532.582:532.5.01

Paper No. 181-12

Conformal Transformation of Row of Flat-plates into Row of Circular Cylinders*

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The present paper deals with a new function which transforms conformally a row of flat-plates into a row of circular cylinders. This new function is the basic function which can be used to obtain the two-dimensional potential flow about blades with arbitrary profiles in cascade. The distortion between the physical plane and the mapped plane, as the result of applying this new transformation, is exceedingly small. Therefore, the flow about blades with arbitrary profiles in cascade can be obtained more easily and accurately by this method than by the other methods; i.e., the conformal transformations of the blades in cascade into a single circular cylinder or into two concentric circular cylinders. Furthermore, the flow through a cascade whose angle of stagger and solidity are large can be obtained easily by this method. In addition, the flow near the leading edge can be obtained accurately.

1. Introduction

In the present paper, a new function which transforms conformally a region exterior to a row of flat-plates into a region exterior to a row of circular cylinders is introduced. This can be applied, as it is, to the conformal mapping of a cascade of flat-plates into a row of circular cylinders. Furthermore, based on this function and adding a certain series of complex variables in the mapped plane, the transformation of a region exterior to a cascade of arbitrary blades into a region exterior to a row of circular cylinders can be performed. The ultimate purpose of the author is to obtain the two-dimensional potential flow through a cascade of arbitrary blades by using the above-mentioned transformation.

The distortion between the physical plane and the mapped plane, as the result of applying this transformation, is exceedingly small even around the blades. Therefore, the flow about the cascade of arbitrary blades can be obtained more easily and accurately by this method than by the other methods, i.e., the conformal transformations of the blades in cascade into a single circular cylinder⁽¹⁾ or into two concentric circular cylinders⁽²⁾⁻⁽⁸⁾. Accordingly, the flow through the cascade whose stagger angle and solidity are large can be obtained easily by this method. In addition, the flow near the leading edge can be obtained accurately. Numerical examples will be reported at another time.

2. Mapping Function

2.1 Row of flat-plates and a row of circular cylinders

The conformal transformation of a row of flat-plates in the complex plane

$z = x + iy$ into a row of circular cylinders in the complex plane $\zeta = \xi + i\eta$ is discussed. As the former row, we consider an infinite row of flat-plates whose chord lengths are l 's and which are arranged at regular intervals of s in the direction of stagger angle γ as illustrated in Fig.1. If the flat-plates are regarded as blades, we can regard l as the chord length of the blade and s as the spacing of a cascade, then the cascade of flat-plates is formed with a stagger angle of γ and solidity of l/s . The real x -axis runs parallel to the flat-plates and the origin is taken at the middle of a flat-plate in the row.

On the other hand, in the ζ -plane, we consider an infinite row of circular cylinders composed of the cylinders whose radii are a 's and which are infinitely arranged at regular intervals of t along the imaginary η -axis, as illustrated in Fig.2. The origin in this plane is taken at the center of a circular cylinder in the row.

2.2 Outline of the method of transformation

The mapping function between the row of flat-plates in the z -plane and the row of circular cylinders in the ζ -plane is obtained by comparing the complex potentials

Fig.1 Row of flat-plates in z -plane.

Fig.2 Row of circular cylinders in ζ -plane.

* Received 6th October, 1978.

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of the corresponding flows in the both planes: in obtaining the mapping function, we consider a flow in the z -plane and the corresponding flow in the ζ -plane. Of course, the flow is inviscid, incompressible, two-dimensional and irrotational. The complex potentials of each flow are expressed by

$$W_z = f(z) \quad \text{and} \quad W_\zeta = g(\zeta) \dots\dots(1).$$

Since the flows on both planes correspond to each other, these two complex potentials are equal. Therefore,

$$W_z = W_\zeta = W \dots\dots\dots(2).$$

Henceforth, let W be the complex potential of the flow on both planes; and substituting Eq.(1) into Eq.(2), we obtain

$$f(z) = g(\zeta) \dots\dots\dots(3).$$

If z is expressed by an explicit function of ζ ; e.g., Eq.(4) described below; we can derive, in simple forms, the differential coefficient between both planes and other requisite relations.

$$z = f^{-1} \{g(\zeta)\} = h(\zeta) \dots\dots\dots(4)$$

As mentioned above, we obtain the mapping function between the row of flat-plates and the row of circular cylinders; i.e., Eq.(3) or Eq.(4).

2.3 Complex potential

At the onset, the complex potentials of the flow on each plane are derived in order to obtain the above-mentioned Eq.(3) or Eq.(4).

2.3.1 Complex potential of the flow in the z -plane

We may arbitrarily choose one of the flows in the two planes, the z -plane and the ζ -plane. Thus, we choose a uniform flow parallel to the flat-plates in the z -plane because we can easily obtain its complex potential W . Let V be the magnitude of the velocity of the flow; then, the complex potential of this flow is given by

$$W = Vz \dots\dots\dots(5).$$

2.3.2 Complex potential of the flow in the ζ -plane

The flow in the ζ -plane which corresponds to the flow in the z -plane, whose complex potential is expressed by Eq.(5), is considered. The flow near the circular cylinders is not clear at the first step, but it is easily concluded that the flow must be uniform at upstream infinity and downstream infinity of the circular cylinders. Let U represent the magnitude of the velocity of this flow and β represent the angle formed with the real ξ -axis.

As stated above, we had settled the velocity vector at upstream infinity and downstream infinity of the row of circular cylinders; thus, we take this velocity vector as the starting point (0th approximation) and derive the complex potential in the ζ -plane by applying the perturbation method.

The symbols about the complex potential of the flow in the ζ -plane are settled

as follows.

(Symbols about the complex potential)

W_0 : 0th approximation to the complex potential

W_N : N th approximation to the complex potential, where N is an integer and $N \geq 1$

w_N : Supplementary complex potential of the order of N

$w_{N,n}$: Supplementary complex potential of the order of N about the n th circular cylinder, where for the central circular cylinder, $n=0$; for those above and below the central one, $n=\pm 1, \pm 2, \pm 3 \dots$ respectively in that order.

Here, we have

$$W_N = W_{N-1} + w_N \dots\dots\dots(6),$$

$$W = W_0 + \sum_{N=1}^{+\infty} w_N \dots\dots\dots(7),$$

$$w_N = \sum_{n=-\infty}^{+\infty} w_{N,n} \dots\dots\dots(8).$$

(i) 0th approximation. The uniform flow, whose magnitude of velocity is U and which forms the angle β with the real ξ -axis, is taken as the flow of the 0th approximation.

$$W_0 = U\zeta e^{-i\beta} = Ua e^{-i\beta} \left(\frac{\zeta}{a}\right) \dots\dots\dots(9)$$

(ii) Supplementary complex potential of the order of I about the central circular cylinder.

$$w_{I,0} = Ua e^{i\beta} \left(\frac{a}{\zeta}\right) \dots\dots\dots(10)$$

The flow expressed by $(W_0 + w_{I,0}) = Ua \{e^{-i\beta}(\zeta/a) + e^{i\beta}(a/\zeta)\}$ satisfies the boundary condition (streamlines proceed along the surface of circular cylinders) on the surface of the central circular cylinder ($\zeta = a e^{i\beta}$).

(iii) Supplementary complex potential of the order of I about the n th circular cylinder. This complex potential is obtained from Eq.(10) as

$$w_{I,n} = Ua e^{i\beta} \left(\frac{a}{\zeta - n\zeta_s}\right) \dots\dots\dots(11),$$

where

$$\zeta_s = it \dots\dots\dots(12).$$

(iv) Supplementary complex potential of the order of I . This complex potential is obtained from Eq.(8) and Eq.(11) as follows.

$$w_I = \sum_{n=-\infty}^{+\infty} w_{I,n} = Ua e^{i\beta} \sum_{n=-\infty}^{+\infty} \left(\frac{a}{\zeta - n\zeta_s}\right) \dots\dots(13)$$

Suppose $|\zeta| < |\zeta_s| = t$, an ascending power of ζ is obtained as

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} \frac{1}{(\zeta - n\zeta_s)} &= \frac{\pi}{\zeta_s} \cot \left(\frac{\pi\zeta}{\zeta_s}\right) \\ &= \frac{\pi}{\zeta_s} \left\{ \frac{\zeta_s}{\pi\zeta} - \sum_{m=1}^{\infty} \frac{2^{2m} B_m}{(2m)!} \left(\frac{\pi\zeta}{\zeta_s}\right)^{2m-1} \right\} \\ &= \frac{1}{\zeta} - \frac{1}{a} \sum_{m=1}^{\infty} \frac{(2\pi)^{2m} B_m}{(2m)!} \left(\frac{a}{\zeta_s}\right)^{2m} \left(\frac{\zeta}{a}\right)^{2m-1} \dots(14) \end{aligned}$$

where B_m is Bernoulli number.

If we set

$$1 > (a/t)^2 = \mu \dots\dots\dots(15),$$

we obtain

$$\left(\frac{a}{\zeta_s}\right)^2 = - \left(\frac{a}{t}\right)^2 = - \mu \dots\dots\dots(16).$$

Further, we set

$$\frac{(2\pi)^{2m} B_m}{(2m)!} = \alpha_m \dots\dots\dots(17),$$

and substituting Eq.(16) and Eq.(17) into Eq.(14), we obtain

$$\sum_{n=-\infty}^{+\infty} \left(\frac{a}{\zeta - n\zeta_s}\right) = \frac{a}{\zeta} - \sum_{m=1}^{\infty} (-1)^m \alpha_m \mu^m \left(\frac{\zeta}{a}\right)^{2m-1} \dots\dots\dots(18).$$

Since the values of α_m for $m=1-8$ are obtained as follows, Eq.(19), it is confirmed numerically that $\lim_{m \rightarrow +\infty} \alpha_m = 2$.

$$\left. \begin{aligned} \alpha_1 &= \frac{(2\pi)^2 B_1}{2!} = 2\pi^2 \times \frac{1}{6} = \frac{1}{3} \pi^2 = 3.2899, & \alpha_2 &= \frac{(2\pi)^4 B_2}{4!} = \frac{2}{3} \pi^4 \times \frac{1}{30} = \frac{1}{45} \pi^4 = 2.1646 \\ \alpha_3 &= \frac{(2\pi)^6 B_3}{6!} = \frac{4}{45} \pi^6 \times \frac{1}{42} = \frac{2}{945} \pi^6 = 2.0347, & \alpha_4 &= \frac{(2\pi)^8 B_4}{8!} = \frac{2}{315} \pi^8 \times \frac{1}{30} = \frac{1}{725} \pi^8 = 2.0082 \\ \alpha_5 &= \frac{(2\pi)^{10} B_5}{10!} = \frac{4}{14 \cdot 175} \pi^{10} \times \frac{5}{66} = \frac{2}{93 \cdot 555} \pi^{10} = 2.0020 \\ \alpha_6 &= \frac{(2\pi)^{12} B_6}{12!} = \frac{4}{467 \cdot 775} \pi^{12} \times \frac{691}{2730} = \frac{1 \cdot 382}{467 \cdot 775 \times 1 \cdot 365} \pi^{12} = 2.0005 \\ \alpha_7 &= \frac{(2\pi)^{14} B_7}{14!} = \frac{8}{467 \cdot 775 \times 91} \pi^{14} \times \frac{7}{6} = \frac{4}{467 \cdot 775 \times 39} \pi^{14} = 2.0001 \\ \alpha_8 &= \frac{(2\pi)^{16} B_8}{16!} = \frac{2}{467 \cdot 775 \times 1 \cdot 365} \pi^{16} \times \frac{3 \cdot 617}{510} = \frac{3 \cdot 617}{467 \cdot 775 \times 1 \cdot 365 \times 255} \pi^{16} = 2.0000 \end{aligned} \right\} \dots\dots\dots(19)$$

If we substitute Eq.(18) into Eq.(13), the supplementary complex potential of the order of I is rewritten as

$$\begin{aligned} w_I = \sum_{n=-\infty}^{+\infty} w_{I,n} &= U a e^{i\beta} \sum_{n=-\infty}^{+\infty} \left(\frac{a}{\zeta - n\zeta_s}\right) = U a e^{i\beta} \frac{\pi a}{\zeta_s} \cot\left(\frac{\pi \zeta}{\zeta_s}\right) = U a e^{i\beta} \left[\left(\frac{a}{\zeta}\right) + \left\{ \alpha_1 \mu \left(\frac{\zeta}{a}\right) \right. \right. \\ &\quad \left. \left. - \alpha_2 \mu^2 \left(\frac{\zeta}{a}\right)^3 + \alpha_3 \mu^3 \left(\frac{\zeta}{a}\right)^5 - \alpha_4 \mu^4 \left(\frac{\zeta}{a}\right)^7 + \alpha_5 \mu^5 \left(\frac{\zeta}{a}\right)^9 - \alpha_6 \mu^6 \left(\frac{\zeta}{a}\right)^{11} + \alpha_7 \mu^7 \left(\frac{\zeta}{a}\right)^{13} - \dots \right\} \right] \dots\dots\dots(20) \end{aligned}$$

provided that $|\zeta| < t$.

The complex potential $(w_0 + w_{I,0})$ satisfies the boundary condition on the surface of the central circular cylinder, but the flow expressed by $(\sum_{n=-\infty}^{\infty} w_{I,n} - w_{I,0})$, namely, $(w_I - w_{I,0})$, does not satisfy the boundary condition on the surface of the central circular cylinder. Here, $(w_I - w_{I,0})$ is obtained as follows from Eq.(10) and Eq.(20).

$$\begin{aligned} w_I - w_{I,0} &= U a e^{i\beta} \left[\alpha_1 \mu \left(\frac{\zeta}{a}\right) - \alpha_2 \mu^2 \left(\frac{\zeta}{a}\right)^3 + \alpha_3 \mu^3 \left(\frac{\zeta}{a}\right)^5 \right. \\ &\quad \left. - \alpha_4 \mu^4 \left(\frac{\zeta}{a}\right)^7 + \alpha_5 \mu^5 \left(\frac{\zeta}{a}\right)^9 - \alpha_6 \mu^6 \left(\frac{\zeta}{a}\right)^{11} + \alpha_7 \mu^7 \left(\frac{\zeta}{a}\right)^{13} - \dots \right] \dots\dots\dots(21) \end{aligned}$$

(v) Supplementary complex potential of the order of II about the central circular cylinder.

$$\begin{aligned} w_{II,0} &= U a e^{-i\beta} \left[\alpha_1 \mu \left(\frac{a}{\zeta}\right) - \alpha_2 \mu^2 \left(\frac{a}{\zeta}\right)^3 + \alpha_3 \mu^3 \left(\frac{a}{\zeta}\right)^5 \right. \\ &\quad \left. - \alpha_4 \mu^4 \left(\frac{a}{\zeta}\right)^7 + \alpha_5 \mu^5 \left(\frac{a}{\zeta}\right)^9 - \alpha_6 \mu^6 \left(\frac{a}{\zeta}\right)^{11} + \alpha_7 \mu^7 \left(\frac{a}{\zeta}\right)^{13} - \dots \right] \dots\dots\dots(22) \end{aligned}$$

Since the flow expressed by $(w_I - w_{I,0}) + w_{II,0}$ satisfies the boundary condition on the surface of the central circular cylinder ($\zeta = a e^{i\theta}$), it is confirmed that Eq.(22) expresses the complex potential mentioned in the title of this paragraph.

(vi) Supplementary complex potential of the order of II about the n th circular cylinder. Replacing ζ with $(\zeta - n\zeta_s)$ in Eq.(22), we obtain the complex potential mentioned in the title of this paragraph.

(vii) Supplementary complex potential of the order of II . This complex potential is obtained from Eq.(8) as follows.

$$w_{II} = \sum_{n=-\infty}^{+\infty} w_{II,n} \dots\dots\dots(23)$$

In deriving w_{II} from Eq.(23) and $w_{II,n}$ expressed in the paragraph(vi), an expansion of $\sum_{n=-\infty}^{+\infty} \left(\frac{a}{\zeta - n\zeta_s}\right)^{2M-1}$ is required. This expansion is obtained from Eq.(24), namely, $\sum_{n=-\infty}^{+\infty} \frac{1}{(\zeta - n\zeta_s)^{2M-1}}$, which is derived by $(2M-2)$ -times differentiating Eq.(18), multiplied by a^{2M-1} .

$$\sum_{n=-\infty}^{+\infty} \frac{1}{(\zeta - n\zeta_s)^{2M-1}} = \frac{1}{(2M-2)!} \frac{d^{2M-2}}{d\zeta^{2M-2}} \sum_{n=-\infty}^{+\infty} \frac{1}{(\zeta - n\zeta_s)} = \frac{1}{(2M-2)!} \frac{d^{2M-2}}{d\zeta^{2M-2}} \left[\frac{1}{\zeta} - \frac{1}{a} \sum_{m=1}^{\infty} (-1)^m \alpha_m \mu^m \left(\frac{\zeta}{a}\right)^{2m-1} \right]$$

$$= \frac{1}{a} \frac{1}{(2M-2)!} \frac{d^{2M-2}}{d\zeta^{2M-2}} \left[\frac{a}{\zeta} + \left\{ \alpha_1 \mu \left(\frac{\zeta}{a}\right) - \alpha_2 \mu^2 \left(\frac{\zeta}{a}\right)^3 + \alpha_3 \mu^3 \left(\frac{\zeta}{a}\right)^5 \right. \right.$$

$$\left. \left. - \alpha_4 \mu^4 \left(\frac{\zeta}{a}\right)^7 + \dots \right\} \right] \dots \dots \dots (24)$$

If we give concrete examples of this expansion for the cases of $M=2$ and 3 , we have Eq.(25) and Eq.(26). The case of $M=1$ has already been described in Eq.(18).

$M = 2$

$$\sum_{n=-\infty}^{+\infty} \left(\frac{a}{\zeta - n\zeta_s}\right)^3 = \left(\frac{a}{\zeta}\right)^3 + \left\{ -3\alpha_2 \mu^2 \left(\frac{\zeta}{a}\right) + 10\alpha_3 \mu^3 \left(\frac{\zeta}{a}\right)^3 \right.$$

$$\left. - 21\alpha_4 \mu^4 \left(\frac{\zeta}{a}\right)^5 + 36\alpha_5 \mu^5 \left(\frac{\zeta}{a}\right)^7 - 55\alpha_6 \mu^6 \left(\frac{\zeta}{a}\right)^9 + 78\alpha_7 \mu^7 \left(\frac{\zeta}{a}\right)^{11} + \dots \right\} \dots \dots (25)$$

$M = 3$

$$\sum_{n=-\infty}^{+\infty} \left(\frac{a}{\zeta - n\zeta_s}\right)^5 = \left(\frac{a}{\zeta}\right)^5 + \left\{ 5\alpha_3 \mu^3 \left(\frac{\zeta}{a}\right) - 35\alpha_4 \mu^4 \left(\frac{\zeta}{a}\right)^3 \right.$$

$$\left. + 126\alpha_5 \mu^5 \left(\frac{\zeta}{a}\right)^5 - 330\alpha_6 \mu^6 \left(\frac{\zeta}{a}\right)^7 + 715\alpha_7 \mu^7 \left(\frac{\zeta}{a}\right)^9 + \dots \right\} \dots \dots \dots (26)$$

Substituting these expansions into Eq.(23), we obtain w_{II} . Further, subtracting $w_{II,0}$ from w_{II} and writing it in order of ascending powers of (ζ/a) , we have

$$w_{II} - w_{II,0} = Ua e^{-i\beta} \left[(\alpha_1^2 \mu^2 + 3\alpha_2^2 \mu^4 + 5\alpha_3^2 \mu^6 + \dots) \left(\frac{\zeta}{a}\right) \right.$$

$$\left. - (\alpha_1 \alpha_2 \mu^3 + 10\alpha_2 \alpha_3 \mu^5 + 35\alpha_3 \alpha_4 \mu^7 + \dots) \left(\frac{\zeta}{a}\right)^3 \right.$$

$$\left. + (\alpha_1 \alpha_3 \mu^4 + 21\alpha_2 \alpha_4 \mu^6 + \dots) \left(\frac{\zeta}{a}\right)^5 - (\alpha_1 \alpha_4 \mu^5 + 36\alpha_2 \alpha_5 \mu^7 + \dots) \left(\frac{\zeta}{a}\right)^7 \right.$$

$$\left. + (\alpha_1 \alpha_5 \mu^6 + \dots) \left(\frac{\zeta}{a}\right)^9 - (\alpha_1 \alpha_6 \mu^7 + \dots) \left(\frac{\zeta}{a}\right)^{11} + \dots \right] \dots \dots (27).$$

(viii) Supplementary complex potential of the order of III about the central circular cylinder.

$$w_{III,0} = Ua e^{i\beta} \left[(\alpha_1^2 \mu^2 + 3\alpha_2^2 \mu^4 + 5\alpha_3^2 \mu^6 + \dots) \left(\frac{a}{\zeta}\right) \right.$$

$$\left. - (\alpha_1 \alpha_2 \mu^3 + 10\alpha_2 \alpha_3 \mu^5 + 35\alpha_3 \alpha_4 \mu^7 + \dots) \left(\frac{a}{\zeta}\right)^3 \right.$$

$$\left. + (\alpha_1 \alpha_3 \mu^4 + 21\alpha_2 \alpha_4 \mu^6 + \dots) \left(\frac{a}{\zeta}\right)^5 - (\alpha_1 \alpha_4 \mu^5 + 36\alpha_2 \alpha_5 \mu^7 + \dots) \left(\frac{a}{\zeta}\right)^7 \right.$$

$$\left. + (\alpha_1 \alpha_5 \mu^6 + \dots) \left(\frac{a}{\zeta}\right)^9 - (\alpha_1 \alpha_6 \mu^7 + \dots) \left(\frac{a}{\zeta}\right)^{11} + \dots \right] \dots \dots \dots (28)$$

Since the flow expressed by $(w_{II} - w_{II,0}) + w_{III,0}$ satisfies the boundary condition on the surface of the central circular cylinder ($\zeta = a e^{i\beta}$), it is confirmed that Eq.(28) is the complex potential mentioned in the title of this paragraph.

(ix) Supplementary complex potential of the order of III .

The supplementary complex potential of the order of II expressed in paragraph (vii) was obtained through paragraph (vi) from the complex potential expressed in paragraph (v). Similarly, the supplementary complex potential of the order of III can be obtained from the complex potential expressed in paragraph (viii).

In this way, the supplementary complex potential w_{III} , which is one order higher than the supplementary complex potential of the order of II , w_{II} , is obtained.

(x) Supplementary complex potential of the order of IV or more. Repeating the process described in the previous paragraph (ix), in which w_{III} is derived from w_{II} , we successively obtain the supplementary complex potential of the order of

$IV, V, VI, \dots$

(xi) Complex potential W . Since $\mu < 1$ as shown in Eq.(15), we take into account terms to the order of μ^7 and omit the terms less than the order of μ^8 . Then, it is adequate to take into account the supplementary complex potentials as far as $w_{VIII,0}$, and the complex potential in the ζ -plane, W , is expressed by the following Eq.(29). In other words, the supplementary complex potentials ($w_{VIII} - w_{VIII,0}$), w_{IX} , w_{X} , ... are less than the order of μ^8 .

$$W = W_0 + \sum_{N=1}^{\infty} w_N + w_{VIII,0} \dots \dots \dots (29)$$

Substituting $w_I, w_{II}, \dots, w_{VII}$ and $w_{VIII,0}$ in Eq.(29), and substituting μ by parameter λ defined by the following Eq.(31), we obtain the complex potential as follows:

$$W = \frac{Ua}{1-\lambda^2} K \left(\frac{\zeta}{a}, \lambda, \beta \right) \dots \dots \dots (30).$$

As aforesaid, we substitute μ by λ using the following relation:

$$\lambda = \frac{1}{3} \pi^2 \mu = \frac{1}{3} \left(\pi \frac{a}{t} \right)^2 \dots\dots\dots(31).$$

Dividing the right side of the following Eq.(38) by $a/(1-\lambda^2)$, we obtain the concrete expression of function K .

In this substitution, it is confirmed that μ^m always combines with α_m in the supplementary complex potentials, Eqs.(21)-(28). Now, Eq.(31) consequently gives

$$\alpha_m \mu^m = \frac{\alpha_m}{(\pi^2/3)^m} \lambda^m \dots\dots\dots(32).$$

Substituting Eq.(19) into Eq.(32), we express $\alpha_m \mu^m$ by means of λ^m multiplied by numerical constants:

$$\left. \begin{aligned} \alpha_1 \mu^1 &= \lambda, & \alpha_2 \mu^2 &= \frac{1}{5} \lambda^2, & \alpha_3 \mu^3 &= \frac{2}{35} \lambda^3, & \alpha_4 \mu^4 &= \frac{3}{175} \lambda^4 \\ \alpha_5 \mu^5 &= \frac{2}{385} \lambda^5, & \alpha_6 \mu^6 &= \frac{1}{875} \frac{382}{875} \lambda^6, & \alpha_7 \mu^7 &= \frac{12}{25} \frac{025}{025} \lambda^7, & \alpha_8 \mu^8 &= \frac{10}{74} \frac{851}{449} \frac{375}{375} \lambda^8 \end{aligned} \right\} \dots\dots\dots(33).$$

Considering that the value of λ is practically smaller than 0.5 when solidity is large and even nearly equal to 2, and that the numerical multiplier of λ^8 in Eq.(33) is of the order 10^{-4} , it is evident that the error in the supplementary complex potential taking account of the terms to the order of λ^7 is extremely small.

2.4 Mapping function

If we put the complex potential in the z -plane, Eq.(5), equal to the complex potential in the ζ -plane, Eq.(30), the functional relation between the complex numbers z and ζ is obtained.

$$Vz = \frac{Ua}{1-\lambda^2} K\left(\frac{\zeta}{a}, \lambda, \beta\right) \dots\dots\dots(34)$$

Now, displacement s parallel to the cascade axis at upstream infinity of the row of flat-plates in the z -plane corresponds to displacement t parallel to the η -axis at upstream infinity of the row of circular cylinders in the ζ -plane. Through these displacements, complex potential W increases as much as $Vs \exp\left\{i\left(\frac{\pi}{2}-\gamma\right)\right\}$ in the z -plane and increases as much as $Ut \exp\left\{i\left(\frac{\pi}{2}-\beta\right)\right\}$ in the ζ -plane.

Since these increments must be equal, we have

$$Vs \exp\left\{i\left(\frac{\pi}{2}-\gamma\right)\right\} = Ut \exp\left\{i\left(\frac{\pi}{2}-\beta\right)\right\} \dots\dots\dots(35),$$

so that

$$Vs = Ut$$

and

$$\gamma = \beta$$

Putting $U = V$, we obtain

$$t = s, \quad \beta = \gamma \dots\dots\dots(36)$$

Hence, Eq.(34) reduces to

$$z = \frac{a}{1-\lambda^2} K\left(\frac{\zeta}{a}, \lambda, \gamma\right) \dots\dots\dots(37)$$

and the concrete functional form is expressed by

$$\begin{aligned} z = \frac{a}{1-\lambda^2} \left[\left\{ \left(1 + \frac{3}{25} \lambda^4 + \frac{314}{1} \frac{\lambda^6}{225} \right) \left(\frac{\zeta}{a} \right) - \left(\frac{1}{5} \lambda^3 + \frac{4}{35} \lambda^5 + \frac{72}{875} \lambda^7 \right) \left(\frac{\zeta}{a} \right)^3 + \left(\frac{2}{35} \lambda^4 + \frac{9}{125} \lambda^6 \right) \left(\frac{\zeta}{a} \right)^5 \right. \right. \\ \left. \left. - \left(\frac{3}{175} \lambda^5 + \frac{72}{1} \frac{\lambda^7}{925} \right) \left(\frac{\zeta}{a} \right)^7 + \frac{2}{385} \lambda^6 \left(\frac{\zeta}{a} \right)^9 - \frac{1}{875} \frac{382}{875} \lambda^7 \left(\frac{\zeta}{a} \right)^{11} \right\} \times e^{-i\gamma} \right. \\ \left. + \left\{ \left(\lambda + \frac{6}{25} \lambda^5 + \frac{418}{1} \frac{\lambda^7}{225} \right) \left(\frac{\zeta}{a} \right) - \left(\frac{1}{5} \lambda^2 + \frac{121}{875} \lambda^6 \right) \left(\frac{\zeta}{a} \right)^3 + \left(\frac{2}{35} \lambda^3 + \frac{69}{875} \lambda^7 \right) \left(\frac{\zeta}{a} \right)^5 \right. \right. \\ \left. \left. - \frac{3}{175} \lambda^4 \left(\frac{\zeta}{a} \right)^7 + \frac{2}{385} \lambda^5 \left(\frac{\zeta}{a} \right)^9 - \frac{1}{875} \frac{382}{875} \lambda^6 \left(\frac{\zeta}{a} \right)^{11} + \frac{12}{25} \frac{025}{025} \lambda^7 \left(\frac{\zeta}{a} \right)^{13} \right\} \times e^{i\gamma} \right] \\ + \frac{a}{1-\lambda^2} \left[\left\{ \left(1 + \frac{3}{25} \lambda^4 + \frac{314}{1} \frac{\lambda^6}{225} \right) \left(\frac{a}{\zeta} \right) - \left(\frac{1}{5} \lambda^3 + \frac{4}{35} \lambda^5 + \frac{72}{875} \lambda^7 \right) \left(\frac{a}{\zeta} \right)^3 + \left(\frac{2}{35} \lambda^4 + \frac{9}{125} \lambda^6 \right) \left(\frac{a}{\zeta} \right)^5 \right. \right. \\ \left. \left. - \left(\frac{3}{175} \lambda^5 + \frac{72}{1} \frac{\lambda^7}{925} \right) \left(\frac{a}{\zeta} \right)^7 + \frac{2}{385} \lambda^6 \left(\frac{a}{\zeta} \right)^9 - \frac{1}{875} \frac{382}{875} \lambda^7 \left(\frac{a}{\zeta} \right)^{11} \right\} \times e^{i\gamma} \right] \end{aligned}$$

$$+ \left\{ \left(\lambda + \frac{6}{25} \lambda^5 + \frac{418}{1225} \lambda^7 \right) \left(\frac{a}{\zeta} \right) - \left(\frac{1}{5} \lambda^2 + \frac{121}{875} \lambda^6 \right) \left(\frac{a}{\zeta} \right)^3 + \left(\frac{2}{35} \lambda^3 + \frac{69}{875} \lambda^7 \right) \left(\frac{a}{\zeta} \right)^5 - \frac{3}{175} \lambda^4 \left(\frac{a}{\zeta} \right)^7 + \frac{2}{385} \lambda^5 \left(\frac{a}{\zeta} \right)^9 - \frac{1}{875} \frac{382}{875} \lambda^6 \left(\frac{a}{\zeta} \right)^{11} + \frac{12}{25} \frac{025}{025} \lambda^7 \left(\frac{a}{\zeta} \right)^{13} \right\} \times e^{-i\gamma} \dots (38),$$

where, as mentioned before, the terms of the order of λ^8 are omitted.

Using Eq.(38), we consider the correspondence between the points on one of the flat-plates and the points on one of the circular cylinders. Then, if we consider only the correspondence between the central flat-plate and the central circular cylinder, we can come to a general conclusion. The point on the flat-plate is expressed by $z = x$, and the point on the circular cylinder is expressed by $\zeta = a e^{i\vartheta}$.

Substituting $\zeta = a e^{i\vartheta}$ into the right side of Eq.(38) and also substituting z by x , we obtain Eq.(39). Here, as is shown in the left side of Fig.3, we suggest the argument θ defined by

$$x = \frac{1}{2} l \cos \theta$$

where $0 \leq \theta \leq \pi$ for the points on the upper side of the flat-plate and $-\pi \leq \theta \leq 0$ for the points on the lower side of the flat-plate.

$$\frac{1}{2} l \cos \theta = x$$

$$= \frac{2a}{1-\lambda^2} \left\{ \left(1 + \frac{3}{25} \lambda^4 + \frac{314}{1225} \lambda^6 \right) \cos(\vartheta - \gamma) - \left(\frac{1}{5} \lambda^3 + \frac{4}{35} \lambda^5 + \frac{72}{875} \lambda^7 \right) \cos(3\vartheta - \gamma) + \left(\frac{2}{35} \lambda^4 + \frac{9}{125} \lambda^6 \right) \cos(5\vartheta - \gamma) - \left(\frac{3}{175} \lambda^5 + \frac{72}{1925} \lambda^7 \right) \cos(7\vartheta - \gamma) + \frac{2}{385} \lambda^6 \cos(9\vartheta - \gamma) - \frac{1}{875} \frac{382}{875} \lambda^7 \cos(11\vartheta - \gamma) \right\} + \left\{ \left(\lambda + \frac{6}{25} \lambda^5 + \frac{418}{1225} \lambda^7 \right) \cos(\vartheta + \gamma) - \left(\frac{1}{5} \lambda^2 + \frac{121}{875} \lambda^6 \right) \cos(3\vartheta + \gamma) + \left(\frac{2}{35} \lambda^3 + \frac{69}{875} \lambda^7 \right) \cos(5\vartheta + \gamma) - \frac{3}{175} \lambda^4 \cos(7\vartheta + \gamma) + \frac{2}{385} \lambda^5 \cos(9\vartheta + \gamma) - \frac{1}{875} \frac{382}{875} \lambda^6 \cos(11\vartheta + \gamma) + \frac{12}{25} \frac{025}{025} \lambda^7 \cos(13\vartheta + \gamma) \right\} \dots (39)$$

2.5 Point on the circular cylinder corresponding to the trailing edge of the flat-plate, and solidity

Since the point on the circular cylinder is expressed by $\zeta = a e^{i\vartheta}$, the argument ϑ expresses the location of the point on the circular cylinder. Then, in the ζ -plane, let ϑ_T be the argument which corresponds to the trailing edge of the flat-plate.

We obtain the differential coefficient $dx/d\vartheta$ from Eq.(39); set it equal to zero; and substitute ϑ by ϑ_T . If we solve this equation, $dx/d\vartheta = 0$, ϑ_T is derived as in the following Eq.(40). For want of space, this equation is described in terms of the order of λ^6 :

$$(\gamma - \vartheta_T) = \sin 2\vartheta_T \lambda - \frac{1}{10} \sin 4\vartheta_T \lambda^2 - \left(\frac{3}{5} \sin 2\vartheta_T - \frac{2}{105} \sin 6\vartheta_T \right) \lambda^3 + \left(\frac{2}{7} \sin 4\vartheta_T - \frac{3}{700} \sin 8\vartheta_T \right) \lambda^4 + \left(\frac{53}{175} \sin 2\vartheta_T - \frac{3}{25} \sin 6\vartheta_T + \frac{2}{1925} \sin 10\vartheta_T \right) \lambda^5 - \left(\frac{39}{350} \sin 4\vartheta_T - \frac{18}{385} \sin 8\vartheta_T + \frac{691}{2627625} \sin 12\vartheta_T \right) \lambda^6 \dots (40).$$

Putting $\vartheta = \vartheta_T$ and $x = \frac{1}{2} l$ into Eq.(39) and substituting γ by ϑ_T and λ , we obtain

$$\frac{l}{4a} = \frac{1}{(1-\lambda^2)} \left\{ 1 + \cos 2\vartheta_T \lambda - \left(\frac{3}{4} - \frac{11}{20} \cos 4\vartheta_T \right) \lambda^2 - \left(\frac{1}{8} \cos 2\vartheta_T + \frac{1}{56} \cos 6\vartheta_T \right) \lambda^3 + \left(\frac{1}{1600} \right. \right. \\ \left. \left. - \frac{61}{112} \cos 4\vartheta_T + \frac{213}{11200} \cos 8\vartheta_T \right) \lambda^4 - \left(\frac{81}{1600} \cos 2\vartheta_T - \frac{3797}{22400} \cos 6\vartheta_T + \frac{1261}{246400} \cos 10\vartheta_T \right) \lambda^5 \right. \\ \left. - \left(\frac{137}{313600} - \frac{323}{448000} \cos 4\vartheta_T + \frac{1299}{140800} \cos 8\vartheta_T - \frac{707921}{448448000} \cos 12\vartheta_T \right) \lambda^6 \right\} \dots (41).$$

If we substitute t by s in Eq.(31) and extract its square root, we obtain

$$\frac{l}{s} = \left(\frac{4\sqrt{3}}{\pi} \sqrt{\lambda} \right) \left(\frac{l}{4a} \right) \dots\dots\dots(42).$$

Then, substituting Eq.(41) into Eq.(42), we can represent solidity l/s as the function of λ and $\mathcal{A}_T$.

We obtain the numerical relation between the stagger angle γ and the argument $\mathcal{A}_T$ using Eq.(40), and this relation is plotted for λ as a parameter in Fig.4. We also obtain the numerical relation between angle γ and solidity l/s using Eqs.(40)-(42), and in Fig.5 this relation is plotted for λ as a parameter. Thus, when the arrangements of the row of flat-plates ($\gamma, l/s$) are given, we can obtain the corresponding values of λ and $\mathcal{A}_T$.

3. Mapping Function for Cascade of Arbitrary Blades

Substituting Eq.(41) into Eq.(38), we obtain

$$z = \frac{1}{4} l \sum_{k=0}^{\infty} \left[A_{2k+1} \left(\frac{\zeta}{a} \right)^{2k+1} + \bar{A}_{2k+1} \left(\frac{a}{\zeta} \right)^{2k+1} \right] \dots\dots\dots(43),$$

where $\bar{A}_{2k+1}$ is the conjugate complex number of A_{2k+1} . When we take into account the terms to the order of λ^7 , it is enough to consider the sum from k equals zero to six. As an example of A_{2k+1} , A_3 is described in Eq.(44) to the order of λ^4 :

$$A_3 = \left\{ -\frac{1}{5} \lambda^3 (1 - \cos 2\mathcal{A}_T \lambda) \right\} e^{-i\gamma} - \frac{1}{5} \lambda^2 \left\{ 1 - \cos 2\mathcal{A}_T \lambda + \left(\frac{5}{4} - \frac{1}{20} \cos 4\mathcal{A}_T \right) \lambda^2 \right\} e^{i\gamma} \dots\dots(44).$$

Now, the mapping function for arbitrarily shaped blades in cascade into a row of circular cylinders is proposed as

$$z = \frac{1}{4} l \sum_{k=0}^{\infty} \left[A_{2k+1} \left(\frac{\zeta}{a} \right)^{2k+1} + \bar{A}_{2k+1} \left(\frac{a}{\zeta} \right)^{2k+1} \right] + l \sum_{K=0}^{\infty} C_K \sum_{n=-\infty}^{+\infty} \left(\frac{a}{\zeta - n\zeta_s} \right)^K \dots\dots\dots(45),$$

where the complex coefficients C_K are determined from the shape of the blades and the arrangements of the cascade.

4. Distortion in Transformation of Row of Flat-plates into Row of Circular Cylinders

When the value of $\log|dz/d\zeta|$ is not zero, it is said that there is distortion in transformation, and the distortion becomes large as this value differs from zero. The transformation presented in this paper has the advantage of $|dz/d\zeta| = 1$, namely, $\log|dz/d\zeta| = 0$ at upstream infinity and downstream infinity of the row of flat-plates and the row of circular cylinders. However, distortion is largest on the flat-plates or on the circular cylinders.

In order to represent the location of the point on a circular cylinder, the argument θ , defined by Eq.(46), is used instead of argument $\mathcal{A}$:

$$\theta = \mathcal{A} - \mathcal{A}_T \dots\dots\dots(46).$$

Here, let ε be the difference between the above-mentioned argument θ and the argument θ , which represents the location of the point on a flat-plate:

$$\varepsilon = \theta - \theta \dots\dots\dots(47).$$

We can expand ε as

$$\begin{aligned} \varepsilon = & c_2 \cos 2\theta + c_4 \cos 4\theta + c_6 \cos 6\theta \\ & + s_2 \sin 2\theta + s_4 \sin 4\theta + s_6 \sin 6\theta \\ & - c_2 - c_4 - c_6 \dots\dots\dots(48), \end{aligned}$$

Fig.4 Relation between γ and $\mathcal{A}_T$.

Fig.3 Correspondence between z -plane and ζ -plane.

Fig.5 Relation between l/s and γ .

where the coefficients $c_2, s_2, c_4, s_4, c_6, s_6$ are calculated from Eq.(39) and are expressed in Eq.(49).

$$\left. \begin{aligned} c_2 &= -\frac{2}{5} \sin 4\vartheta_T \lambda^2 + \frac{4}{35} \sin 6\vartheta_T \lambda^3 \\ &\quad + \frac{8}{175} \sin 8\vartheta_T \lambda^4 \\ s_2 &= -\frac{2}{5} \cos 4\vartheta_T \lambda^2 + \frac{4}{35} \cos 6\vartheta_T \lambda^3 \\ &\quad + \left(-\frac{6}{25} + \frac{8}{175} \cos 8\vartheta_T\right) \lambda^4 \\ c_4 &= \frac{4}{35} \sin 6\vartheta_T \lambda^3 - \frac{13}{175} \sin 8\vartheta_T \lambda^4 \\ s_4 &= \frac{4}{35} \cos 6\vartheta_T \lambda^3 - \frac{13}{175} \cos 8\vartheta_T \lambda^4 \\ c_6 &= -\frac{6}{175} \sin 8\vartheta_T \lambda^4 \\ s_6 &= -\frac{6}{175} \cos 8\vartheta_T \lambda^4 \end{aligned} \right\} \quad (49)$$

As an example, the relation between ε and θ is plotted with a solid line in Fig.6 concerning the row of flat-plates whose stagger angle and solidity are 60° and 1.667 respectively. In the same figure, ε - θ curves obtained by using the other transformation methods are also plotted; the broken line represents the case of the transformation of a cascade into two concentric circular cylinders; the dot-dash-line represents the case of the transformation of a cascade into a single circular cylinder. Also, in the case of the latter two methods, the relations $\varepsilon = \theta - \theta$ and $\theta = \vartheta - \vartheta$ are used, and the argument ϑ represents the location of the point on the cir-

Fig.6 Distortion in transformation.
($\gamma = 60^\circ$, $L/s = 1.667$)

cular cylinder. If we make a comparison between these three curves, it is clear that the distortion in the transformation of a row of flat-plates into a row of circular cylinders is much smaller than in the other two methods.

If we carry out the transformation of cascade of arbitrary blades by Eq.(45), the analysis is easily performed because of the small value of ε which indicates that the shape of the curve used for Fourier analysis is similar to the contour describing the blade profile in Cartesian coordinates. On the contrary, when we transform a cascade into a single circular cylinder, the shape of the curve used for Fourier analysis sometimes becomes similar to a rectangular wave; the convergence of the higher orders of the Fourier series becomes poor; and analysis becomes impossible in practice.

5. Conclusions

A new function, Eq.(43), which conformally transforms a row of flat-plates into a row of circular cylinders, is introduced in this paper. Further, it is shown that by using the mapping function, Eq.(45), which is derived on the basis of the above-mentioned new function, the conformal transformation of cascade of arbitrary blades is easily performed and the flow through the cascade whose stagger angle and solidity are large can be obtained much more accurately than through any other conformal transformation method. Numerical examples will be reported at another time.

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