An Asymmetric Mixed Boundary-value Problem of the Elastic Half-space under Shear by an Annular Rigid Punch*

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The problem considered is equivalent to a mixed boundary value problem in the theory of elasticity where the displacement of the contact area is assumed to be translation of an annular rigid punch in a direction parallel to the surface of an elastic half-space. The solution is obtained by expressing the two stress components of the contact area as appropriate Fourier series and then reducing the problem to consideration of an infinite system of algebraic equations. Numerical results are presented for the distributions of displacements and stresses on the plane \( r = 0 \).

1. Introduction

Contact problem of an elastic half-space when a small tangential force is applied across the contact surface was first considered by Mindlin\(^{26}\). Since then some investigators\(^{27-30}\) have discussed the contact problems of the elastic medium under the translation of a rigid circular punch in a direction parallel to the plane and they reduced the problems to the consideration of a pair of coupled Fredholm integral equations of the second kind.

It is the purpose of this paper to consider, within the classical theory of elasticity, a class of asymmetric mixed boundary-value problems concerning a half-space subjected to nonzero surface shear by an annular rigid punch. Making use of the method in our paper\(^{31}\), the problem is reduced to the consideration of an infinite system of algebraic equations.

2. Formulation of problem.

The solution of the elastic equilibrium equations in cylindrical polar coordinates\((r, \theta, z)\) is given by Boussinesq's stress functions \( \varphi_r, \varphi_\theta \) and \( \lambda_r \), and the components of the displacement and stress are expressed by

\[
\begin{align*}
\sigma_r &= \frac{\partial \varphi}{\partial r} + \frac{\partial \lambda_r}{\partial z} - \frac{1}{2} \left( \frac{\partial \varphi_\theta}{\partial r} + \frac{\partial \varphi_r}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial \varphi_r}{\partial r} \right) - \frac{1}{r} \frac{\partial \lambda_r}{\partial \theta} - \frac{1}{r^2} \varphi_\theta \frac{\partial \varphi_\theta}{\partial \theta}, \\
\sigma_\theta &= \frac{\partial \varphi_\theta}{\partial \theta} + \frac{\partial \lambda_r}{\partial z} - \frac{1}{2} \left( \frac{\partial \varphi_\theta}{\partial r} + \frac{\partial \varphi_r}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \varphi_\theta}{\partial \theta} \right) - \frac{1}{r} \frac{\partial \lambda_r}{\partial r} - \frac{1}{r^2} \varphi_r \frac{\partial \varphi_\theta}{\partial r}, \\
\sigma_z &= \frac{\partial \lambda_r}{\partial r} + \frac{\partial \lambda_r}{\partial z} - \frac{1}{2} \left( \frac{\partial \varphi_r}{\partial r} + \frac{\partial \varphi_\theta}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \lambda_r}{\partial r} \right) - \frac{1}{r} \frac{\partial \lambda_r}{\partial \theta} - \frac{1}{r^2} \varphi_\theta \frac{\partial \lambda_r}{\partial r} + \frac{1}{r^2} \varphi_r \frac{\partial \lambda_r}{\partial \theta}.
\end{align*}
\]

(1)

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where \( \varphi_r, \varphi_\theta \) and \( \lambda_r \) are harmonic functions, and \( c \) is the shear modulus and \( \nu \) is Poisson's ratio.

We now consider an elastic half-space in contact with an annular rigid punch as shown in Fig.1. It is assumed that the contact region \((r, r_s, z)\) has a constant displacement, \( u_r \), in the \( x \)-direction.
Fig. 1. Geometry of the problem

We assume the stress functions as follows:

\[ 2G\psi = -(2-2v) \cos \theta \int_0^\theta \frac{1}{r} A(\gamma) J_1(\gamma r) r^{-1} \, d\gamma \]
\[ 2G\varphi = \cos \theta \int_0^\theta \frac{1}{r} A(\gamma) J_0(\gamma r) r^{-1} \, d\gamma \]
\[ 2G\lambda = \sin \theta \int_0^\theta B(\gamma) J_0(\gamma r) r^{-1} \, d\gamma \]

(3)

From Eqs. (1) and (3) the components of displacement and stress are given by

\[ 2G \left( \frac{u_x}{\cos \theta} + \frac{v_y}{\sin \theta} \right) = \int_0^\theta \left[ (2-2v) A(\gamma) J_1(\gamma r) r^{-1} \right] \, d\gamma \]
\[ 2G \left( \frac{u_y}{\cos \theta} + \frac{v_x}{\sin \theta} \right) = \int_0^\theta \left[ (2-2v) A(\gamma) J_0(\gamma r) r^{-1} \right] \, d\gamma \]
\[ 2G \frac{v_y}{\cos \theta} = \int_0^\theta \left[ 2A(\gamma) J_0(\gamma r) r^{-1} \right] \, d\gamma \]

(4)

\[ \frac{\sigma_{xx}}{\cos \theta} = \int_0^\theta \left[ 2A(\gamma) J_0(\gamma r) r^{-1} \right] \, d\gamma \]
\[ \frac{\sigma_{yy}}{\sin \theta} = \int_0^\theta \left[ 2A(\gamma) J_0(\gamma r) r^{-1} \right] \, d\gamma \]

(5)

and it can easily be shown from Eqs. (7) that the boundary conditions (ii) and (iv) are satisfied. The remaining conditions (i) and (iii) lead to the pairs of the integral equations

\[ 2G \left( \frac{u_x}{\cos \theta} + \frac{v_y}{\sin \theta} \right) \bigg|_{r=r_1} = -\int_0^{\varphi_1} \left[ (1-v) A(\gamma) - \frac{1}{2} B(\gamma) \right] J_1(\gamma r) \, d\gamma + \frac{4G}{r} \left( \frac{u_x}{\cos \theta} + \frac{v_y}{\sin \theta} \right) \bigg|_{r=r_2} \]
\[ 2G \left( \frac{u_y}{\cos \theta} + \frac{v_x}{\sin \theta} \right) \bigg|_{r=r_1} = \int_0^{\varphi_1} \left[ (1-v) A(\gamma) + \frac{1}{2} B(\gamma) \right] J_0(\gamma r) \, d\gamma + \frac{2G}{r} \left( \frac{u_y}{\cos \theta} + \frac{v_x}{\sin \theta} \right) \bigg|_{r=r_2} \]

(5)

\[ \frac{\tau_{xx}}{\cos \theta} \bigg|_{r=r_1} = \int_0^{\varphi_1} \frac{1}{2} A(\gamma) J_0(\gamma r) \, d\gamma \]
\[ \frac{\tau_{yy}}{\sin \theta} \bigg|_{r=r_1} = \int_0^{\varphi_1} \frac{1}{2} B(\gamma) J_0(\gamma r) \, d\gamma \]

(5)

It follows that the displacement and stress field of the problem will be given by Eqs. (4) provided that we can find functions \( A(\gamma) \) and \( B(\gamma) \) to satisfy the integral equations (5). The functions \( A(\gamma) \), \( B(\gamma) \) are more difficult to determine from Eqs. (5) directly. In the determination of \( A(\gamma) \) and \( B(\gamma) \) it is convenient to use a similar method to that in Ref. (6).

We now introduce a new variable \( \varphi \) defined by

\[ \varphi = r \cos \theta, \quad \theta = r \sin \theta \]

(6)

where \( r \) and \( 2\theta \) are the average radius and the width of the punch, respectively. Using Eqs. (6) we find that a variable \( r \) in \( r_1 \leq r \leq r_2 \) is replaced with a new one \( \varphi \) in \( 0 \leq \varphi \leq \pi \) and then \( r=r_1 \) corresponds to \( \varphi=0 \) and \( r=r_2 \) to \( \varphi=\pi \).

It may be shown, as in Westmann's result\(^{(3)}\), that the stress components \((\tau_{xx})_{rs} \) and \((\tau_{yy})_{rs} \) have singularities of the form \( (r^2-r_1^2)(r^2-r_2^2) \) at the boundaries of the contact region. Taking the result for a circular punch into account, we can express \((\tau_{xx})_{rs} \) and \((\tau_{yy})_{rs} \) in the form

\[ (\tau_{xx})_{rs} = \frac{2G}{r} \left( \frac{u_x}{\cos \theta} + \frac{v_y}{\sin \theta} \right) \bigg|_{r=r_2} \]

\[ (\tau_{yy})_{rs} = \frac{2G}{r} \left( \frac{u_x}{\cos \theta} + \frac{v_y}{\sin \theta} \right) \bigg|_{r=r_2} \]
\[
\begin{align*}
\frac{r_n \cos \theta}{r_n - r_n^*} \left( \frac{r_n}{r_n^*} \cos \theta \right) & = \frac{1}{(2\pi)^{n}} \sum_{\ell=1}^{\infty} \frac{(a_n^* \cos \phi)}{(r_n^2 - r_n^*^2)} \left( r_n < r_n^* \right) \\
\frac{r_n \sin \theta}{r_n - r_n^*} \left( \frac{r_n}{r_n^*} \sin \theta \right) & = \frac{1}{(2\pi)^{n}} \sum_{\ell=1}^{\infty} \frac{(b_n^* \cos \phi)}{(r_n^2 - r_n^*^2)} \left( r_n < r_n^* \right)
\end{align*}
\]

(7)

where \(a_n^*\) and \(b_n^*\) \((n=0,1,2,\ldots)\) are unknown coefficients.

The following results are to be utilized extensively

\[
\begin{align*}
\int_{r}^{\infty} \frac{J_0(kr)Z_0(z)dz}{z} & = \frac{1}{2\pi} \ln \left( \frac{r}{r} \right) \left( r < r \right)
\end{align*}
\]

(8)

where

\[
Z_0(r) = \frac{1}{2} \frac{d}{dz} \left[ 1 - \frac{1}{2} \frac{d}{dz} Z_0(z) \right] \quad (n=0,1,2,\ldots)
\]

(9)

Inverting the expressions for the stress components in Eqs. (5) and (7) by means of the Hankel inversion theorem, we have

\[
A(z) = \frac{E}{4\pi} \sum_{m=1}^{\infty} (a_m^* Z_0(z) + b_m^* F_0(z)) \quad h(z) = \frac{E}{4\pi} \sum_{m=1}^{\infty} (a_m^* Z_0(z) + b_m^* F_0(z))
\]

(10)

It is easily shown from Eqs. (8) that Eqs. (10) for arbitrary coefficients \(a_n^*\) and \(b_n^*\) satisfy the boundary condition (ii).

In the determination of the unknown coefficients \(a_n^*\) and \(b_n^*\) we take the boundary condition (i) into account. Substituting Eqs. (10) into the first and second representations of Eqs. (5) we obtain

\[
2 \int \frac{r_n \cos \theta}{r_n - r_n^*} \left( \frac{r_n}{r_n^*} \cos \theta \right) \left( \frac{2\pi}{2\pi} \sum_{\ell=1}^{\infty} \left[ (a_n^* Z_0(z) + b_n^* F_0(z)) J_0(kr)dz \right] = 4G
\]

\[
2 \int \frac{r_n \sin \theta}{r_n - r_n^*} \left( \frac{r_n}{r_n^*} \sin \theta \right) \left( \frac{2\pi}{2\pi} \sum_{\ell=1}^{\infty} \left[ (a_n^* Z_0(z) + b_n^* F_0(z)) J_0(kr)dz \right] = 0
\]

(11)

Now by Neumann

\[
J_0(kr) = Z_0(z) + 2 \sum_{n=1}^{\infty} Z_0(z) \cos \phi
\]

(12)

and using Eq. (12) we have

\[
r^2 J_0(kr) = 1 \frac{d}{dr} \left[ \frac{1}{r^2} r J_0(kr) \right] = \frac{1}{2} \frac{d}{dz} \left[ \frac{1}{2} \frac{d}{dz} F_0(z) \right] = F_0(z) + 2 \sum_{n=1}^{\infty} F_0(z) \cos \phi
\]

(13)

Substituting Eqs. (12) and (13) into Eqs. (11) we find

\[
2 \int \frac{r_n \cos \theta}{r_n - r_n^*} \left( \frac{r_n}{r_n^*} \cos \theta \right) \left( \frac{2\pi}{2\pi} \sum_{\ell=1}^{\infty} \left[ (a_n^* Z_0(z) + b_n^* F_0(z)) J_0(kr)dz \right] \right)
\]

\[
\times \left( Z_0(z) + 2 \sum_{n=1}^{\infty} Z_0(z) \cos \phi \right) \right) = 4G
\]

\[
2 \int \frac{r_n \sin \theta}{r_n - r_n^*} \left( \frac{r_n}{r_n^*} \sin \theta \right) \left( \frac{2\pi}{2\pi} \sum_{\ell=1}^{\infty} \left[ (a_n^* Z_0(z) + b_n^* F_0(z)) J_0(kr)dz \right] \right)
\]

\[
\times \left( F_0(z) + 2 \sum_{n=1}^{\infty} F_0(z) \cos \phi \right) \right) = 0
\]

(14)

and equating the coefficient of \(\cos \phi\) in both sides of Eqs. (14), respectively, we obtain

\[
\sum_{m=0}^{\infty} \int_{r}^{\infty} (a_m Z_0(z) + b_m F_0(z)) Z_0(z) dz = 2\pi
\]

\[
\sum_{m=0}^{\infty} \int_{r}^{\infty} (a_m Z_0(z) + b_m F_0(z)) F_0(z) dz = 0
\]

(15)

where
\[ \varepsilon = \frac{\nu}{2 - \nu}, \quad a_0 = \frac{(2-\nu)\pi}{8G\alpha}, \quad b_0 = \frac{(2-\nu)\pi}{8G\alpha} \]  

and \( \delta \) is Kronecker's delta.

Eqs. (15) can be used to determine the unknown coefficients \( a_0 \) and \( b_0 \). Therefore,

\[ A(0) = -\frac{2G\alpha}{2-\nu} \sum_n \left( a_n Z_n(2) + b_n F_n(2) \right), \quad B(0) = -\frac{4G\alpha}{2-\nu} \sum_n \left( a_n Z_n(2) + b_n F_n(2) \right) \]

and using Eqs. (17) we find

\[ \frac{1}{\rho_0} \sin \theta \frac{\partial}{\partial \theta} \int_{0}^{2\pi} \left\{ \frac{(24-2\nu)}{2-\nu} \sum_n \left( a_n Z_n(2) + b_n F_n(2) \right) J_n(\rho_0) \right\} d\theta d\rho \]

3. The components of displacement and stress on the plane \( z = 0 \).

As the integrands in Eqs. (18) decrease rapidly to zero with an increasing \( \rho \) when \( z = 0 \), the infinite integrals with respect to \( \rho \) can easily be evaluated by numerical method.

In a similar way as shown in Ref. (6) we find that the components of displacement and stress on the plane \( z = 0 \) are given by

\[ \frac{1}{\rho} \sin \theta \frac{\partial}{\partial \theta} \int_{0}^{2\pi} \left\{ \frac{(24-2\nu)}{2-\nu} \sum_n \left( a_n Z_n(2) + b_n F_n(2) \right) J_n(\rho_0) \right\} d\theta d\rho \]

where

\[ I_0 = \int_{0}^{2\pi} Z_n(2) J_n(\rho_0) d\theta \]

and these results are given in Ref. (7).

We now compute the total force \( S_x \) on the contact region. It is in the \( x \)-direction and is

\[ S_x = \int_{0}^{2\pi} \int_{0}^{r_c} r \sin \theta \frac{\partial}{\partial \theta} \left[ (r_c) \sin \theta - (r_c) \cos \theta \right] d\theta dr = -\frac{2G\alpha}{2-\nu} \]

4. The special case of \( r_c \rightarrow 0 \).

Now \( r_c = \text{const.} \) and \( r_c \rightarrow 0 \) in Eqs. (6) show that

\[ r_c = r_c \sin (\theta/2) \quad (0 \leq \theta \leq \pi) \]

and using these results we may reduce Eqs. (15) to

\[ \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{r_c} \left[ a_n J_n(2r_c/2) + b_n J_n(2r_c/2) \right] J_n(2r_c/2) d\rho d\theta = 0 \]

and

\[ \frac{1}{2} \int_{0}^{2\pi} \int_{0}^{r_c} \left[ a_n J_n(2r_c/2) + b_n J_n(2r_c/2) \right] J_n(2r_c/2) d\rho d\theta = -\frac{4G\alpha}{2-\nu} \]

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Eqs. (22) and (23) can be used to determine the coefficients \(a_1\) and \(\lambda_1\) for the special case. Now put
\[
a_n = \frac{2}{\pi} r_n \quad a_n = -\frac{4}{\pi} r_n \left[ \phi_n - (\pi - \phi_n) \right] \quad (n \geq 1), \quad \lambda_n = 0 \quad (n \geq 0)
\]
(24)

Then, the following results \(a_n\) are to be utilized extensively
\[
\begin{align*}
J_n(\lambda_n) &= \frac{2}{\pi} \int_0^{2\pi} J_n(\lambda_n \sin \theta) \cos (\mu \lambda_n \sin \theta) d\theta \\
J_n^2(\lambda_n) &= \frac{2}{\pi} \int_0^{2\pi} J_n^2(\lambda_n \sin \theta) d\theta \\
\sum_{m=0}^{\infty} a_n &\sum_{m=0}^{\infty} J_n(\lambda_n) J_m(\lambda_m) \sum_{m=0}^{\infty} J_n(\lambda_n) J_m(\lambda_m) \sum_{m=0}^{\infty} J_n(\lambda_n) J_m(\lambda_m)
\end{align*}
\]
(25)

and using these results we find that the summation on the left hand side of Eq. (22) is reduced in a closed form as
\[
\sum_{n=0}^{\infty} a_n J_n^2(\lambda_n) = \frac{2}{\pi} \int_0^{2\pi} \sin \theta J_n(\lambda_n \sin \theta) d\theta = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin \lambda_n}{\lambda_n} \sum_{n=0}^{\infty} \frac{\sin \lambda_n}{\lambda_n}
\]
(26)

Making use of Eq. (26) and the formula
\[
\int_0^{\pi} \frac{1}{2} \sin \lambda_n \sin \lambda_m \sin \theta d\lambda = \frac{1}{2} \delta_{nm}
\]
then we see that
\[
\sum_{n=0}^{\infty} a_n J_n^2(\lambda_n) J_m(\lambda_m) d\lambda = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin \lambda_n}{\lambda_n} J_n^2(\lambda_n) d\lambda = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin \lambda_n}{\lambda_n}
\]
(27)

and that Eq. (22) is satisfied.

Upon substituting Eq. (22) directly into Eq. (23), for the case in which \(m \geq 1\), we obtain
\[
\begin{align*}
\sum_{n=0}^{\infty} a_n J_n^2(\lambda_n) \sum_{m=0}^{\infty} \left[ \frac{d}{d\lambda} \frac{1}{\lambda} \frac{d}{d\lambda} J_n(\lambda_n) \right] d\lambda \\
= -\frac{\epsilon \lambda_1}{2\pi} \sum_{n=0}^{\infty} \int_0^{2\pi} J_n(\lambda_n \sin \theta) d\lambda
\end{align*}
\]
(28)

From Eqs. (25) it is seen that
\[
\frac{1}{2} \frac{d}{d\lambda} (J_n^2(\lambda_n)) = -\frac{\epsilon \lambda_1}{4} \left[ J_n^2(\lambda_n) + J_n(\lambda_n) J_n(\lambda_n) \right]
\]
\[
= -\frac{\epsilon \lambda_1}{4} \sum_{n=0}^{\infty} \left[ J_n(\lambda_n \sin \theta) + J_n(\lambda_n \sin \theta) \cos \lambda_n \sin \theta \right] d\theta
\]
(29)

and using the result, for the case in which \(m = 0\), we find
\[
\begin{align*}
\sum_{n=0}^{\infty} a_n &\sum_{m=0}^{\infty} J_n(\lambda_n) \left[ \frac{1}{\lambda} \frac{d}{d\lambda} J_m(\lambda_m) \right] d\lambda \\
= \frac{\epsilon \lambda_1}{2\pi} \sum_{n=0}^{\infty} \int_0^{2\pi} J_n(\lambda_n \sin \theta) d\lambda
\end{align*}
\]
(30)

Accordingly, it may be shown that \(a_n\) and \(\lambda_n\) in Eqs. (24) are the necessary forms to satisfy Eqs. (22) and (23). On the plane \(\pi = 0\), the displacements and stress components for the special case are then written as follows:

\[
\begin{align*}
\frac{H(r_s)}{r_s} &= \frac{2}{\pi} \left[ \frac{r_s}{2 - \nu^2} \sqrt{1 - \left( \frac{r_s}{r} \right)^2} + (2 - \nu) \sin^{-1} \frac{r_s}{r} \right] \\
\times \left[ H(r_s) - \frac{r_s}{2 - \nu} \frac{d}{dr} H(r_s - r) \right]
\end{align*}
\]
(31)

\[
\begin{align*}
\frac{H(r_s)}{r_s} &= \frac{2}{\pi} \left[ \frac{r_s}{2 - \nu} \sqrt{1 - \left( \frac{r_s}{r} \right)^2} - (2 - \nu) \sin^{-1} \frac{r_s}{r} \right] \\
\times \left[ H(r_s) - \frac{r_s}{2 - \nu} \frac{d}{dr} H(r_s - r) \right]
\end{align*}
\]
(32)

\[
\begin{align*}
\frac{H(r_s)}{r_s} &= \frac{2}{\pi} \left( 1 - \frac{r_s}{r} \right) \left[ \frac{r_s}{2 - \nu} \sqrt{1 - \left( \frac{r_s}{r} \right)^2} + \frac{r_s}{r} \frac{d}{dr} H(r_s - r) \right]
\end{align*}
\]
(33)
These agree with the results for a circular punch.  

5. Numerical results.

First we determine the coefficients a, b, defined in Eqs.(15). In the determination of a, b, we must compute the value of the infinite integrals in which the integrands have the multiplication of four Bessel functions.

We now have 2N simultaneous linear algebraic systems for the 2N unknowns a, b, (m=1,2,⋯,N). The symmetrical matrix \( A_{m,n} \) is then given by

\[
\begin{align*}
A_{m,n} &= \sum_{k=1}^{2N} \int_0^1 Z_k(\rho)F_k(\rho)\rho d\rho + A_{m,n} \quad (m \leq N, n \leq N) \\
&= \sum_{k=1}^{2N} \int_0^1 Z_k(\rho)F_k(\rho)\rho d\rho + A_{m,n} \quad (m \leq N, n \geq N+1) \\
&= \sum_{k=1}^{2N} \int_0^1 Z_k(\rho)F_k(\rho)\rho d\rho + A_{m,n} \quad (m \geq N+1, n \leq N) \\
&= \sum_{k=1}^{2N} \int_0^1 Z_k(\rho)F_k(\rho)\rho d\rho + A_{m,n} \quad (m \geq N+1, n \geq N+1)
\end{align*}
\]

where

\[
A_{m,n} = \begin{cases} 
\sum_{k=1}^{2N} \int_0^1 Z_k(\rho)Z_k(\rho)\rho d\rho & (m \leq N, n \leq N); \\
\sum_{k=1}^{2N} \int_0^1 Z_k(\rho)F_k(\rho)\rho d\rho & (m \leq N, n \geq N+1) \\
\sum_{k=1}^{2N} \int_0^1 F_k(\rho)Z_k(\rho)\rho d\rho & (m \geq N+1, n \leq N); \\
\sum_{k=1}^{2N} \int_0^1 F_k(\rho)F_k(\rho)\rho d\rho & (m \geq N+1, n \geq N+1)
\end{cases}
\]

and \( b \) is a large value. The first integrals in Eqs.(31) can be evaluated by Simpson’s rule. Making use of Hankel’s result, we can evaluate \( A_{m,n} \) by

\[
\begin{align*}
A_{m,n} &= \frac{1}{2\pi} \left[ \frac{\cos 2\alpha_r \sin 2\beta + r_n \sin(2\alpha_r) + (-1)^n \frac{1 - \cos 2\alpha_r}{2\pi} - r_n \sin(2\beta)\right] \\
&\quad + \left( (-1)^n m - (-1)^{n+1} m \right) \left[ \frac{\sin 2\alpha_r + \sin 2\beta - r_n \cos(2\alpha_r) - \cos(2\beta)\right] \\
&\quad + \left( (-1)^n + (-1)^{n+1} m \right) \left[ \frac{\sin 2\alpha_r + \sin 2\beta - r_n \cos(2\alpha_r) - \cos(2\beta)\right]
\end{align*}
\]

and \( \alpha(\rho) \) and \( \cos(\rho) \) are the integral sine and cosine functions, respectively.

As expected, the convergence of the coefficients \( a, b \), giving the displacement and stress field turned out to be sensitive to the value of \( r \). The coefficients \( a, b \), for \( r=0.25, 0.5 \) and \( 0.75 \) are shown in Table 1. The convergence becomes slower and more terms are required in Eqs.(15) as \( r \) becomes smaller. In order to investigate the effect of \( N \) on the accuracy in the numerical examples the values of \( \{a_{m,n}\} \) near both edges of the punch for \( N=10,15,20 \) and \( r=0.5 \) are shown in Table 2. The accuracy in these results could easily be controlled by adjusting the number of terms, \( N \). Hence, the value of \( N \) for each case of \( r \) is determined, as in Table 1.

In Fig.2 the results are presented for the distributions of displacements and stresses on the plane \( \rho=0 \). The figure also shows that the results for a circular punch of radius \( r=1.0 \) are illustrated by the cross-mark, and that the curves for \( r_{m,n} \) and \( r_{m,n} \) plotted here are based on the absolute values. The \( a_{m,n} \) and \( b_{m,n} \) in each case of \( r \) are, for \( 0\leq\rho \leq\pi \), related approximately as
### Table 1 Coefficients $a_n$ and $b_n$

<table>
<thead>
<tr>
<th>$r_i$</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$a_n$</th>
<th>$b_n$</th>
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<td>0.05668586</td>
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<td>0.05822409</td>
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<td>0.05957967</td>
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<td>-0.06161830</td>
<td>0.08197802</td>
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<td>-0.00712547</td>
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<td>0.06422129</td>
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</tr>
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<td>-0.00785515</td>
<td>0.04144457</td>
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<td>-0.00007656</td>
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<td>5</td>
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</tr>
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</tr>
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<td>0.00000253</td>
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<td>0.00006625</td>
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<td>-0.00002893</td>
<td>0.00015196</td>
<td>0.00000253</td>
<td>0.00006625</td>
</tr>
</tbody>
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Table 2 The variations of $(r_{m})_{rs}$ with $N$ and $r$

<table>
<thead>
<tr>
<th>$r_{m}/r_{0}$</th>
<th>$N = 10$</th>
<th>$N = 15$</th>
<th>$N = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.51</td>
<td>2.32845</td>
<td>2.32801</td>
<td>2.32872</td>
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<tr>
<td>0.55</td>
<td>1.74119</td>
<td>1.74072</td>
<td>1.74264</td>
</tr>
<tr>
<td>0.59</td>
<td>1.49883</td>
<td>1.49871</td>
<td>1.49852</td>
</tr>
<tr>
<td>0.63</td>
<td>1.32334</td>
<td>1.32235</td>
<td>1.32628</td>
</tr>
<tr>
<td>0.67</td>
<td>1.20515</td>
<td>1.20424</td>
<td>1.20855</td>
</tr>
<tr>
<td>0.71</td>
<td>1.01323</td>
<td>1.01306</td>
<td>1.01787</td>
</tr>
</tbody>
</table>

Both slopes of $(w_{1})_{rs}$ and $(w_{2})_{rs}$ become infinite as $r_{m}/r_{0}$ and $r_{m}/r_{n} = 0$. For $r_{m}/r_{0}$ these curves for $r_{m}/r_{0} = 0.25$ and 0.5 come close to those for a circular punch while the values of $(w_{1})_{rs}$ and $(w_{2})_{rs}$ when $r_{m}/r_{n} = 0.75$ are smaller than those when $r_{m}/r_{n} = 0.25$ and 0.5.

The values of $(w_{1})_{rs}$ for all cases of $r_{m}/r_{0}$ are much smaller than those of other displacement components, and for $a_{rs}$ and $b_{rs}$ $(w_{1})_{rs} = 0$.

The shear components of $(r_{m})_{rs}$ and $(r_{n})_{rs}$ in all cases of $r_{m}/r_{0}$ are given by the inequality.
\[
\frac{(t_{r_0})_{xz}}{\cos \theta} + \frac{(t_{r_0})_{xz}}{\sin \theta} = 2\frac{(t_{r_0})_{xz}}{\sin 2\theta} > 0
\]

Fig. 3 The variation of \( S_a \) with \( r_i/r_0 \)

and the value of \( (t_{r_0})_{xz}/\sin 2\theta \) increases with an increasing \( r_i \). The curves for the shear components come close to those of a circular punch as \( r_i \rightarrow r_0 \).

Fig. 3 shows the variation of \( S_a \) with \( r_i/r_0 \). It may be seen that \( S_a \), when \( r_i/r_0 = \text{const.} \) and \( r_i/r_0 < 0.6 \), is nearly equal to the value for a circular punch.

References