Out-of-plane Vibration of Arc Bar of Variable Cross-section*

By

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The free out-of-plane vibration of an arc bar of variable cross-section is analyzed by use of the transfer matrix approach. For this purpose, the equations of out-of-plane vibration of an arc bar are written as a coupled set of first-order differential equations by using the transfer matrix of the bar. Once the matrix has been determined by the numerical integration of the equations, the natural frequencies and the mode shapes of the vibration can be calculated numerically in terms of the elements of the matrix for a given set of boundary conditions at the ends of the bar. This method is applied to bars of linearly, parabolically and exponentially varying rectangular cross-sections, and the effects of the varying cross-section and slenderness ratio are studied.

1. Introduction

Early investigations of out-of-plane vibration of uniform circular arc bars perpendicular to the plane of initial curvature of the bars are done by Love1 and Federhofer2, who derived the fundamental equations governing the vibration. Later, Takahashi3 studied the out-of-plane vibration of free-free arc bars and Chang and Volterra4 studied clamped-clamped arc bars. Volterra and Morel5 obtained the fundamental frequencies of curved beams having the center line in the forms of a cycloid, catenary or parabola, and also Takahashi and Suzuki6 studied elliptically curved beams. However, these studies have been all confined to uniform beams, and no papers have been presented on arc bars with variable cross-section.

This paper presents an analysis of the free out-of-plane vibration of arc bars having a cross-section expressed by an arbitrary function, in which the transfer matrix method is used. For this purpose, the governing equations of out-of-plane vibration of an arc bar are written as a coupled set of first-order differential equations. By introducing the transfer matrix of the bar, these equations can be expressed in a matrix differential equation. Once the transfer matrix has been determined by integrating numerically the matrix equation, the natural frequencies and the mode shapes of vibration are calculated in terms of the elements of the matrix for a given set of boundary conditions at the ends.

2. Fundamental Equations

We consider a circular arc bar of radius R with variable cross-section.

With the angular co-ordinate denoted by \( \theta \) and the opening angle by \( \phi \), the \( r \)-, \( y \)- and \( \phi \)-axes are taken in radial, transverse and tangential directions, respectively, as shown in Fig.1. Assuming that the shear center of the cross-section coincides with the centroid and not taking into account the shear deformation, we can write the equations of out-of-plane vibration of the bar as

\[
\begin{align*}
\frac{dM_\phi}{d\theta} + \phi A(\phi) R \psi \frac{d\phi}{d\theta} & = 0 \\
\frac{dT^*}{d\theta} - RQ^* & = 0 \\
\frac{dM_*}{d\theta} + \phi J(\phi) R \psi & = 0 \\
\frac{dT^*}{d\theta} - M_* & = 0
\end{align*}
\]

where \( \rho \) is the mass per unit volume, \( A(\phi) \) is the cross-sectional area of the bar and \( \phi \) is the circular frequency. The vari-

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able $G^*$ denotes the shearing force. The bending moment $M^*$ and torsional moment $T^*$, respectively, are given by

$$
M^* = -\frac{E_1}{R^2} \left( \frac{dv}{dy} - \frac{R}{R^2} \varphi \right)
$$

$$
T^* = \frac{G_1}{R} \left( \frac{dv}{dy} + \frac{2R}{dy} \varphi \right)
$$

(2)

in terms of the transverse deflection $v^*$ and the angle of torsion $\varphi$. Here, $E_1(\theta)$ is the bending rigidity and $G_1(\theta)$ is the torsional rigidity. The slope $\phi$ of the bar is expressed as

$$
\phi = -\frac{1}{R} \frac{dv}{dy}
$$

(3)

Equations (1)-(3) can be written in a matrix differential equation as follows:

$$
\begin{bmatrix}
\frac{d}{dy}v \\
\frac{d}{dy}\phi \\
\frac{d}{dy}\varphi
\end{bmatrix}
= \begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & 1 \\
-2A(\theta) & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & \frac{1}{S_0(\theta)} & 0 \\
0 & 0 & \frac{1}{S_0(\theta)}
\end{bmatrix}
\begin{bmatrix}
v \\
\phi \\
\varphi
\end{bmatrix}
$$

(4)

For simplicity of the analysis, the following dimensionless variables have been introduced:

$$
\begin{align*}
\varsigma &= \frac{v}{R^2}, \\
Q_r &= \frac{R^2}{E_1(\theta)}G^*, \\
(M_r, T) &= \frac{R}{E_1(\theta)}(M^*, T^*)
\end{align*}
$$

(5)

where $A_r$ is the cross-sectional area, $I_m$ are the second moments of area, $J_\theta$ is the polar moment of area at one end ($\vartheta = 0$). The quantities $S_0$ and $S_\eta$ express the slenderness ratios of the bar at the end, and $\lambda$ denotes the frequency parameter.

3. Analysis by Use of the Transfer Matrix Approach

Since the analytical solution can not be obtained for a non-uniform arc bar, the transfer matrix approach is adopted here. For this purpose, Eq. (4) is written as

$$
\frac{d}{d\vartheta}[\xi(\vartheta)] = [M(\vartheta)]\xi(\vartheta)
$$

(6)

by using the state vector $[\xi(\vartheta)] = [v \phi \varphi Q_r M_r T_r]$. and the coefficient matrix $[M(\vartheta)]$ given in Eq. (4). In general, $[\xi(\vartheta)]$ can be expressed as

$$
[\xi(\vartheta)] = [T(\vartheta)]\xi(0)
$$

(7)

by using the transfer matrix $[T(\vartheta)]$. From Eqs. (6) and (7), the following equation is derived:

$$
\frac{d}{d\vartheta}[T(\vartheta)] = [M(\vartheta)] [T(\vartheta)]
$$

(8)

For an arc bar with variable cross-section, the matrix $[T(\vartheta)]$ is obtained by integrating Eq. (8) numerically with the starting value

$$
[T(0)] = [I] \quad \text{(unit matrix)}
$$

(9)

which is given by taking $\vartheta = 0$ in Eq. (7). In the calculation, the elements of the transfer matrix are determined by using the Runge-Kutta-Gill integration method.

The present method can be applied to any combination of boundary conditions of an arc bar. Here, two examples will be explained:

1. A free-clamped arc bar
   When one end ($\vartheta = 0$) of the bar is clamped

$$
v = 0, \quad \phi = \varphi = 0
$$

(10)

and Eq. (7) is written as

$$
\begin{align*}
\xi &= \begin{bmatrix} T_{0r} & T_{0s} & T_{0t} \\
T_{sr} & T_{sr} & T_{sr} \\
T_{tr} & T_{tr} & T_{tr}
\end{bmatrix}
\end{align*}
$$

(11)

with only the elements of $[T(\vartheta)]$ necessary for the calculation.

2. The free end ($\vartheta = a$), the boundary conditions

$$
Q_r = 0, \quad M_r = T = 0
$$

(12)

must be satisfied, and therefore the following frequency equation is obtained:

$$
\begin{align*}
0 &= T_{ar} T_{sr} T_{tr} \\
0 &= T_{ar} T_{sr} T_{sr} \\
0 &= T_{ar} T_{sr} T_{sr}
\end{align*}
$$

(13)

Since $[M(\vartheta)]$ depends upon the frequency parameter $\lambda$ in addition to $\vartheta$, $[T(\vartheta)]$ is also a function of $\lambda$. Equation (13) gives a set of linear homogeneous equations with respect to unknown coefficients $Q_r, M_r$ and $T$. The natural frequencies of the bar are determined by calculating the eigenvalues $\lambda$ of Eq. (13), and the mode shapes of the vibration are determined by calculating the eigenvectors $[Q_r, M_r, T] \omega^2$, corresponding to the eigenvalues.
(2) A clamped-clamped arc bar
For an arc bar with clamped ends, the following equation is derived by the same procedure as in example (1):

\[
\begin{bmatrix}
0 & T_n & T_n & T_n & Q_n \\
0 & -T_n & T_n & T_n & M_n \\
0 & T_n & -T_n & T_n & 0 \\
0 & T_n & T_n & -T_n & 0
\end{bmatrix} \begin{bmatrix}
T_n \\
T_n \\
T_n \\
T_n
\end{bmatrix} = \begin{bmatrix}
Q_n \\
M_n \\
Q_n \\
M_n
\end{bmatrix} = 0
\]

\[\theta = \alpha (m, n > 0) \ldots (14)\]

4. Numerical Calculation and Discussion

In this section, the present method is applied to some arc bars with varying rectangular cross-section, and the natural frequencies and the mode shapes of the vibration are calculated numerically.

(A) An arc bar with breadth and depth varied as power functions
Consider an arc bar whose breadth and depth, respectively, are expressed as

\[
b(\theta) = b_0 - (b_0 - b_1) \left( \frac{\theta}{\alpha} \right)^n \]

\[
h(\theta) = h_0 - (h_0 - h_1) \left( \frac{\theta}{\alpha} \right)^n \quad (m, n > 0) \ldots (15)
\]

where \( b_0, h_0 \) denote the breadth and depth at one end and \( b_1, h_1 \) denote the breadth and depth at the other end, respectively. In this case, \( a(\theta) \), \( i(\theta) \) and \( j(\theta) \) are written as

\[
a(\theta) = \left[ 1 - \left( \frac{b_1}{b_0} \right)^n \right] \left[ 1 - \left( \frac{h_1}{h_0} \right)^n \right]^{-1}
\]

\[
i(\theta) = \left[ 1 - \left( \frac{b_1}{b_0} \right)^n \right] \left[ 1 - \left( \frac{h_1}{h_0} \right)^n \right]^{-1}
\]

\[
j(\theta) = \frac{1}{1 + \left( \frac{b_1}{b_0} \right)^n} \left[ 1 - \left( \frac{b_1}{b_0} \right)^n \right] \left[ 1 - \left( \frac{h_1}{h_0} \right)^n \right]^{-1}
\]

\[+ \frac{1}{1 + \left( \frac{b_1}{b_0} \right)^n} \left[ 1 - \left( \frac{b_1}{b_0} \right)^n \right] \left[ 1 - \left( \frac{h_1}{h_0} \right)^n \right]^{-1} \ldots (16)
\]

The breadth and depth vary linearly for \( m = n = 1 \), and parabolically for \( m = n = 2 \).

(B) An arc bar with exponentially varying rectangular cross-section
When the breadth and depth are expressed as

\[b(\theta) = b_0 \left( \frac{b_1}{b_0} \right) e^{-\alpha \theta}, \quad h(\theta) = h_0 \left( \frac{h_1}{h_0} \right) e^{-\alpha \theta} \ldots (17)\]

\[a(\theta), i(\theta) \] and \( j(\theta) \) are written as

\[
a(\theta) = \left( \frac{b_1}{b_0} \right) e^{-\alpha \theta}, \quad i(\theta) = \left( \frac{b_1}{b_0} \right) e^{-\alpha \theta}
\]

\[
j(\theta) = \frac{1}{1 + \left( \frac{b_1}{b_0} \right) e^{-\alpha \theta}} \left[ 1 - \left( \frac{b_1}{b_0} \right) e^{-\alpha \theta} \right]^{-1}
\]

\[
+ \frac{1}{1 + \left( \frac{b_1}{b_0} \right) e^{-\alpha \theta}} \left[ 1 - \left( \frac{b_1}{b_0} \right) e^{-\alpha \theta} \right] \ldots (18)
\]

All of these quantities have a unit value for an arc bar with uniform cross-section.

In the following, the numerical results are shown for arc bars with square cross-section at one end ( \( \theta = 0 \) ).

In Table 1, the frequency parameters of a clamped-clamped uniform arc bar obtained by the method are compared with the results obtained by Chang and Volterra using the Ritz method without taking the rotary inertia into account. In general, the frequency parameters obtained using the Ritz method are higher than the exact values, and therefore the results obtained here are considered to be sufficiently

| Table 1 Frequency parameters of a clamped-clamped uniform arc bar \( (\alpha = 0.3, \alpha = 80°, S_0 = S_0 = 20°) \) |
|---|---|---|---|---|
|\( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) |
| Present method | 3.259 | 5.476 | 5.619 | 7.644 |
| Chang et al. | 3.294 | 5.477 | 5.627 | 7.658 |

![Fig. 2 Frequency parameters of free-clamped arc bars with variable cross-section \( (\alpha = 80°, S_0 = S_0 = 20°) \) ]

(a) Straight line
(b) \( b_1/b_0 = 1/4 \)
(c) \( h_1/h_0 = 1 \)
accurate. One can obtain accurate values in practice, if the transfer matrix \[ T(\bar{b}) \] is integrated numerically with fifty steps at most on \([0,1]\), which takes no plenty of time.

Figures 2 and 3 show the frequency parameters of free-clamped arc bars with slenderness ratio \( S_{\alpha}=S_{\beta}=50 \) and the opening angles \( \alpha = 90^\circ \) and \( 180^\circ \), respectively. Figures (a) of these figures show the frequency parameters of tapered bars versus the depth ratio \( h/h_a \), where the breadth ratio \( b_i/b_e \) is taken as a parameter. The frequency parameters become smaller, with an increase of the ratio \( h/h_a \), and the values increase monotonically except for the lowest modes, with an increase of the ratio \( h_i/h_e \).

Figures (b) show the frequency parameters of bars with uniform breadth and linearly, parabolically or exponentially varying depth. The bar of \( h_i/h_e=1 \) represents a uniform bar. For the bars of \( h_i/h_e<1 \), the parameters of bar with exponentially varying depth are the smallest, and the parameters increase in the order of bars with linearly and parabolically varying depths. On the contrary, the parameters of bars of \( h_i/h_e>1 \) increase in the order of bars with parabolically exponentially and linearly varying depths. These orders are the same as those of the depth of bars for the specified ratio of \( h_i/h_e \), as shown in Fig.4. This means physically that the effect of an increase of the flexural rigidity is larger than that of the mass. This tendency is more remarkable in higher modes.

Figures (c) show the frequency parameters of bars with uniform depth and non-uniform breadth expressed by the above-mentioned three functions. In general, these values become smaller, with an increase of the ratio \( h_i/h_e \). The parameters hardly depend upon the function expressing the property of the cross-section.

Figure 5 shows the frequency parameters of free-clamped quadrangular pyramid-shaped arc bars with square cross-section.
Fig. 6 Frequency parameters of free-clamped arc bars with tapered depth (r = 1.3, h₀/hₐ = 2, S₀ = Sₐ = 20)

For various slenderness ratios. The parameters of bars with an infinite slenderness ratio increase monotonically except for the lowest modes, with an increase of the breadth and depth ratios. In general, the parameters become smaller with a decrease of the slenderness ratio. This tendency is also more remarkable in higher modes.

Figure 6 shows the frequency parameters of bars with uniform breadth and tapered depth versus the opening angle. With an increase of the angle, the parameters decrease monotonically. With an increase of the ratio h₀/hₐ, the values increase except for the lowest modes (see Figs. 2 and 3) and also for the angle in a certain range shown on the top of Fig. 6.

Figures 7 show the frequency parameters of clamped-clamped bars with the same dimensions as free-clamped bars shown in Fig. 2. In this case, the parameter λₚ of the torsion-type (T-type) vibration where the angle of torsion is dominant in value appear in the vicinity of the value λ, in addition to the parameters λ₁, λ₄, λ₆ of the usual bending-type (B-type) vibration. As shown in Fig. (a), the parameters of the B-type vibration become larger with an increase of the ratio h₀/hₐ, and become slightly larger with an increase of the ratio h₀/hₐ. On the contrary, the parameters of the T-type vibration become smaller, with an increase of the ratio h₀/hₐ. As shown in Fig. (b), the parameters of the B-type vibration of the bars of h₀/hₐ < 1 become larger in the order of bars with exponentially, linearly and parabolically varying depths. Meanwhile the order of the parameters λₚ is opposite to that of these values. For the bars with uniform depth, the parameters of the B-type vibration are not so much affected by the ratio h₀/hₐ, but those of the T-type vibration are slightly affected by it, as shown in Fig. (c).

Figure 8 shows the frequency parameters and the mode shapes of clamped-clamped quadrangular pyramid-shaped arc bars with the slenderness ratio S₀ = Sₐ = 20, having the same dimensions as the free-clamped bars shown in Fig. 5. In general, the parameters of the B-type vibration become...
larger, with an increase of the ratios \( b_1/b_0 = h_1/h_0 \). The effects of the slenderness ratio are remarkable in higher modes, though they are not so large in lower modes. In the frequency range presented in this figure, the lowest parameters \( \lambda_1 \), \( \lambda_2 \), and \( \lambda_3 \) of the T-type vibration appear in the bars with the ratio \( S_{10} = S_{20} = 20 \), and the parameters \( \lambda_4 \), \( \lambda_5 \), \( \lambda_6 \) of the T-type vibration appear in the bar with the ratio \( S_{10} = S_{20} = 50 \). The parameters of the vibration of this type are not so much affected by the ratio \( b_1/h_0 \) and \( b_2/h_2 \). In the figures showing the mode shapes, the maximum deflection \( \varphi \) of the bars is taken to have a unit value for both of the B- and T-type vibrations. It is found from these figures that the transverse deflection is dominant in the B-type vibration, while the angle of torsion is dominant in the T-type vibration.

5. Conclusions

The free out-of-plane vibration of arc bars with variable cross-section has been studied by using the transfer matrix approach.

The governing equations of out-of-plane vibration of a non-uniform arc bar have been expressed in a matrix differential equation by using the transfer matrix of the bar, the elements of which are calculated numerically by the Runge-Kutta-Gill integration method. From the elements of the transfer matrix thus obtained and the boundary conditions, the natural frequencies and the mode shapes have been determined numerically.

The frequency parameters of uniform arc bars obtained by the method have been compared with the values obtained by other authors and found to be in good agreement with the latter.

The frequency parameters of free-clamped or clamped-clamped arc bars with linearly, parabolically and exponentially varying rectangular cross-section were calculated numerically by applying the present method, from which the effects of the slenderness ratio and varying cross-section on the vibration have been clarified quantitatively.

Examples of the vibration mode shapes of clamped-clamped arc bars have been shown in figures.

This method is very simple and clear as an analytical process, and has significant computational advantages.

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