Simulated Analysis about Swing Motion of Lower Limb Prosthesis
with Knee-angle Feedback Device

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In this paper, stability has been considered of a prosthetic gait obtained
with an artificial knee-angle feedback system.
For a biped gait, the swing leg has a more important role than the
stance leg. But, in an above-knee prosthetic gait, the learning control
of a prosthetic knee is the most difficult task for an amputee. Then,
swing motion was observed, and its performance was described with four
parameters.

Based upon the theory of learning control, simulated experiments
were performed, assuming the process of amputee's progress in his gait to
be an optimal feedback gain adjusting process.
From simulated experiments, validity of our modeling and analysis
was confirmed.

1. Introduction

In order to improve the conventional lower prosthesis and make it alive and
physiological, and thus give it an adaptive capability to respond to environmental
walking conditions, we intend to supplement some useful sensory feedback devices to a
lower prosthesis. The authors for the first time tried to take up this problem and
successfully developed a practically useful prostheses with artificial sensory
devices, and we call such a total hardware system "the biological feedback control
prosthesis"[1]. A series of experimental works have been done in parallel to some
mathematical research works based upon the theory of learning control in order to
describe the stabilizing process of prosthetic walking synergies in the course of rehabilita-
tional training.

As stated above, stability has been considered of a prosthetic gait obtained
with our artificial knee-angle feedback system in this paper. Learning control of
a prosthetic knee to walk is the most diff-
cult task for an amputee. Therefore,
prosthetic swing motion was observed mainly
in this paper, and its performance was
described with four parameters.

Then, based upon the theory of learning control, simulated experiments were
performed, assuming the process of amputee's progress in his gait to be an optimal feed-
back gain adjusting process in his motor
control system. Furthermore, stochastic
fluctuation has been superposed on a model
in this paper considering uncertainty intro-
duced into gait state variable in each gait
cycle.

From simulated experiments, the validity of our modeling and analysis was
confirmed.

2. Derivation of dynamic model

We will derive a dynamic model for the swing phase behavior of an above knee pro-
thesis as shown in Fig.1(b), in which a carriage substitutes the body and double
pendulums substitute the residual thigh and a prosthesis. In the model, friction at a
knee joint is neglected and dynamics is restricted to a sagittal plane.
The equation of motion for the system in Fig.1(b), can be derived by Lagrange's
method, and the total kinetic energy is expressed as,

\[
\tau = \frac{1}{2} \int (f_1 \frac{d\theta_1}{dt} + \frac{1}{2} l_1 (\theta_2 \frac{d\theta_2}{dt})^2 + \frac{1}{2} l_2 \frac{d\theta_2}{dt})^2 + \frac{1}{2} \frac{d\theta_1}{dt})^2 + \frac{1}{2} \frac{d\theta_2}{dt})^2 + \frac{1}{2} \frac{d\theta_3}{dt})^2 + \frac{1}{2} \frac{d\theta_4}{dt})^2 + l_1 \frac{d\theta_2}{dt}) \cos \theta_1 + \frac{1}{2} \frac{d\theta_3}{dt}) \cos \theta_2 + \frac{1}{2} \frac{d\theta_4}{dt}) \cos \theta_3
\]

(1)

Similarly, the total potential energy is,

\[
U = -mg \frac{d\theta_1}{dt} \cos \theta_1 - \frac{1}{2} g \cos \theta_1 + \frac{1}{2} \cos \theta_2 \cos \theta_3
\]

(2)

Then, the equation of motion is obtained by Lagrange function, \(L=\mathcal{T}-\mathcal{U}\), as,

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System motion is described with three non-linear second-order differential equations. Such a high order model is, however, very complex to handle, so some simplification to render a problem tractable is necessary.

We first linearize equations via

\[ \dot{\theta}_i, \ddot{\theta}_i, \ddot{x}_i = (0, 0, 0, 0) \]

and next eliminate the differential terms of \( x \). Therefrom, we have the following two differential equations.

\[ \begin{align*}
\dot{\theta}_1 &= b_1 \dot{\theta}_1 + b_2 \dot{\theta}_2 + b_3 \dot{\theta}_3 + b_4 \theta_1 - \ddot{x}_1 \\
\dot{\theta}_2 &= b_5 \dot{\theta}_1 + b_6 \dot{\theta}_2 + b_7 \dot{\theta}_3 - M_1 \ddot{x}_1
\end{align*} \]

and the relation, \( \theta_1 = \Theta_1 \theta_2 \) is employed in eq. (b).

Next, we will define the following converted variables, \( \tau \) as,

\[ \tau = \frac{1}{\sqrt{\dot{\theta}_1^2 + \dot{\theta}_2^2}} \times (K_1^2 + \dot{x}^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2) \]

Assuming that the notation "-" means differentiation by \( \tau \) instead of real time, \( t \), we obtain our ultimate equations as,

\[ \begin{align*}
\ddot{\theta}_1 &= b_1 \dot{\theta}_1 \theta_2 + b_2 \dot{\theta}_2 \theta_2 + b_3 \dot{\theta}_3 \theta_2 + b_4 \theta_1 - \ddot{x}_1 \\
\ddot{\theta}_2 &= b_5 \dot{\theta}_1 \theta_2 + b_6 \dot{\theta}_2 \theta_2 + b_7 \dot{\theta}_3 \theta_2 + b_4 \theta_1 - \ddot{x}_1
\end{align*} \]

Equation (6) is rewritten in compact form utilizing vector notation such as,

\[ x = A \theta + b \theta^T + c \dot{\theta} + d + u \]

where,

- \( x = (x_1, x_2, x_3, x_4)^T \) (eq. (3)),
- \( \theta = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)^T \)
- \( b = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12})^T \)
- \( c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12})^T \)
- \( d = (d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12})^T \)

From eq. (7) and Appendix I, we can verify that the dynamic system as shown in Fig. 1 is theoretically controllable, and the solution of eq. (7) is given as,

\[ x(t) = e^{A(t)}x(0) + \int_0^t e^{A(t)} \dot{x}(t) dt \]

where, \( x(0) \) is the initial vector of \( x(t) \).

Although we have had tentative solutions to the equation in our model as cited above, we will present other solutions making use of Laplace transform techniques in the following chapter.

3. Transfer function of control object

It is a matter of course that we utilize Laplace transformation in order to depict a system block diagram which is available to grasp the system construction.

After Laplace transformation of eq. (6), \( \hat{\theta}_1(s) \) and \( \hat{\theta}_2(s) \) are presented as,

\[ \begin{align*}
\hat{\theta}_1(s) &= G_1(s) \hat{\theta}_1(s) + G_2(s) \hat{\theta}_2(s) + G_3(s) (s) + G_4(s) (s) + G_5(s) (s) + G_6(s) (s) \]
\[ \hat{\theta}_2(s) &= G_7(s) \hat{\theta}_1(s) + G_8(s) \hat{\theta}_2(s) + G_9(s) (s) + G_{10}(s) (s) + G_{11}(s) (s) + G_{12}(s) (s) \]

where, \( \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4 \) are the initial values respectively, and \( G_{ij}(s) \), which means \( G_{ij}(s) \), \( i=1,2 \) \( j=1,2,3,4 \), is presented as,

\[ \begin{align*}
G_1(s) &= \left[ b_1 g_1(s) + b_2 g_2(s) + b_3 g_3(s) + b_4 g_4(s) \right] / G(s) \\
G_2(s) &= \left[ b_5 g_1(s) + b_6 g_2(s) + b_7 g_3(s) + b_8 g_4(s) \right] / G(s) \\
G_3(s) &= b_1 g_1(s) + b_2 g_2(s) + b_3 g_3(s) + b_4 g_4(s) \\
G_4(s) &= b_5 g_1(s) + b_6 g_2(s) + b_7 g_3(s) + b_8 g_4(s) \\
G_5(s) &= b_1 g_1(s) + b_2 g_2(s) + b_3 g_3(s) + b_4 g_4(s) \\
G_6(s) &= b_5 g_1(s) + b_6 g_2(s) + b_7 g_3(s) + b_8 g_4(s) \\
G_7(s) &= b_1 g_1(s) + b_2 g_2(s) + b_3 g_3(s) + b_4 g_4(s) \\
G_8(s) &= b_5 g_1(s) + b_6 g_2(s) + b_7 g_3(s) + b_8 g_4(s) \\
G_9(s) &= b_1 g_1(s) + b_2 g_2(s) + b_3 g_3(s) + b_4 g_4(s) \\
G_{10}(s) &= b_5 g_1(s) + b_6 g_2(s) + b_7 g_3(s) + b_8 g_4(s) \\
G_{11}(s) &= b_1 g_1(s) + b_2 g_2(s) + b_3 g_3(s) + b_4 g_4(s) \\
G_{12}(s) &= b_5 g_1(s) + b_6 g_2(s) + b_7 g_3(s) + b_8 g_4(s)
\end{align*} \]

Equation (9) is depicted with mutual interacting system as shown in Fig. 2. In Fig. 2, the block framed with dotted line is a functional one describing a terminal impact phenomenon. In case this phenomenon occurs at the moment, \( \tau \), after toeing off, angular velocities with respect to hip and knee joints, i.e., \( \dot{\theta}_1(\tau) \) and \( \dot{\theta}_2(\tau) \), change instantaneously to,

\[ \begin{align*}
\dot{\theta}_1 &= \frac{1}{\dot{A} + AB + m \dot{L}} \{ (A^2 + AB + m \dot{L}) \dot{\theta}_1(\tau) + 2AB \dot{\theta}_2(\tau) \} \\
\dot{\theta}_2 &= \frac{1}{\dot{A} + AB + m \dot{L}} \{ 2(A + B) m \dot{L} \dot{\theta}_1(\tau) - (A^2 + AB + m \dot{L}) \dot{\theta}_2(\tau) \}
\end{align*} \]

where, \( A = \dot{L} + m \dot{L} \), \( B = \dot{L} + m \dot{L} \).
Fig. 2 Block diagram of control object

While, the extending angle of a normal knee denoted by $\theta_2(s)$ in Fig. 1 does not exceed 180 degrees, a prosthetic knee angle exceeds 180 degrees in order to prevent "the knee collapse" for duration of body weight bearing. As to this angle denoted by $\theta_3$, the following relations hold,

$$h_2=\theta_2-\theta_3'=\theta_3$$  \hspace{1cm}  (11)

where, $\theta_3$ is defined later in eq. (16).

4. Control system of prosthetic swing motion

In this chapter, we discuss a biological control system including a prosthetic mechanical system.

Though a concrete expression of human nervous system is next to impossible, we dare to construct such a feasible feedback control system as shown in Fig. 3. In Fig. 3, we assume that prosthetic knee angle information is afferently well perceived, and comparative computation between such "afferent sensory information" and "efferent motor command" is performed perfectly by an amputee's brain.

Fig. 3 Block diagram of control system

We have omitted the blocks concerning the initial values and terminal impact from Fig. 3, since they are irrelevant to our later analysis.

The driving force of carriage (body stem) $F(x(s))$ may influence on prosthetic swing motion, but it has never a dominant role to control prosthetic swing motion. It will be better to treat $F(x(s))$ as a kind of disturbing input to our system.

Therefrom, the system outputs denoted by $\theta_3(s)$ and $\theta_2(s)$ in Fig. 3 are given by,

$$\theta_3(s)=G_3K_3G_3(s)G_2(s)\theta_2(s)$$  \hspace{1cm}  (12)

and, the equation $R(s)=0$, is a characteristic equation for closed loop transfer function.

From the assumption that the control system of Fig. 3 is linear, the feedback elements $K_1$ and $K_2$ are expressed with polynomial equations of Laplace transform variable, $S$. For the system outputs, $\theta_3(s)$ and $\theta_2(s)$ to be imaginary functions, $K_1$ and $K_2$ must be at the most the first-order equations of $S$. Hence, $K_1$ and $K_2$ are defined as,

$$K_1=k_1+k_2s$$  \hspace{1cm}  (14)

$$K_2=k_1+k_3s$$

The disturbance, $F(x(s))$ is expressed with the following lump form function from actual measured data using a forcing plate,

$$F(x(s))=f(s)\sqrt{s^2+\omega^2}$$

(15)

And, system inputs for control strategy are expressed with the following step functions,

$$\theta_3(s)=R_3/s$$  \hspace{1cm}  (16)

where, $\theta_3$ and $R_3$ are constants.

After some algebraical computation and Laplace inverse transformation described in Appendix II, the solutions to eq. (16), $\dot{\theta}_3(t)$ and $\dot{\theta}_2(t)$ are given as,

$$\dot{\theta}_3(t)=-\omega^2r\theta_3(t)\sin(\omega t)$$

$$\dot{\theta}_2(t)=-\omega^2r\theta_2(t)\sin(\omega t)$$

(17)

and $\dot{\theta}_3(t)$ and $\dot{\theta}_2(t)$ are also given as,\n
$\dot{\theta}_3(t)=-\omega^2r\theta_3(t)\sin(\omega t)$

$\dot{\theta}_2(t)=-\omega^2r\theta_2(t)\sin(\omega t)$

(18)

where, notations in left sides of eqs. (17) and (18) like $\omega$, $R_3$, $\omega$, $\omega$, etc. are defined in Appendix II.

5. Optimal feedback gain adjustment via learning control method

In this chapter, a performance criterion to evaluate gait formats is presented. As human gait is an extremely complicated dynamic phenomenon, there have been proposed various expressions to evaluate gait formats. Dynamic stability and energy consumption have been examined mainly from a technical point of view, while the aesthetic formats of gait have been regarded as important from a clinical point of view. Our study in this paper is rather different in standpoint from the above, and it also principally at a smooth swing motion of a prosthetic knee. Further, we use the term "stability" in the sense of statistical control.

Figure 4 shows typical diagrams of time responses in eqs. (17) and (18). From Fig. 4, we define the following four parameters.

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Terminal conditions of prosthetic swing motion are expressed with the above four parameters. Then, the following relation,
\[ e_1 = e_2 = e_3 = e_4 = 0 \]  
(20)
means the realization of our ideal regardless of gait cadence and step length. On the contrary, it is so difficult for an amputee to swing out a conventional prosthetic shank satisfying the condition in eq.(20), that he is obliged to limp with a sharp asymmetrical and unbalanced feature. Then, we will proceed to examine the problem in which feedback gain vector k is adjusted to establish error vector e as zero vector 0.

Where,
\[ k = (k_1, k_2, k_3, k_4)^T \]
\[ e = (e_1, e_2, e_3, e_4)^T \]  
(21)

Equations (17) and (18) are too complicated to compute the optional values of k so as to make e equal to 0. Therefore, we cannot obtain an analytical solution of k. But we may obtain a numerical solution using the trial and error method. Here, we have conceived a learning control system incorporating cerebellum functions inferred in biological motor control system. We have constructed a hierarchy system as shown in Fig.5, where the block for feedback gain adjustment has been added at a tier higher than the feedback loop in Fig.3.

Fig.3 Time response of system output

\[ \theta_1 - \theta_2 = c_1 \quad \theta_3 - \theta_4 = c_2 \]
\[ \dot{\theta}_1 = \dot{\theta}_2 \quad \dot{\theta}_3 = \dot{\theta}_4 \]  
(22)

6. Algorithm for optimum searching

There are different approaches to an extrema searching problem including Box's hill climbing method.

Box's hill climbing method in this case is reduced to a searching problem for the summits of four-dimensional Euclid hyper space, i.e. R^4, using the following quadratic performance index.
\[ p(t) = \sum_{i=1}^{4} \omega_i \]  
(22)

We must repeat 2^4 (=16) times experiments for one time correction of k. Further, convergence to the extremum depends on an ascending distance along gradient vector. In the case of this paper, each optimal element of k is known to be 0, and each element of k can be observed independently in one time experiment. Therefore, our performance indexes may be defined separately such as,
\[ p_i = \sum_{j=1}^{4} \omega_j \]  
(23)

Our conception of feedback gain adjusting procedure is as follows. First, we will relate feedback gains, (k_j), to performance indexes, (e_j), with a linear regression equation. Next, using this regression equation, we work out such (k_j) that satisfies the condition, (e_j)=0, and employ such (k_j) for the next gait cycle. Further, we correct a form of our linear regression equation in each gait cycle one after another. At the n-th gait cycle, we can obtain data e_n+1 where subscript n means the n-th data. Now, we will assume e_n to be expressed as follows,
\[ r_{n+1} = r_n k_1 + r_n k_2 + r_n k_3 + r_n k_4 \]  
(2k)

where, Z_n is a normal distribution random variable, i.e., N(0, σ^2), and k_n is feedback gain k_n employed at the n-th gait cycle. From eq.(2k), we have the following expression too,
\[ k_n = r_{n+1} - r_n k_1 - r_n k_2 - r_n k_3 - r_n k_4 \]  
(25)

where, Z_n is the same random variable as Z_n in eq.(2k). In eqs.(2k) and (25), N means the total number of gait cycles in experiment. Equation (24) has been utilized usually in the theory of experimental design, but we must regard eq.(25) as important. Because eq.(25) means a regression model printed on cerebellum, in contrast to such regression model of real movement as eq.(24).

Next, we will compute a regression coefficient, (\delta_g) in eq.(25). Substituting the estimate of \delta_g for \delta_g, we obtain the value of \delta_g from the method of least square by
\[ \delta_{g_0} = 0 \quad \delta_{g_1} = 0 \quad \delta_{g_2} = 0 \quad \delta_{g_3} = 0 \]  
(26)

In the sequel, from the condition (24),
\[ \sum_{n=1}^{N} r_{n+1} k_m = \sum_{n=1}^{N} (r_{n+1} k_m) + \sum_{n=1}^{N} \delta_{g_m} k_m \]  
(27)

is obtained. And from the condition, \delta_{g_m} = 0

Fig.5 Block diagram of hierarchy control system
is obtained. In order to have the matrix expression including both eqs. (27) and (28) in the lump, we define the following matrices.

\[
\mathbf{a} = (a_{ij}, \mathbf{z} = (z_{ij})
\]

\[
\begin{bmatrix}
   a_{11} & a_{12} & \cdots & a_{1d} \\
   a_{21} & a_{22} & \cdots & a_{2d} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{d1} & a_{d2} & \cdots & a_{dd}
\end{bmatrix}
\]

Then, eqs. (27) and (28) are expressed as,

\[
\mathbf{z} = \mathbf{a} \mathbf{a}^T f
\]

If \( \mathbf{a}^T \mathbf{a} \) is a regular matrix, then \( \mathbf{a} \) is computed as,

\[
\mathbf{a} = (\mathbf{z}^T \mathbf{z}^T)^{-1} \mathbf{z}^T f
\]

Therefore, \( \mathbf{a} = \mathbf{a}^T \).

Hence,

\[
\begin{bmatrix}
   a_{11} & a_{12} & \cdots & a_{1d} \\
   a_{21} & a_{22} & \cdots & a_{2d} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{d1} & a_{d2} & \cdots & a_{dd}
\end{bmatrix}
\]

Putting \( c_i(N+1) \) for \( \theta \), feedback gain values which should be employed at the \( (N+1) \)-th cycle are computed as,

\[
h_{i(N+1)} = \theta_i \quad (i = 1, 2, 1, 4)
\]

While the number of matrix columns in eq. (33) increases in accordance with the gait cycle number, so many data need not be stored provided a real nonlinear relation between \( (k_i) \) and \( (e_i) \) is approximated to a linear regression hyperplane. Thus, we will employ \( N \)-fold Markov property in our model, and eliminate the head column elements, i.e. \( k_{d2} \) and \( c_{d2} \), each time a new set of \( k_{i(N+1)} \) and \( e_{i(N+1)} \) is obtained.

In case a terminal impact phenomenon occurs during swing phase, the performance indexes described in eq. (19) must be modified. An amputee may perceive such information as the occurring time of terminal impact, \( t_a \), and knee angle \( \theta_a(t_a) \) and knee angle velocity, \( \dot{\theta}_a(t_a) \) at that moment. Further, as a second-hand information by the click sound, an amputee may perceive knee angle acceleration, \( \ddot{\theta}_a(t_a) \) at that moment. Therefore, we employ such pseudo observed data \( e_{3a} \) and \( e_{4a} \) instead of \( e_3 \) and \( e_4 \). (see Fig.6)

\[
\begin{align*}
   e_{3a} &= (e_{3} - \dot{e}_{3}(t_a) - \ddot{e}_{3}(t_a)) \\
   e_{4a} &= (e_{4} - \dot{e}_{4}(t_a) - \ddot{e}_{4}(t_a))
\end{align*}
\]

where, \( \dot{\theta}_a = \tan^{-1}(\dot{\theta}_a(t_a)) \), and the preceding data on \( e_3 \) and \( e_4 \) are employed again.
7-2. Criterion for convergence to statistical control

System outputs, $\theta_i(t)$ and $\theta_j(t)$, and accordingly the performance index, $e_i$, are random variables, because the initial value and disturbance accompany a normal distributing fluctuation. Then, in a computer simulated analysis, we must decide on a stopping rule in consideration of uniformity of each value, $e_i$, as well as proximity of $e_i$ to $0$. We will employ $X-R$ control chart in the theory of quality control, introducing such boundaries that $e_i$ does not exceed in smaller sides.

Using measured data, $e_{in}$, we can introduce the control limits of lot number $n$ and four sample sizes. If the mean values and variances with respect to $e_{in}$ continue to fall into our control limits during $m$ cycles after the $n$-th cycle, we can conclude that our theoretical gait training has been completed and our control system has accomplished the state of statistical control at the $n$-th cycle.

7-3. Simulation results

We set $m$ at 20 in our simulation. The computational flow chart is shown in Fig.8. We employed the values of actual measurement but we employed estimated values about the residual thigh. These numerical values are shown in Table 1.

Typical examples of $X-R$ control chart are shown in Figs.9, 10 and 11. Figures 9, and 10 are the results with our knee-angle feedback system on conditions that $N=5$, and $N=10$, where $N$ means the number of matrix columns in eq.(33). Figure 11 is the result without feedback system. From Fig.11, we know that the state of statistical control is attainable even with a conventional prosthesis. But, too long rehabilitational training is required and a gait format after training is miserable at that. For an amputee with a conventional prosthesis, it is impossible to walk more aesthetically irrespective of his walking ability, because in case sensory information cannot be utilized, the learning control algorithm cited above proves to be futile.

Table 2 shows the rearranged results of Figs.9, 10 and 11 in the lump.

Figure 12 shows examples about transition process of movements, $\theta_1$ and $\theta_2$. From Fig.12, we can conclude that movement patterns are improved gradually through our theoretical training.

8. Concluding remarks

Sensory information is not only an one-sided information from receptors to brain, but a so-called indirect motor command. Our sensory device supplemented to a prosthesis may be called "feedback control system", because it is included into the motor control system of an amputee. We would like to emphasize that our sensory feedback system should be utilized for improvement of prosthetic gait dynamics. Our opinion has been verified to be legitimate theoretically in this paper.
The summarized results of this paper are as follows.
(1) Dynamic model of prosthetic swing motion was derived, and eqs.(10) and (23) were proposed for its ideal performance indexes.
(2) Mastering process of prosthetic gait for rehabilitational training was simulated by an optimal feedback gain adjusting process using the theory of learning control.
(3) Stochastic fluctuation was superposed on a model, considering uncertainty introduced into gait state variable in each gait cycle, in order to reproduce a real gait format faithfully.
(4) X-R control chart in the theory of quality control was employed to depict stabilizing process of gait synergies.
(5) Usefulness of our sensory feedback system was verified theoretically.
(6) Conception of learning control in this paper would be applicable to control strategies of robots, owing to possibilities to give them adaptable capability.

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References

Appendix I
(Controllability of the control object)
The control object represented by Fig.2 is described with a fourth-order differential equation. From eq.(7), the necessary and sufficient condition that the control object is perfectly controllable is as follows,
\[ \text{rank}(D_{14},A^2D_{14},A^3D_{14}) = 4 \]  \hspace{1cm} (A.1)
The Gram matrix as shown in eq.(A.1) is not a squar matrix, and its determinant does not exist. But, it is possible to prove eq.(A.1), using matrix \( b \) denoted in eq.(7) instead of \( D \). Namely, the following equation is obtained.
\[ \text{Det}(b_{14},A_{14}b_{14},A_{14}^2b_{14},A_{14}^3b_{14}) = \begin{vmatrix} b_{14} & b_{14}b_{14} & b_{14}b_{14}^2 & b_{14}b_{14}^3 \end{vmatrix}^2 = 0 \]  \hspace{1cm} (A.2)
Then,
\[ \text{rank}(D_{14},A^2D_{14},A^3D_{14}) = \text{rank}(b_{14},A_{14}b_{14},A_{14}^2b_{14},A_{14}^3b_{14}:C_{14},A_{14}^2C_{14},A_{14}^3C_{14}) \geq 4 \]  \hspace{1cm} (A.3)
Where, eq.(A.1) is the 4x6 type matrix, and,
\[ \text{rank}(D_{14},A^2D_{14},A^3D_{14}) = 4 \]  \hspace{1cm} (A.4)
holds. From eqs.(A.3) and (A.4), eq.(A.1) is proved.[]
Appendix II

(Derivation of Laplace inverse transformation)

Substituting eqs. (14), (15) and (16) into eq. (12), the following equation is obtained.

\[ \Theta(s) = \frac{\beta_1 s^6 + \beta_2 s^5 + \beta_3 s^4 + \beta_4 s^3 + \beta_5 s^2 + \beta_6 s + \beta_7}{s^6 + \alpha_1 s^5 + \alpha_2 s^4 + \alpha_3 s^3 + \alpha_4 s^2 + \alpha_5 s + \alpha_6} \]  \hspace{1cm} (A.5)

\[ \Theta(s) = \frac{\beta_1 s^6 + \beta_2 s^5 + \beta_3 s^4 + \beta_4 s^3 + \beta_5 s^2 + \beta_6 s + \beta_7}{s^6 + \alpha_1 s^5 + \alpha_2 s^4 + \alpha_3 s^3 + \alpha_4 s^2 + \alpha_5 s + \alpha_6} \]

where,

\[ \beta_1 = 2b_1 \]
\[ \beta_2 = b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7 \]
\[ \beta_3 = b_1 k_1 k_2 + b_2 k_2 k_3 + b_3 k_3 k_4 + b_4 k_4 k_5 + b_5 k_5 k_6 + b_6 k_6 k_7 + b_7 k_7 k_8 \]
\[ \beta_4 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_0 \]
\[ \beta_5 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_1 \]
\[ \beta_6 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_2 \]
\[ \beta_7 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_3 \]

\[ \alpha_1 = b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7 \]
\[ \alpha_2 = b_1 k_1 k_2 + b_2 k_2 k_3 + b_3 k_3 k_4 + b_4 k_4 k_5 + b_5 k_5 k_6 + b_6 k_6 k_7 + b_7 k_7 k_8 \]
\[ \alpha_3 = b_1 k_1 k_2 k_3 + b_2 k_2 k_3 k_4 + b_3 k_3 k_4 k_5 + b_4 k_4 k_5 k_6 + b_5 k_5 k_6 k_7 + b_6 k_6 k_7 k_8 + b_7 k_7 k_8 k_9 \]
\[ \alpha_4 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_0 \]
\[ \alpha_5 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_1 \]
\[ \alpha_6 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_2 \]
\[ \alpha_7 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_3 \]
\[ \alpha_8 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_4 \]
\[ \alpha_9 = (b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5 + b_6 k_6 + b_7 k_7) \eta_5 \]

Factoring the denominators in eq. (A.5), such as,

\[ s^6 + \alpha_1 s^5 + \alpha_2 s^4 + \alpha_3 s^3 + \alpha_4 s^2 + \alpha_5 s + \alpha_6 = (s^2 + a_1 s + b_1)(s^2 + a_2 s + b_2)(s^2 + a_3 s + b_3) \]

then, eq. (A.7) is obtained from eq. (A.5).

\[ \Theta(s) = \frac{\alpha_1 s^6 + \alpha_2 s^5 + \alpha_3 s^4 + \alpha_4 s^3 + \alpha_5 s^2 + \alpha_6 s + \alpha_7}{s^6 + \beta_1 s^5 + \beta_2 s^4 + \beta_3 s^3 + \beta_4 s^2 + \beta_5 s + \beta_6} \]  \hspace{1cm} (A.6)

\[ \Theta(s) = \frac{\alpha_1 s^6 + \alpha_2 s^5 + \alpha_3 s^4 + \alpha_4 s^3 + \alpha_5 s^2 + \alpha_6 s + \alpha_7}{s^6 + \beta_1 s^5 + \beta_2 s^4 + \beta_3 s^3 + \beta_4 s^2 + \beta_5 s + \beta_6} \]

where, the relation between \( \alpha_{ij} \) and \( \beta_{ij} \) is expressed as,

\[ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix} \times (\alpha_{ij})^T = (\alpha_{ij})^T \]

\[ \alpha_{ij} \] is computed as follows,