A Study on Facilities Location Using the Properties of Cut Set Matrix*

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This paper deals with a problem of facilities location. Fundamental suggestion is making an individual location algorithm for combination of "Presupposition" and "Objective function" in each location problem.

The presupposition of this paper is "that facilities locate straight", and the objective function is "to minimize Backward Distance Intensity". This new facilities location algorithm to select "minimum Backward Distance $\times$ Intensity location" is developed by using a property on Cut Set matrix that is a part of graph theory. The feature of this algorithm is ability to select the optimal solution by only using easy matrices manipulation. Trial of this algorithm in regard to an actual 6 facilities location is described here in.

1. Introduction

The facilities used to located by a skillful layout planner. But now, a new algorithm which makes satisfactory facilities location is demanded. And some effective "Mathematical Methods" have been developed.

One of the properties of mathematical method is clear objective function. It is disadvantage to replace the problem of facilities location involving many complex factors with a simple problem handling only one factor to be minimized or maximized. When we reflect on the complexity of facilities location, development of an allround algorithm to solve every kind of layout problem is very difficult. Therefore, we think that we should make an individual algorithm for combination of "Presupposition" and "Objective function".

This paper develops a new facilities location algorithm to minimize "Backward Distance $\times$ Intensity" without tryanderror, supposing that facilities locate straight, by use of property of Cut Set matrix that is a part of graph theory. The presupposition of this algorithm is "that facilities locate straight", and the objective function is "to minimize Backward Distance $\times$ Intensity".

2. Outline of algorithm

Given input information is a "from-to chart" that describes materials flows

among facilities, and "Width of facilities".

Nodes are defined as facilities and arrows are defined as materials flows, the information of materials movement is described by use of oriented graph. We can make Cut Set matrix from materials flow information by using the property of Cut Set, that is a set of minimal branches needed for open elimination in dividing graph into two parts. Adding an information about existence and direction of arrow, that is coming from Cut Set matrix, to an information about materials flow and width of facilities (distance between centers of facilities i to j), that is coming from input information, we can calculate "Backward Distance $\times$ Intensity (Backward $D \times I$)". Then, select the optimal solution according to the heuristic method. To avoid an increase in number of trial runs, that is incidental to heuristic way, we make "minimum Backward Intensity facilities location", that can be obtained by applying Tie Set algorithm to a temporary initial solution. Because "Minimum Backward Intensity" is a part of "Minimum Backward Distance $\times$ Intensity", which is objective function of this paper.

3. Fundamental theory

This developed algorithm has two phases, one being [1] Preparative phase, the other [2] Judgement phase. The preparative phase is matrices manipulation of "Backward Distance $\times$ Intensity", that is the objective function of this paper. The judgement phase is selection of an optimal solution. The fundamental theory of this algorithm is explained below;

Input information

The information about state of materials flow is described on a from-to chart (Fig.1). Fig.2 is information on width of facilities. The existence of materials

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flow is shown on oriented graph (Fig.3).

	A	B	C	D
A		10	1	0
B	4		2	8
C	0	7		3
D	2	0	0	

Fig.1 From-to chart

Machine	A	B	C	D
Width	6	2	4	2

Fig.2 Width of facilities

Fig.3 Oriented graph

[1] Preparative phase

- (1) Distance between centers of facilities i and j .

When facilities locate straight, we grasp the distance of materials flow and as center-to-center distance of the facilities which is shown in Fig.4. When the width of facilities is given as in Fig.2, we can make distance matrix (Fig.5) by use of iD_j , that is coming from Fig.4. Fig.5 describes the matrix for the distance between adjacent facilities.

Fig.4 Distance between centers of facilities i and j

	A	B	C	D
A		4	5	4
B	4		3	2
C	5	3		3
D	4	2	3	

Fig.5 iD_j Matrix

(2) Calculated Cut Set ⁽⁴⁾

Fundamental Cut Set is calculated by matrices manipulation. All Cut Set matrices can be obtained as combinations of fundamental Cut Set.

$$A = A_t, A_1$$

$$Q_f = [U, (A_t^{-1}, A_1)]$$

A : Reduced incidence matrix
 A_t : Tree of reduced incidence matrix
 A_1 : Co-tree of reduced incidence matrix
 Q_f : Fundamental Cut Set matrix
 U : Unity matrix
 A_t^{-1} : Inverse matrix

(3) Proposed site and Cut Set

We number the proposed site as in Fig. 6, 1 being the nearest site from IN, 2 the next one and so on. If A facility locate at No.1 site, any one of B, C, D facilities can be located at No.2 site (see Fig.6).

Fig.6 Proposed site and Cut i

In this case, arrows (materials flow) between No.1 site and No.2 site can be seen on A column of calculated fundamental Cut Set matrix as " (2) Calculate the Cut Set ". Similarly, the existence of arrows for the case of B facility being located at No.1 site, is described on B column of above-mentioned fundamental Cut Set matrix. " Backward movement of materials " is indicated by a negative sign (-) of Cut Set matrix. We call a set of negative arrows between proposed sites i and $i+1$ Cut No.1.

In Fig.6, A facility locates at No.4 site, that is the nearest site from OUT side, and B, C, D facilities locate at No. 3 site, the backward materials flows being positive (+) arrows of fundamental Cut Set. And then, we name the set of these backward materials flows Cut No.3 (see Fig.6 and 7). The same holds with Cut No.2 (Fig.8).

	10	1	4	2	8	7	3	2	material flow	
a	b	c	d	e	f	g	h		Cut.1	Cut.3
A	1	1	-1	0	0	0	0	-1	6	11
B	-1	0	1	1	1	-1	0	0	17	14
C	0	-1	0	-1	0	1	1	0	3	10
D	0	0	0	0	-1	0	-1	1	11	2

Fig.7 Cut 1 and Cut 3

	10	1	4	2	8	7	3	2	material flow	
a	b	c	d	e	f	g	h		cut.2	
AB	0	1	0	1	1	-1	0	-1	9	11
AC	1	0	-1	-1	0	1	1	-1	8	20
AD	1	1	-1	0	-1	0	-1	0	15	11
										BC

Fig.8 Cut 2

- (4) Calculate "the amount of backward materials flows "

Columns of Cut Set matrix, that is calculated in (2), correspond to the arrows. So an amount of backward material flows can be calculated as the sum of negative arrows. Here, Cut Set matrix and the calculated amount of backward material flows of Cuts 1, 2 and 3 are described as Fig.7 and 8.

- (5) Calculate " Backward Distance $\times$ Intensity "

" Backward Distance $\times$ Intensity " is calculated by multiplying " (4) the amount of backward materials flow " by " (1) Distance information ". The results are shown in Fig.9~11. Further, pick up the minimum one [= min (Cut.i)] from each matrix for [2] Judgement phase.

cut.1	A	B	C	D
6	A	24	30	24
17	B	68	51	34
3	C	15	9	9
11	D	44	22	33

min.(cut.1)=9

C-B
C-D

Fig.9 Cut 1

A	B	C	D	cut.3
44	55	44	A	11
56	42	28	B	14
50	30	30	C	10
8	4	6	D	2

min.(cut.3)=4

B-D

Fig.10 Cut 3

cut.2	A	B	C	D
9	A B	-	-	27 18
	B A	-	-	45 36
8	A C	-	24	- 24
	C A	-	32	- 32
15	A D	-	30 45	-
	D A	-	60 75	-
11	B C	55	-	33
	C B	44	-	22
20	B D	80	-	60
	D B	80	-	60
11	C D	44	22	-
	D C	55	33	-

min.(cut.2)=18

A-B-D

Fig.11 Cut 2

[2] Judgement phase

" Backward Distance $\times$ Intensity " is obtained by above-mentioned matrices manipulation. Next step is the judgement phase, that means selection of an optimal solution. The fundamental judgement is made as follows;

Set the initial solution (K_0) to the temporary minimum, and then, eliminate the Backward Distance $\times$ Intensity exceeding (K_0) from each Cut Set matrix. To calculate the upper limit of each Cut by use of the initial solution and the other minimum [= min (Cut i)]. If any " Backward Distance $\times$ Intensity " exceeds the up-

per limit, there is no possibility of correcting the initial solution (K_0). So eliminate it from the subject of judgement, to avoid an increase in number of trial runs. If a smaller one than the initial solution (K_0) is obtained, we take this one as a new initial solution. And continue the above steps until the optimal solution is selected.

- (1) Initial solution (K_0)

We make " minimum backward materials flow location ", by using the property of Tie Set, and we make " Backward Distance $\times$ Intensity " of this location the initial solution (K_0) (see Fig.12).

$$K_0 = 5 \times (4+2) + 3 \times (4+2+2) + 2 \times 2 = 58 \quad (1)$$

Fig.12 Initial solution

- (2) Upper limit of Cut i

This step is an attempt to decrease the number of trials.

- Upper limit of Cut 1

$$= K_0 - \min(\text{Cut } 2) - \min(\text{Cut } 3) = 58 - 18 - 4 = 36 \quad (2)$$

Numbers over this upper limit never reduce initial solution (K_0), so they are excluded from the object of selection. Other upper limits of Cut Set are as follows;

- Upper limit of Cut 2

$$= 58 - 9 - 4 = 45 \quad (3)$$

- Upper limit of Cut 3

$$= 58 - 9 - 18 = 31 \quad (4)$$

Using these upper limits of Cut i, the matrix which is calculated in " [1] Preparative phase of (5) " is modified as Figs.13 15.

- (3) Select the optimal solution

The optimal solution is selected from the combinations of modified Cut i (Backward Distance $\times$ Intensity). This step remarkably decreases the number of trials, as compared with traditional complete enumeration method. Generally, min(Cut i) is most effective to improve the initial solution, therefore the selection starts with a comparison between min(Cut i) and the initial solution (K_0).

Thinking about Cut 1, that is a connection of proposed sites 1 and 2, the smallest number is 9. And the facilities locations in this case are (i) C-B, (ii) C-D (see Fig.9). The facility that can be located at the proposed site 3 is shown as row of CB and CD on modified Cut 2 (Fig.14). There are only 4 cases, as follows (see Fig.14);

In case of (i) : C - B - A, C - B - D

In case of (ii) : C - D - A, C - D - B

CUT.1				
	A	B	C	D
A		24	30	24
B	v		v	34
C	15	9		9
D	v	22	33	

Fig.13 Modified Cut 1

CUT.2				
	A	B	C	D
AB	-	-	27	18
BA	-	-	v	36
AC	-	24	-	24
CA	-	32	-	32
AD	-	30	v	-
DA	-	v	v	-
BC	v	-	-	33
CB	44	-	-	22
BD	v	-	v	-
DB	v	-	v	-
CD	44	22	-	-
DC	v	33	-	-

Fig.14 Modified Cut 2

CUT.3				
A	B	C	D	
	v	v	v	A
v		v	28	B
v	30		30	C
8	4	6		D

Fig.15 Modified Cut 3

Next, think about the proposed site 4. In both (i) and (ii), location of the rest of facilities is described on modified Cut 3 matrix (Fig.15). Every column of Fig.15 is, then none of combinations used in the min(Cut 1) improve the initial solution.

Applying the same procedure to min (Cut 2), we cannot obtain any better locations.

For min(Cut 3), the following two locations are selected.

$$\begin{aligned} A - C - B - D \\ \text{Backward Distance} \times \text{Intensity} \\ = 52 < K_0 \quad (5) \end{aligned}$$

$$\begin{aligned} C - A - B - D \\ \text{Backward Distance} \times \text{Intensity} \\ = 51 < K_0 \quad (6) \end{aligned}$$

then, the initial solution (K_0) is corrected by Eq.(6). Initial solution is renewed to 51.

In the above-mentioned procedure, we finish trial of min(Cut i). Then, the second smallest value in each Cut is set as a new min(Cut i).

$$\min(\text{Cut } 1) = 24 \rightarrow 15$$

$$\min(\text{Cut } 2) = 22 \rightarrow 18$$

$$\min(\text{Cut } 3) = 6 \rightarrow 4$$

Upper limit of Cut i is calculated by using the renewed initial solution (K_0) and min(Cut i). Eliminate nonimprovable co-

mbinations from Fig.13 15, and so on.

When the initial solution (K_0) can be improve or all columns on the matrix of Cut i are excluded, the algorithm is stopped. And the very same facilities location is the optimal solution. The selected optimal solution of this example is shown in Fig.16. Fig.17 is a conceptional sketch of Fig.16 using the information on facilities width shown in Fig.2.

Fig.16 Sketch of the optimal solution

Total Backward Distance $\times$ Intensity = 51

Fig.17 Conceptional sketch of the optimal location

4. Developed algorithm which minimizes " Backward Distance $\times$ Intensity " in facilities location.

The overall procedure of this algorithm is outlined in " 3. Fundamental theory ". Here, a detailed procedure is described.

[Step 1]

Grasp the information of materials flow and width of facilities in the shape of From-to chart and matrix for input of algorithm.

[Step 2]

Make the From-to chart into Node to Node matrix and change the information about width of facilities into d_{ij} matrix (distance between centers of facilities i and j).

[Step 3]

Label the existing arrows of Node to Node matrix.

[Step 4]

Make incidence matrix (A_a), that describes materials flow, by using the labeled arrows.

[Step 5]

Make reduced incidence matrix (A). To eliminate an arbitrary row from the incidence matrix (A_a).

[Step 6]

Separate the reduced incidence matrix (A) into Tree part and Co-tree part, and make a new matrix in the shape of $A = A_t, A_l$

[Step 7]

Calculate the fundamental Cut Set matrix (Q_f) by using the result of Step 6.

$$Q_f = [U, (A_t^{-1} \cdot A_l)]$$

[Step 8]

Calculate the whole Cut Set by using the combinations of fundamental Cut Sets.

[Step 9]

Correlate Cut, between facilities, with Cut Set (see Fig.6) and make the matrix of " Backward Distance $\times$ Intensity ".

[Step 10]

Calculate $\min(\text{Cut } i)$ by use of each Cut matrix, and extract the facilities location of this case.

[Step 11]

Using the algorithm on " Facilities location with Tie Set ⁽²⁾ " to make " the minimum backward materials flow facilities location ". Add the information on distance between centers of facilities i and j , that is shown as d_{ij} matrix (Step 2) to above-mentioned location, and calculate Backward Distance $\times$ Intensity of this case and put it as the initial solution (K_0).

[Step 12]

Calculate upper limit of each Cut by use of the initial solution (K_0) and $\min(\text{Cut } i)$.

[Step 13]

Using the upper limit of Cut i (see Step 12), eliminate the combination of facilities, which never correct the initial solution (K_0) and make a modified Cut i matrix (Fig.13 ~ 15. The initial solution (K_0) after all combinations eliminated, is the optimal solution.

[Step 14]

Enumerate the possible facilities locations in the case of $\min(\text{Cut } i)$, that is most effective to improve the initial solution (K_0) by using the modified Cut i .

[Step 15]

Calculate the " Backward Distance $\times$ Intensity " for possible facilities locations that is selected by Step 14.

[Step 16]

Compare the minimum value that is selected by Step 15 with the initial solution (K_0).

minimum value $<$ initial solution
go to Step 17

minimum value $>$ initial solution
the facilities location
of the initial solution
is optimum.

[Step 17]

Make the minimum value of Step 15 a new initial solution.

[Step 18]

After checking $\min(\text{Cut } i)$, take the second smallest value in each Cut as a new $\min(\text{Cut } i)$.

[Step 19]

Go back to Step 12, and continue this algorithm until the optimal solution is reached by using the renewed initial solution (K_0) and the renewed $\min(\text{Cut } i)$.

Above-mentioned procedure is a developed algorithm which minimizes " Backward Distance $\times$ Intensity " in facilities location.

When in Step 13, every column of modified Cut i is eliminated or in Step 16 the minimum value, that is selected in Step 15, is greater than the initial solution (K_0), we stop this algorithm. Then this facilities

location is the optimum solution.

5. Conclusions

As a fundamental suggestion on the facilities location an individual algorithm for combination of " Presupposition " and " Objective function " is proposed.

This paper describes a new facilities location algorithm to minimize " Backward Distance $\times$ Intensity " by use of property on Cut Set matrix that is a part of graph theory. The developed algorithm is shown in Chapter 4. The special features of the developed algorithm are as follows;

- (1) Each Cut Set is calculated by easy matrices manipulation.
- (2) The existence and direction of arrows are checked by Cut Set.
- (3) Use of the initial solution (K_0) and the upper limit of Cut i to eliminate impossible combinations of facilities.

6. Supplementary note

Here is described an investigation about the efficiency of this developed algorithm. Examined is a 6 facilities problem and input information is described in Fig.18 and 19. The oriented graph of this example is shown in Fig.20.

	A	B	C	D	E	F
A		9	8	0	0	17
B	0		0	11	2	0
C	3	0		0	14	0
D	10	9	0		0	5
E	16	4	0	0		0
F	0	5	2	0	0	

Fig. 18 From-to chart

Machine	A	B	C	D	E	F
Width	4	2	4	6	6	8

Fig.19 Width of facilities

Fig.20 Oriented graph

The traditional complete enumeration method needs $6! = 720$ trials to select " the minimum Backward Distance $\times$ Intensity facilities location ". The application of the developed algorithm to this 6 facilities problem is made as follows;

The result of Step 9, that is a final part of " Preparative phase ", is shown in Fig.21.

Cut 1							Cut 2	
	A	B	C	D	E	F	E	F
29	A	87	116	145	174	188	235	
27	B	81	81	108	135	235	232	
10	C	40	30	50	60	140	168	
11	D	55	44	55	66	77	140	168
16	E	80	64	80	96	112	190	210
22	F	132	110	132	154	154	160	210

Cut 3							Cut 4	
	A	B	C	D	E	F	E	F
28	D	F	168	140	168	—	196	—
	F	D	140	112	140	—	168	—
38	E	F	228	190	228	—	—	—
	F	E	190	152	190	—	—	—
32	C	F	E	160	128	—	192	—
	E	F	C	128	96	—	160	—

Cut 5							Cut 6	
	A	B	C	D	E	F	E	F
39	A	102	136	170	204	238	272	306
38	B	39	52	65	78	91	104	117
68	C	51	65	78	91	104	117	130
102	D	96	120	144	168	192	216	240
100	E	80	100	120	140	160	180	200
42	F	35	42	49	56	63	70	77

Fig.21 " Backward Distance $\times$ Intensity " Matrix (Step 9)

The result of Step 11, that is reduction of trial number by using the initial solution (K_0) and the upper limit, is shown in Fig.22.

Trial	K_0	Cut 1		Cut 2		Cut 3		Cut 4		Cut 5	
		Min	Limit	Min	Limit	Min	Limit	Min	Limit	Min	Limit
1	360	30	202	36	208	42	214	45	217	35	207
2	275	40	81	48	89	56	97	51	92	39	80
3	275	44	50	54	60	69	75	60	66	42	48

Fig.22 Initial solution (K_0) and reduction of upper limit of Cut 1

In Fig.22, the optimal solution ($K_0 = 275$) is selected through three times repetition of Steps 12 to 19. A sketch of the optimal solution is given in Fig.23.

In the course of applying the developed algorithm to several actual examples, we could gain a remarkable reduction of trial number and always select the optimal solution.

Fig.23 Selected optimal solution

In the future, we will develop other algorithm and other " Objective functions " .

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