda_m^2 \right) \alpha_j^{2+1-} - \frac{3-v}{1-v} \lambda_m^2 + \frac{2}{1-v} \lambda_m^4$$

The values of the poles p_m are determined from the roots of the transcendental equation $|A|=0$ and they coincide with the natural frequencies of a finite circular cylindrical shell with both ends clamped. The roots α_j ($j=1, \dots, 8$) satisfy the following characteristic equation of eighth degree obtained by equating λ to $i\lambda_m$.

$$\alpha^8 + a_6 \alpha^6 + a_4 \alpha^4 + a_2 \alpha^2 + a_0 = 0 \dots\dots\dots (9)$$

The solutions for the other displacement components (U,V) are easily obtained by the same procedure and the solutions for the stress resultants are easily obtained from the representations of the differentiated formulas of the displacement components. The end motions are prescribed by three kinds of loading functions, i.e., the cosine-like, ramp-like and exponential functions as shown in Fig.2, whose velocity histories in time are half sine, rectangular and

exponentially decaying pulses, respectively. Therefore, the loading functions $f(t)$ and the integral including Eq.(8) are written in the following equations.

(a) *Cosine-like Function in Fig.2(a) Having Half Sine Pulse as Velocity History*

$$f(\tau) = \begin{cases} \frac{1}{2} \left(1 - \cos \frac{\pi}{\tau_0} \tau \right) & (0 < \tau \leq \tau_0) \\ 1 & (\tau_0 < \tau) \end{cases} \dots\dots\dots (10)$$

(i) If $\pi / \tau_0 \neq \lambda_m$

$$F(\tau) = \begin{cases} \frac{f(\tau)}{\lambda_m} - \frac{1}{2\lambda_m} \frac{\pi^2 / \tau_0^2}{\pi^2 / \tau_0^2 - \lambda_m^2} \left(\cos \lambda_m \tau - \cos \frac{\pi}{\tau_0} \tau \right) & (0 < \tau \leq \tau_0) \\ \frac{f(\tau)}{\lambda_m} - \frac{1}{2\lambda_m} \frac{\pi^2 / \tau_0^2}{\pi^2 / \tau_0^2 - \lambda_m^2} \left\{ \cos \lambda_m \tau + \cos \lambda_m (\tau_0 - \tau) \right\} & (\tau_0 < \tau) \end{cases} \dots\dots\dots (11)$$

(ii) If $\pi / \tau_0 = \lambda_m$

$$F(\tau) = \begin{cases} \frac{f(\tau)}{\lambda_m} - \frac{\tau}{4} \sin \lambda_m \tau & (0 < \tau \leq \tau_0) \\ \frac{f(\tau)}{\lambda_m} - \frac{\tau_0}{4} \sin \lambda_m \tau & (\tau_0 < \tau) \end{cases} \dots\dots\dots (12)$$

(a) *Cosine-like Function Having Half Sine Pulse as Velocity History*

(b) *Ramp-like Function in Fig.2(b) Having Rectangular Pulse as Velocity History*

$$f(\tau) = \begin{cases} \tau / \tau_0 & (0 < \tau \leq \tau_0) \\ 1 & (\tau_0 < \tau) \end{cases} \dots\dots\dots (13)$$

$$F(\tau) = \begin{cases} \frac{f(\tau)}{\lambda_m} - \frac{1}{\tau_0 \lambda_m^2} \sin \lambda_m \tau & (0 < \tau \leq \tau_0) \\ \frac{f(\tau)}{\lambda_m} - \frac{1}{\tau_0 \lambda_m^2} \{ \sin \lambda_m \tau - \sin \lambda_m (\tau - \tau_0) \} & (\tau_0 < \tau) \end{cases} \dots\dots\dots (14)$$

(b) *Ramp-like Function Having Rectangular Pulse as Velocity History*

(c) *Exponential Function in Fig.2(c) Having Exponentially Decaying Pulse as Velocity History*

$$f(\tau) = 1 - e^{-\Omega \tau}, \quad \Omega = R\omega / c \dots\dots\dots (15)$$

$$F(\tau) = \frac{f(\tau)}{\lambda_m} - \frac{1}{\lambda_m} \frac{\Omega^2}{\Omega^2 + \lambda_m^2} \times \left(\cos \lambda_m \tau - e^{-\Omega \tau} + \frac{\lambda_m}{\Omega} \sin \lambda_m \tau \right) \dots\dots\dots (16)$$

(c) *Exponential Function Having Exponentially Decaying Pulse as Velocity History*

where
$$F(\tau) = \int_0^\tau f(s) \sin \lambda_m (\tau - s) ds$$

Fig.2 Loading Functions and Their Differentiations with Respect to Time

The solutions for the bending moment M_x , the membrane force N_x and the effective radial and circumferential shearing forces (T_x, S_x) are represented by the equations

$$\left. \begin{aligned} \frac{R^2 M_x}{D \delta_0 \cos \theta} &= -2 \sum_{m=1}^{\infty} \frac{F(\tau)}{\lambda_m \sum_{j=1}^8 |B_j|} \sum_{j=1}^8 (|A_{2j}| + |A_{3j}|) (\alpha_j f_j + \nu g_j - \nu + \alpha_j^2) e^{\alpha_j \xi} \\ \frac{R^3 N_x}{D \delta_0 \cos \theta} &= \frac{2}{\kappa^2} \sum_{m=1}^{\infty} \frac{F(\tau)}{\lambda_m \sum_{j=1}^8 |B_j|} \sum_{j=1}^8 (|A_{2j}| + |A_{3j}|) (\alpha_j f_j + \nu g_j - \nu + \kappa^2 \alpha_j^2) e^{\alpha_j \xi} \\ \frac{R^3 T_x}{D \delta_0 \cos \theta} &= -2 \sum_{m=1}^{\infty} \frac{F(\tau)}{\lambda_m \sum_{j=1}^8 |B_j|} \sum_{j=1}^8 (|A_{2j}| + |A_{3j}|) \left\{ \alpha_j^3 - (2 - \nu) \alpha_j + \left(\alpha_j^2 + \frac{1 - \nu}{2} \right) f_j + \frac{3 - \nu}{2} \alpha_j g_j \right\} e^{\alpha_j \xi} \\ \frac{R^3 S_x}{D \delta_0 \sin \theta} &= -\frac{1 - \nu}{\kappa^2} \sum_{m=1}^{\infty} \frac{F(\tau)}{\lambda_m \sum_{j=1}^8 |B_j|} \sum_{j=1}^8 (|A_{2j}| + |A_{3j}|) \{ f_j - (1 + 3\kappa^2) \alpha_j g_j + 3\kappa^2 \alpha_j \} e^{\alpha_j \xi} \end{aligned} \right\} \dots\dots\dots (17)$$

3. Solution for the Static Problem

Substituting the first terms of Eqs.(11), (12), (14) or (16) into Eq.(8), the infinite series coincide with the quasi-static solution, i.e., the product of the solution for the corresponding static problem and the loading function $f(\tau)$. However, this infinite series converges so slowly that this static solution must be replaced by a closed form solution. Therefore, another analysis for the static problem is made.

In the static problem, the displacement components (U,V,W) in the form of separation of variables, i.e., Eqs.(3), are functions of the axial coordinate x only. Then, the general solutions of the functions (U,V,W) are given by the equations

$$\left. \begin{aligned} U &= \sum_{j=1}^4 f_j c_j e^{\alpha_j \xi} + 3c_5 \xi^2 + 2c_6 \xi + 6(2+\nu)c_5 + c_7 \\ V &= \sum_{j=1}^4 g_j c_j e^{\alpha_j \xi} + c_5 \xi^3 + c_6 \xi^2 + (c_7 - 6\nu c_5)\xi + c_8 - 2\nu c_6 \\ W &= \sum_{j=1}^4 c_j e^{\alpha_j \xi} + c_5 \xi^3 + c_6 \xi^2 + c_7 \xi + c_8 \end{aligned} \right\} \dots\dots\dots (18)$$

where $f_j = \frac{-\kappa^2 \alpha_j^5 + \nu \alpha_j^3 + \alpha_j}{(\alpha_j^2 - 1)^2}$, $g_j = \frac{2\kappa^2 \alpha_j^4 - (2+\nu)\alpha_j^2 + 1}{(\alpha_j^2 - 1)^2}$

and α_j ($j=1, \dots, 4$) are the roots of the following characteristic equation of fourth degree.

$$\alpha^4 - 2(2 - \nu)\alpha^2 + J^2 = 0 \dots\dots\dots (19)$$

With use of the boundary conditions, i.e., Eqs.(1) and (2), equating the loading function $f(t)$ to unity, a set of eighth linear algebraic equations in the eight undetermined coefficients c_j ($j=1, \dots, 8$) are obtained as follows :

$$\left. \begin{aligned} \sum_{j=1}^4 f_j c_j + 6(2+\nu)c_5 + c_7 &= 0, \quad \sum_{j=1}^4 g_j c_j - 2\nu c_6 + c_8 = \delta_0, \quad \sum_{j=1}^4 c_j + c_8 = \delta_0 \\ \sum_{j=1}^4 \alpha_j c_j + c_7 &= 0, \quad \sum_{j=1}^4 f_j e^{\alpha_j L} c_j + 3(L^2 + 4 + 2\nu)c_5 + 2Lc_6 + c_7 = 0 \\ \sum_{j=1}^4 g_j e^{\alpha_j L} c_j + (L^3 - 6\nu L)c_5 + (L^2 - 2\nu)c_6 + Lc_7 + c_8 &= 0 \\ \sum_{j=1}^4 e^{\alpha_j L} c_j + L^3 c_5 + L^2 c_6 + Lc_7 + c_8 &= 0, \quad \sum_{j=1}^4 \alpha_j e^{\alpha_j L} c_j + 3L^2 c_5 + 2Lc_6 + c_7 = 0 \end{aligned} \right\} \dots\dots\dots (20)$$

Solving Eqs.(20), the solutions of the displacement components (U,V,W) in the static problem are determined. The solutions for the bending moment M_x , the membrane force N_x and the effective radial and circumferential shearing forces (T_x, S_x) are represented by the equations

$$\left. \begin{aligned} \frac{R^2 M_x}{D \cos \theta} &= - \sum_{j=1}^4 \frac{(1-\kappa^2)\alpha_j^4 - 2(1-\nu\kappa^2)\alpha_j^2 + 2-\nu^2}{(\alpha_j^2 - 1)^2} \alpha_j^2 e^{\alpha_j \xi} c_j - 2(2-\nu^2)(3\xi c_5 + c_6) \\ \frac{R^3 N_x}{D \cos \theta} &= - \sum_{j=1}^4 \frac{2(1-\nu)\alpha_j^2 - J^2 - 1}{(\alpha_j^2 - 1)^2} \alpha_j^2 e^{\alpha_j \xi} c_j + (1+J^2)(6\xi c_5 + 2c_6) \\ \frac{R^3 T_x}{D \cos \theta} &= \sum_{j=1}^4 \frac{\kappa^2 \alpha_j^4 - \kappa^2(5-\nu)\alpha_j^2/2 + J^2 - 3 + 2\nu}{(\alpha_j^2 - 1)^2} \alpha_j^3 e^{\alpha_j \xi} c_j - 6(3-2\nu)c_5 \\ \frac{R^3 S_x}{D \sin \theta} &= \sum_{j=1}^4 \frac{3(1-\nu)\kappa^2 \alpha_j^2 - J^2 - 3\nu(1-\nu)/2}{(\alpha_j^2 - 1)^2} \alpha_j^3 e^{\alpha_j \xi} c_j - 3\{2J^2 + 3\nu(1-\nu)\}c_5 \end{aligned} \right\} \dots\dots\dots (21)$$

where $D = Eh^3/12(1-\nu^2)$, $T_x = Q_x - \partial M_x / \partial \theta$, $S_x = N_x \theta + M_x / R$

Fig. 3 Comparison of Deflection Curves ($H=0.02$)

Fig. 4 Static Distributions of the Membrane Forces and the Bending Moments Along the Generator ($H=0.02$, $\theta=0$)

Fig. 5 Static Distributions of the Effective Shear Forces Along the Generator ($H=0.02$, $T_x: \theta=0$, $S_x: \theta=\pi/2$)

4. Numerical Results

In numerical calculations, $\nu=0.3$ (Poisson's ratio) is used and the dimensionless parameters $H(=h/R)$ and $L(=l/R)$ are introduced.

4.1 Results for the Static Problem

First, static distributions of the displacement components and the stress resultants are calculated on the basis of Eqs. (18) and (21). The relation between the deflection (radial displacement) curve and the length of the shell is shown in Fig. 3. At the situation $x/l=0.5$, all the deflection curves indicate the value 0.5 and are symmetric about the point $x/l=w/\delta_0=0.5$. If the shell is long, the curve coincides nearly with that from the elementary beam theory neglecting the shear deformation. In Figs. 4 and 5, the distributions of the membrane forces, the bending moments and the effective shear forces are shown. The distributions of the stress resultants are also symmetric about the point $x/l=w/\delta_0=0.5$. The distributions of the membrane forces N_x are represented by straight lines and they coincide with those of the bending stresses given from the elementary beam theory. The bending moment M_x and the radial shear force T_x appear in the vicinity of both clamped ends and vanish as the distance from the ends increases.

Contrarily, the absolute values of the circumferential shear force S_x decrease as the ends are approached. The resultant force toward the translational direction of the base is calculated in the following way

$$F_x = \int_0^{2\pi} (T_x \cos\theta + S_x \sin\theta) R d\theta$$

and then, obtained by the equation

$$F_x / \pi R = (T_x)_{\theta=0} + (S_x)_{\theta=\pi/2}$$

Therefore, the sum of the values of T_x and S_x indicates a constant value independent of the axial coordinate.

4.2 Results for the Dynamic Problem

The roots of the transcendental equation $|A|=0$, i.e., the natural frequencies λ_m , are numerically found out with use of the trial and error procedure and Newton's method.

In Table 1, the lowest ten frequencies are listed to show the effect of the shape of the shell. The natural frequencies $(\lambda_m)_B$ given from the Bernoulli-Euler beam equation of motion are calculated by the equation [10]

$$(\lambda_m)_B = (\delta_m/L)^2 \sqrt{(1-\nu^2)/2},$$

where $\delta_1=4.730$, $\delta_2=7.853$, $\delta_3=10.996$

For the sake of comparison, the lowest three frequencies given from the beam theory are also listed in the parentheses of Table 1.

As the shell becomes shorter, the fre-

quencies by the beam theory become higher than those by the shell theory and this inclination becomes more prominent in higher modes. However, this difference would decrease if the Timoshenko beam equation including the shear deformation and the rotatory inertia effect were employed instead of the Bernoulli-Euler equation.

Displacement and stress variations in time are numerically calculated with use of those frequencies. The relation between the radial displacement and the membrane force variations in time and the type of the loading function is shown in Figs. 6 and 7.

Thereby the initial velocity of the end motion is equated between the ramp-like and exponential loading functions, i.e., $1/t_0 = \omega$, and the rise time t_0 is equated between the ramp-like and cosine-like loading functions.

The dimensionless times $\tau_1 (= 2\pi/\lambda_1)$ and $\tau_0 (= ct_0/R)$ mean the period of the lowest natural frequency of the shell and the rise time of the end motion, respectively, and the chain lines denote the results of the corresponding static problem. The maximum values increase in the case of the exponential, ramp-like and cosine-like functions in that order.

Namely the time history of the translational velocity of the end motion affects notably the dynamic maximum responses of the shell. In Figs. 8 and 9, the radial displacement and the membrane force variations in time are shown at three different points in the case of the cosine-like loading function.

There is a time lag of the half period at the time when the dynamic maximum values of the membrane force at both clamped ends yield. In Figs. 10, 11 and 12, the bending moment and the resultant force toward the translational direction of the base variations in time are shown and the effect of the rise time of the end motion is examined in the case of the cosine-like loading function.

As the rise time τ_0 becomes shorter than the period τ_1 , the dynamic maximum values increase notably over the static results.

By modifying Eqs. (11) and (14) for the time region $\tau_0 < \tau$ in the case of the cosine-like and ramp-like loading functions and Eq.

(16) ignoring the term $e^{-\Omega\tau}$ in the case of the exponential loading function, the upper bounds of the dynamic maximum values in the time variations are derived from the equations:

Fig. 6 Radial Displacement Variations in Time ($H=0.02$, $L=8$, $\xi=4$, $\tau_0=40$, $\tau_1=46.4$; Chain Line Denotes Static Result)

Fig. 7 Membrane Force Variations in Time ($H=0.02$, $L=\xi=16$, $\tau_0=40$, $\tau_1=131$; Chain Line Denotes Static Result)

Table 1 Comparison of Natural Frequencies

m	H = 0.02			H = 0.01
	L = 4	L = 8	L = 16	L = 8
1	0.306 287 (0.943 21)	0.135 319 (0.235 80)	0.048 061 4 (0.058 95)	0.135 175 (0.235 80)
2	0.573 871 (2.599 90)	0.277 350 (0.649 98)	0.111 535 (0.162 49)	0.277 168 (0.649 98)
3	0.761 393 (5.097 48)	0.434 041 (1.274 37)	0.187 309 (0.318 59)	0.433 805 (1.274 37)
4	0.846 958	0.573 479	0.268 324	0.573 165
5	0.893 017	0.683 629	0.350 652	0.683 079
6	0.902 502	0.692 536	0.430 651	0.692 244
7	0.927 132	0.760 675	0.505 583	0.759 774
8	0.942 049	0.812 421	0.573 235	0.810 951
9	0.965 489	0.848 011	0.619 833	0.845 891
10	0.987 888	0.871 582	0.632 647	0.868 263

(Values in the Parentheses: Elementary Beam Theory)

Fig. 8 Radial Displacement Variations in Time (Cosine-Like Function, $H=0.02$, $L=8$, $\tau_0=20$, $\tau_1=46.4$; Chain Lines Denote Static Results)

(a) Cosine-like Loading Function

$$\left(\frac{W}{\delta_0}\right)_{\max} = 2 \sum_{m=1}^{\infty} |W_m| \left\{ 1 + \frac{\pi^2 |\cos(\lambda_m \tau_0/2)|}{|\pi^2 - \lambda_m^2 \tau_0^2|} \right\} \dots (22)$$

(b) Ramp-like Loading Function

$$\left(\frac{W}{\delta_0}\right)_{\max} = 2 \sum_{m=1}^{\infty} |W_m| \left(1 + \frac{2}{\tau_0 \lambda_m} \left| \sin \frac{\lambda_m \tau_0}{2} \right| \right) \dots (23)$$

(c) Exponential Loading Function

$$\left(\frac{W}{\delta_0}\right)_{\max} = 2 \sum_{m=1}^{\infty} |W_m| (1 + \lambda_m \Omega \sqrt{\Omega^2 + \lambda_m^2}) \dots (24)$$

where

$$W_m = \frac{\sum_{j=1}^8 (|A_{2j}| + |A_{3j}|) e^{\alpha_j \xi}}{\lambda_m^2 \sum_{j=1}^8 |B_j|}$$

The equations for the other displacement components and the stress resultants corresponding to Eqs. (22), (23) and (24) are similarly deduced and numerical calculations for the upper bounds are made. The effect of the type of the loading function on the dynamic maximum values of the membrane force is shown in Fig. 13, where the solid, broken and chain lines are the upper bounds given by the equations corresponding to Eqs. (22), (23) and (24), respectively. The circles and triangles represent the maximum values from among the initial five peaks of the time variations and ()_{st} means the results of the corresponding static problem. The upper bounds increase in the case of the exponential, ramp-like and cosine-like loading func-

Fig. 9 Membrane Force Variations in Time (Cosine-Like Function, Chain Lines Denote Static Results; $H=0.02$, $L=8$, $\tau_0=20$, $\tau_1=46.4$)

Fig. 10 Bending Moment Variations in Time (Cosine-Like Function, Chain Line Denotes Static Result; $H=0.02$, $L=4$, $\xi=0$, $\tau_1=20.5$)

Fig. 11 Bending Moment Variations in Time (Cosine-Like Function, Chain Line Denotes Static Result; $H=0.02$, $L=8$, $\xi=0$, $\tau_1=46.4$)

Fig. 12 Shear Force Variations in Time (Cosine-Like Function, Chain Line Denotes Static Result; $H=0.02$, $L=8$, $\xi=0$, $\tau_1=46.4$)

Fig. 13 Dynamic Maximum Values of the Membrane Force ($H=0.02$, $L=\xi=8$, $\tau_1=46.4$)

tions in that order. In Figs.14 and 15, the effect of the length of the shell on the dynamic maximum values of the bending moment and the radial displacement is shown in the case of cosine-like loading function.

Normalizing the results with use of the corresponding static results and the period of the lowest natural frequency of the shell, it is clearly shown that the dynamic maximum values increase notably over the static results, as the rise time becomes shorter than the period. Numerical calculations for the infinite series with respect to m are carried out to an accuracy of three figures by the sum of 20 terms for the displacement components in the case of the cosine-like loading function and by the sum of 250 terms for the shear forces in the case of the ramp-like loading function.

5. Conclusions

The following conclusions are made as a result of this study.

- (1) The maximum values of the dynamic deformations and stresses in the cylindrical shell are much affected by the velocity histories of the end motions.
- (2) As the rise time of the end motions becomes shorter than the period of the lowest natural frequency of the cylindrical shell, the maximum values of the dynamic deformations and stresses increase more and more over those of the static results.
- (3) Applying the method of solution shown in this paper, the dynamic responses of circular cylindrical shells subjected to translational excitations may be exactly analyzed under arbitrary boundary conditions.

References

- [1] Jacobsen, L. S., Bull. Seism. Soc. Amer., Vol.39, No.3 (1949-7), p.189.
- [2] Epstein, H. I., Proc. Amer. Soc. Civ. Engr., Vol.102, No.ST9 (1976-9), p.1659.
- [3] Sogabe, K., et al., Jour. Japan Soc. Mech. Engrs. (in Japanese), Vol.79, No.689 (1976-4), p.305.
- [4] Miles, J. W., J. Appl. Mech., Vol.25, No.2 (1958-6), p.277.
- [5] Chu, W. -H., Trans. ASME, Ser. E, Vol.30, No.4 (1963-12), p.532.
- [6] Bauer, H. F. and Siekmann, J., Ing.-Arch., Vol.40, No.4 (1971-8), p.266.
- [7] Veletsos, A. S. and Yang, J. Y., Proc. U. S. -Japan Seminar on Earthq. Engng. Research with Emphasis on Lifeline System (1976), p.317.
- [8] Flügge, W., Statik und Dynamik der Schalen, (1962), p.273, Springer-Verlag.
- [9] Yuan, S. W. and Ting, L., J. Appl. Mech., Vol.24, No.2 (1957-6), p.278.
- [10] Timoshenko, S., et al., Vibration Problems in Engineering, (1974), p.415, John Wiley & Sons.

Fig.14 Dynamic Maximum Values of the Bending Moment (Cosine-Like Function, $H=0.02$, $\xi=0$)

Fig.15 Dynamic Maximum Values of the Radial Displacement (Cosine-Like Function, $H=0.02$, $x/l=0.5$)