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Study on Deflection of Spur Gear Teeth*
 (1st Report, Calculation of Tooth Deflection by
 Two-dimensional Elastic Theory)

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The authors calculated the deflection of spur gear teeth by using a two-dimensional elastic theory and the mapping function, and compared the calculated results with the values calculated by the well-known tooth deflection formulas.

Consequently, it was found that all values of tooth deflection calculated by this method are smaller than that by the well-known tooth deflection formulas and the measured value, because of the Hertzian contact deformation, which is smaller than the value obtained from Lundberg's method.

Furthermore, it was also found that the tooth deflection is nearly proportional to the tooth load.

1. Introduction

There have already been many studies of spur gear tooth deflection reported in the literature, and the calculation formulas for tooth deflection also have been proposed by many investigators. However, the results obtained by these formulas differ from each other. This is attributed to the differences in how to approximate the gear tooth profile and of the evaluations of the deformation produced by Hertzian contact pressure and the deflection due to the inclination of tooth foundation. For the gear tooth, we may consider that the deflection due to the inclination of tooth foundation has the same order of quantity as both the deflections by the bending moment and the shearing force, because the ratio of length to thickness for the gear tooth is small. Therefore, a difference between the formulas for the deflection due to the inclination of tooth foundation has direct effect upon the evaluation of the total tooth deflection, and also a difference in way of evaluating the deformation by Hertzian contact pressure is a great factor.

Both the tooth deflection formulas proposed by Weber et al.⁽¹⁾ and Ishikawa⁽²⁾ have been often used. Both formulas were obtained under a few assumptions. One of the assumptions is that the tooth is a cantilever beam with a simplified nominal stress distribution. However, in fact, since the true stress distribution differs from the simplified distribution due to the stress concentration arising on the root fillet, a question is raised about whether

the assumptions used in those formulas are justified or not. It is desirable that the deflection of the tooth considering the exact involute-curve and the root fillet be calculated including the deflection due to the inclination of tooth foundation and the deformation due to the Hertzian contact pressure, without using the above assumption. From this point of view, Tobe et al.⁽³⁾ treated the tooth deflection as a two-dimensional elastic problem, and they calculated it by using the finite element method. The application of the finite element method enables one to obtain the tooth deflection according to the stress concentration on the root fillet and the true stress of the inside of tooth. However, the finite element method has a problem in that the difference in the number of the elements which are obtained by dividing the continuous structure influences the calculation accuracy of the tooth deflection.

Therefore, in the present study, we calculate the tooth deflection by applying the method developed by Aida and Terauchi⁽⁴⁾, which was used for obtaining the theoretical solution of the true stress at the root fillet using the two-dimensional elastic theory and the conformal mapping function. In the first step of the calculation, we choose such a mapping function that its mapped figure coincides with the actual tooth profile, and calculate the tooth deflection by approximating the normal tooth load with a group of many concentrated loads. In the next step, the calculation result is compared with the values obtained from the well-known tooth deflection formulas, and the calculated Hertzian contact deformation is also compared with the values obtained from Lundberg's formula⁽⁵⁾ used in Ishikawa's method⁽²⁾. The details will be reported below.

2. Calculation method

2.1 Calculation procedure for tooth deflection

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The tooth deflection has been calculated by applying the theory of elasticity using complex variables and transforming the convex curve of tooth profile on the boundary line of a semi-infinite plate. In this case, assumptions were made that the distance between the root circle and the arborhole is large enough in comparison with the tooth height and that the influence of neighbouring teeth can be neglected, because for the most gears in actual use, the hub is usually thicker than the tooth height. Considering the above fact and the result of the experiment of photo-elasticity, these assumptions can be justified.

Let Airy stress function on the $z(x, y)$ plane be U that is expressed by using $z = x + iy$, $\bar{z} = x - iy$. Then the stress function will be as follows:

$$2U = \bar{z}\varphi_1(z) + z\bar{\varphi}_1(\bar{z}) + \psi_1(z) + \bar{\psi}_1(\bar{z}) \\ = 2\Re[\bar{z}\varphi_1(z) + \psi_1(z)], \quad (1)$$

where φ_1, ψ_1 are analytical functions of z ; $\bar{\varphi}_1, \bar{\psi}_1$ are expressed as the conjugate of $\varphi_1(z), \psi_1(z)$ and they are functions of $\bar{z}$.

As shown in Fig.1, when the concentrated load P is acting at the point w on the boundary of a semi-infinite plate in the $\textcircled{2}$ -plane, and making an angle β with the x -axis, U is expressed as

$$U = -\frac{P}{\pi} r \theta \sin(\theta - \beta), \quad (2)$$

and Eq. (2) is rewritten by using the complex equation $z - w = re^{i\theta}$ as follows:

$$U = \frac{P}{2\pi} \Re[(z - w) \log(z - w) e^{-i\beta} - (\bar{z} - \bar{w}) \log(\bar{z} - \bar{w}) e^{i\beta}], \quad (3)$$

The analytical functions φ_1 and ψ_1 can be determined from Eqs. (1) and (3):

$$\left. \begin{aligned} \varphi_1(z) &= -\frac{P}{2\pi} e^{i\beta} \log(z - w), \\ \psi_1(z) &= \frac{P}{2\pi} [e^{-i\beta} (z - w) \log(z - w) + \bar{w} e^{i\beta} \log(z - w)] \end{aligned} \right\} \quad (4)$$

By using Eq. (4), the deflection based on the plane stress in the $\textcircled{2}$ -plane is given as

$$u - iv = \frac{1}{2G} \left[\frac{3-\nu}{1+\nu} \bar{\varphi}_1(\bar{z}) - \{ \bar{z} \varphi_1'(z) + \psi_1'(z) \} \right], \quad (5)$$

where u, v are the displacements along the coordinates x and y respectively; G is the shear modulus; and ν is Poisson's ratio. When Eq. (4) is rewritten by using $\psi_1(z) = \psi_1'(z)$, we obtain

$$\psi_1(z) = \frac{P}{2\pi} \left[e^{-i\beta} \log(z - w) + e^{-i\beta} + \frac{\bar{w}}{z - w} e^{i\beta} \right], \quad (6)$$

then Eq. (5) can be transformed into

Fig.1 Applied load at a point on boundary of semi-infinite plate

$$u - iv = \frac{1}{2G} \left[\frac{3-\nu}{1+\nu} \bar{\varphi}_1(\bar{z}) - \{ \bar{z} \varphi_1'(z) + \psi_1'(z) \} \right], \quad (7)$$

In order to apply Eq. (7), which is for the straight boundary, to the semi-infinite plate ($\textcircled{2}$ -plane) having a boundary of tooth profile as shown in Fig.2, we choose a conformal mapping function which transforms the curved boundary C on the $\textcircled{2}$ -plane into the imaginary axis Γ' on the $\textcircled{1}$ -plane, and also transforms the domain D into the domain Δ . The mapping function is expressed as

$$z = \omega(\zeta) = c\zeta - \frac{m}{\zeta+a} - \frac{n}{\zeta+b} - \frac{p}{\zeta+d}, \quad (8)$$

Equation (8) has the properties of $\omega'(\zeta) \neq 0$ and $\lim_{\zeta \rightarrow \infty} |\omega(\zeta) - c\zeta| = 0$. When $z = \omega(\zeta)$ is substituted into Eqs. (4), (6), and if $\varphi_1[\omega(\lambda)] = \varphi(\lambda)$, $\psi_1[\omega(\lambda)] = \psi(\lambda)$ (λ is a point on Γ'), by taking account of the boundary condition, $\varphi(\zeta)$ and $\psi(\zeta)$ can be calculated as follows:

$$\varphi(\zeta) = \varphi_0(\zeta) + \frac{1}{2\pi i} \int_{\Gamma'} \frac{\omega(\lambda)}{\omega'(-\lambda)} \frac{\bar{\omega}'(-\lambda)}{\lambda - \zeta} d\lambda, \quad (9)$$

where

$$\varphi_0(\zeta) = -\frac{P}{2\pi} e^{i\beta} \log(\zeta - \tau), \quad (10)$$

and

$$\psi(\zeta) = \frac{P}{2\pi} e^{-i\beta} \log(\zeta - \tau) + \frac{1}{2\pi i} \int_{\Gamma'} \frac{\bar{\omega}(-\lambda)}{\omega'(-\lambda)} \frac{\varphi'(\lambda)}{\lambda - \zeta} d\lambda, \quad (11)$$

where Γ, Γ' are the paths of integration surrounding Δ, Δ' respectively; τ is a point on the imaginary axis Γ' corresponding to w . Moreover, if $z = \omega(\zeta)$, $\bar{z} = \bar{\omega}(\bar{\zeta})$, $\bar{\varphi}_1(\bar{z}) = \bar{\varphi}_1[\bar{\omega}(\bar{\zeta})] = \bar{\varphi}(\bar{\zeta})$, $\varphi_1'(z) = d\varphi_1/dz = \varphi'(\zeta)/\omega'(\zeta)$ and $\psi_1'(z) = \psi'(\zeta)/\omega'(\zeta) = \psi'(\zeta)$, Eq. (7) can be rewritten as

$$u - iv = \frac{1}{2G} \left[\frac{3-\nu}{1+\nu} \bar{\varphi}(\bar{\zeta}) - \{ \bar{\omega}(\bar{\zeta}) \frac{\varphi'(\zeta)}{\omega'(\zeta)} + \psi'(\zeta) \} \right], \quad (12)$$

Therefore, the deflection of the semi-infinite plate having the boundary of tooth profile can be obtained from Eq. (12).

It can be accepted in general case that if there is no friction between two engaging teeth, a load acts on tooth profile at right angles to it, and this load acts as a distributed load on the very small area of tooth surface based on Hertzian contact pressure distribution. In the present study, the distributed load was approximated with a group of many rectangular loads having widths equal to each other as shown in Fig.3. Moreover, these rectangular loads were replaced with the concentrated loads which are equal to each area of rectangle, and act at the centers ($1 \sim m$) of the widths of rectangles. The acting directions of these concentrated loads are parallel to the

Fig.2 Conformal transformation

Fig.3 Approximation of contact pressure distribution

direction of the center of Hertzian pressure distribution which is at right angles to the tooth profile. Putting φ_0 of Eq.(10) as $\varphi_0(\zeta; \tau)$, and $\varphi(\zeta)$, $\psi(\zeta)$ of Eqs.(9) and (10) as $\varphi(\zeta; \tau)$, $\psi(\zeta; \tau)$ respectively, and superposing them, φ and ψ for many concentrated loads are expressed as follows:

$$\varphi_0(\zeta) = \sum \varphi_0(\zeta; \tau), \quad \varphi(\zeta) = \sum \varphi(\zeta; \tau),$$

$$\psi(\zeta) = \sum \psi(\zeta; \tau) \quad (13)$$

The deflection produced by many concentrated loads can be calculated by substituting Eq.(13) into Eq.(12).

2.2 Mapping function and analytical function

Though it is difficult to obtain such a function that its mapped figure coincides exactly with the actual tooth profile, we choose the mapping function of Eq.(8). The mapped figure of this function has such a contour as shown in Fig.2. By choosing the more suitable constants $c, a, m, b, n, d, p, \dots$ in Eq.(8) and taking the larger number of fractional terms, the degree of approximation to the actual tooth profile can be improved further. Closer approximation is required for the calculation of deflection than for the calculation of root fillet stress. Therefore, in the present study, we used a mapping function having seven fractional terms, of which the mapped figures are in good agreement with the tooth profiles having various numbers of teeth and addendum modification coefficients. This mapping function is expressed as

$$z = \omega(\zeta) = c\zeta - \frac{m}{\zeta+a} - \frac{n}{\zeta+b} - \frac{p}{\zeta+d} - \frac{q}{\zeta+e} - \frac{r}{\zeta+f} - \frac{s}{\zeta+g} - \frac{t}{\zeta+h} \quad (14)$$

By using Eq.(14), $\varphi(\zeta)$ and $\psi(\zeta)$ can be calculated; substituting Eq.(14) into Eq.(9), $\varphi(\zeta)$ is obtained as

$$\varphi(\zeta) = \varphi_0(\zeta) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{\varphi}'(-\lambda)}{\omega'(-\lambda)} \times \frac{c\lambda - \frac{m}{\lambda+a} - \frac{n}{\lambda+b} - \frac{p}{\lambda+d} - \frac{q}{\lambda+e} - \frac{r}{\lambda+f} - \frac{s}{\lambda+g} - \frac{t}{\lambda+h}}{\lambda - \zeta} d\lambda \quad (15)$$

In the integral on the right-hand side of Eq.(15), the singular points stand at seven points $(-a), (-b), (-d), (-e), (-f), (-g)$ and $(-h)$, and using the residue of each point in Eq.(15), $\varphi(\zeta)$ is written as follows:

$$\varphi(\zeta) = \varphi_0(\zeta) + \frac{m}{\zeta+a} \frac{\bar{\varphi}'(a)}{\bar{\omega}'(a)} + \frac{n}{\zeta+b} \frac{\bar{\varphi}'(b)}{\bar{\omega}'(b)} + \frac{p}{\zeta+d} \frac{\bar{\varphi}'(d)}{\bar{\omega}'(d)} + \frac{q}{\zeta+e} \frac{\bar{\varphi}'(e)}{\bar{\omega}'(e)} + \frac{r}{\zeta+f} \frac{\bar{\varphi}'(f)}{\bar{\omega}'(f)} + \frac{s}{\zeta+g} \frac{\bar{\varphi}'(g)}{\bar{\omega}'(g)} + \frac{t}{\zeta+h} \frac{\bar{\varphi}'(h)}{\bar{\omega}'(h)}, \quad (16)$$

where $\bar{\omega}'(a), \dots, \bar{\omega}'(h)$ are obtained by substituting $\bar{\zeta} = a, \dots, h$ in

$$\bar{\omega}'(\bar{\zeta}) = c + \frac{m}{(\bar{\zeta}+a)^2} + \frac{n}{(\bar{\zeta}+b)^2} + \frac{p}{(\bar{\zeta}+d)^2} + \frac{q}{(\bar{\zeta}+e)^2} + \frac{r}{(\bar{\zeta}+f)^2} + \frac{s}{(\bar{\zeta}+g)^2} + \frac{t}{(\bar{\zeta}+h)^2}.$$

In order to evaluate $\bar{\varphi}'(a), \bar{\varphi}'(b), \dots, \bar{\varphi}'(h)$, we differentiate $\varphi(\zeta)$ in Eq.(16) with respect to ζ , and first of all putting $\zeta = a$, we obtain $\varphi'(a) = \varphi'_0(a) - A_1\bar{\varphi}'(a) - A_2\bar{\varphi}'(b) - A_3\bar{\varphi}'(d) - A_4\bar{\varphi}'(e) - A_5\bar{\varphi}'(f) - A_6\bar{\varphi}'(g) - A_7\bar{\varphi}'(h)$, where $A_1 = m/[4a^2\bar{\omega}'(a)]$,

$$A_2 = n/[(a+b)^2\bar{\omega}'(b)], \quad A_3 = p/[(a+d)^2\bar{\omega}'(d)],$$

$$A_4 = q/[(a+e)^2\bar{\omega}'(e)], \quad A_5 = r/[(a+f)^2\bar{\omega}'(f)],$$

$$A_6 = s/[(a+g)^2\bar{\omega}'(g)], \quad A_7 = t/[(a+h)^2\bar{\omega}'(h)]$$

$\varphi'_0(a)$ in Eq.(10) can also be obtained by substituting $\zeta = a$ in differentiated Eq.(10). In a similar manner, $\varphi'(b), \varphi'(d), \varphi'(e), \varphi'(f), \varphi'(g)$ and $\varphi'(h)$ can be evaluated, and then their coefficients are $B_1, \dots, B_7, D_1, \dots, D_7, E_1, \dots, E_7, F_1, \dots, F_7, G_1, \dots, G_7, H_1, \dots, H_7$, respectively. Expressing these in a lump, we obtain

$$\left. \begin{aligned} \varphi'(a) &= \varphi'_0(a) - A_1\bar{\varphi}'(a) - A_2\bar{\varphi}'(b) - A_3\bar{\varphi}'(d) - A_4\bar{\varphi}'(e) - A_5\bar{\varphi}'(f) - A_6\bar{\varphi}'(g) - A_7\bar{\varphi}'(h) \\ \varphi'(b) &= \varphi'_0(b) - B_1\bar{\varphi}'(a) - B_2\bar{\varphi}'(b) - B_3\bar{\varphi}'(d) - B_4\bar{\varphi}'(e) - B_5\bar{\varphi}'(f) - B_6\bar{\varphi}'(g) - B_7\bar{\varphi}'(h) \\ \varphi'(d) &= \varphi'_0(d) - D_1\bar{\varphi}'(a) - D_2\bar{\varphi}'(b) - D_3\bar{\varphi}'(d) - D_4\bar{\varphi}'(e) - D_5\bar{\varphi}'(f) - D_6\bar{\varphi}'(g) - D_7\bar{\varphi}'(h) \\ \varphi'(e) &= \varphi'_0(e) - E_1\bar{\varphi}'(a) - E_2\bar{\varphi}'(b) - E_3\bar{\varphi}'(d) - E_4\bar{\varphi}'(e) - E_5\bar{\varphi}'(f) - E_6\bar{\varphi}'(g) - E_7\bar{\varphi}'(h) \\ \varphi'(f) &= \varphi'_0(f) - F_1\bar{\varphi}'(a) - F_2\bar{\varphi}'(b) - F_3\bar{\varphi}'(d) - F_4\bar{\varphi}'(e) - F_5\bar{\varphi}'(f) - F_6\bar{\varphi}'(g) - F_7\bar{\varphi}'(h) \\ \varphi'(g) &= \varphi'_0(g) - G_1\bar{\varphi}'(a) - G_2\bar{\varphi}'(b) - G_3\bar{\varphi}'(d) - G_4\bar{\varphi}'(e) - G_5\bar{\varphi}'(f) - G_6\bar{\varphi}'(g) - G_7\bar{\varphi}'(h) \\ \varphi'(h) &= \varphi'_0(h) - H_1\bar{\varphi}'(a) - H_2\bar{\varphi}'(b) - H_3\bar{\varphi}'(d) - H_4\bar{\varphi}'(e) - H_5\bar{\varphi}'(f) - H_6\bar{\varphi}'(g) - H_7\bar{\varphi}'(h) \end{aligned} \right\} \quad (17)$$

and form seven conjugate equations from Eq.(17). By solving the simultaneous equations consisting of the conjugate equations and Eq.(17), $\bar{\varphi}'(a), \bar{\varphi}'(b), \bar{\varphi}'(d), \bar{\varphi}'(e), \bar{\varphi}'(f), \bar{\varphi}'(g)$ and $\bar{\varphi}'(h)$ are obtained. Therefore, $\varphi(\zeta)$ can be evaluated by substituting $\bar{\varphi}'(a), \dots, \bar{\varphi}'(h)$ in Eq.(16).

In the next place, $\psi(\zeta)$ should be evaluated. $\bar{\omega}(-\lambda)$ and $\omega'(\lambda)$ in the integral on the right-hand side of Eq.(11) are expressed as

$$\bar{\omega}(-\lambda) = -c\lambda - \frac{m}{-\lambda+a} - \frac{n}{-\lambda+b} - \frac{p}{-\lambda+d} - \frac{q}{-\lambda+e} - \frac{r}{-\lambda+f} - \frac{s}{-\lambda+g} - \frac{t}{-\lambda+h}, \quad (18)$$

$$\omega'(\lambda) = c + \frac{m}{(\lambda+a)^2} + \frac{n}{(\lambda+b)^2} + \frac{p}{(\lambda+d)^2} + \frac{q}{(\lambda+e)^2} + \frac{r}{(\lambda+f)^2} + \frac{s}{(\lambda+g)^2} + \frac{t}{(\lambda+h)^2}, \quad (19)$$

and since the path of integration of Γ is in the domain $\mathcal{D}$,

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\bar{\omega}(-\lambda)}{\omega'(\lambda)} \frac{\varphi'(\lambda)}{\lambda - \zeta} d\lambda$$

has eight singular points of $\lambda \rightarrow a, \lambda \rightarrow b, \lambda \rightarrow d, \lambda \rightarrow e, \lambda \rightarrow f, \lambda \rightarrow g, \lambda \rightarrow h$ and $\lambda \rightarrow \zeta$. When the above expression is integrated by taking account

of these singular points, the following equation is derived:

$$\begin{aligned} \psi(\zeta) = & \frac{P}{2\pi} e^{-i\theta} \log(\zeta - \tau) - \frac{m}{a-\zeta} \frac{\varphi'(a)}{\omega'(a)} - \frac{n}{b-\zeta} \frac{\varphi'(b)}{\omega'(b)} \\ & - \frac{p}{d-\zeta} \frac{\varphi'(d)}{\omega'(d)} - \frac{q}{e-\zeta} \frac{\varphi'(e)}{\omega'(e)} - \frac{r}{f-\zeta} \frac{\varphi'(f)}{\omega'(f)} \\ & - \frac{s}{g-\zeta} \frac{\varphi'(g)}{\omega'(g)} - \frac{t}{h-\zeta} \frac{\varphi'(h)}{\omega'(h)} - \frac{\bar{\omega}(-\zeta)\varphi'(\zeta)}{\omega'(\zeta)}, \end{aligned} \quad (20)$$

where $\omega'(a), \dots, \omega'(\zeta)$ are obtained by replacing λ in Eq.(19) with $a, \dots, \zeta$, respectively, and $\bar{\omega}(-\zeta)$ is also obtained by using Eq.(18). $\varphi'(a), \dots, \varphi'(h)$ have been already calculated in Eq.(17), and $\varphi'(\zeta)$ can be obtained by differentiating Eq.(16).

3. Calculation results and consideration

3.1 Gear tooth profile and mapped figure

The figure obtained from the mapping function of Eq.(14) approximates the actual tooth profile when suitable coefficients $a, b, c, d, e, f, g, h, m, n, p, q, r, s$ and t are chosen. Figure 4 shows the actual tooth profiles (full lines) of the spur gears having the addendum modification coefficients $x=0, 0.5$ and -0.5 and their mapped figures (broken lines) obtained from Eq.(14). The dimensions of the spur gear are of the module $m=4$, the pressure angle $\alpha_0=20^\circ$ and the number of teeth $z=28$. It is seen from Fig.4 that the mapped figure is in a good agreement with the tooth profile both at tooth surface and at root fillet. Since the mapped figure is of a continuous curve, it has a protuberance outside of tooth tip,

Fig.4 Gear tooth profile and mapped figure

which the actual tooth profile has not. However, it can be considered that the influence of the protuberance on the calculation result of tooth deflection is negligibly small, because the protuberance stands beyond the working surface.

3.2 Comparison between results obtained from well-known tooth deflection formulas and results calculated in present study

Figure 5 shows an example of the calculated tooth deflection for the gear of $m=4$, $\alpha_0=20^\circ$, $z=28$ and $x=0$. Here the applied load per unit tooth width is $P_n/b=10$ kgf/mm. The tooth deflection shown in this figure was obtained by magnifying the deflections u (in x direction) and v (in y direction) calculated by Eq.(12) both for each point on the tooth surface and for the center line of the tooth. The calculation for Fig.5 was carried out assuming that there are no displacement and no inclination at the position on x -axis corresponding to the gear center, because a gear has not any deflection at its center (calculation A). It is seen from Fig.5 that the gear tooth deforms to lift up the tooth bottom on the loaded side and to sink in it on the opposite side, and that the load point w moves to deviate from the direction of the line of action (point w'). The hollow at the load point may be attributed to a deformation resulting from Hertzian contact. In the calculation for Fig.5 a distributed load which was based on Hertzian contact pressure distribution in the contact between the tooth profile and the edge of a plate was used, and the distributed load was approximated with a group of ten concentrated loads (Fig.3, $m=10$). Then the deflection at point P (the center of the distributed load) in Fig.3 was adopted as the deflection at the load point. The tooth deflections calculated by using both $m=30$ and $m=50$ decrease respectively by 0.4% and 0.5% as compared with the deflection for $m=10$, and there is little difference between them. For this reason, $m=10$ will be used in the following calculation.

Figure 6 shows a deflection (calculation A) at the load point versus different meshing positions. This deflection was obtained by converting the deflection of Fig.5 into a deflection in the direction of the line of action. The abscissa of Fig.6 denotes the meshing position which is expressed with the roll distance at mesh

Fig.5 Example of calculation result of tooth deflection

$$m=4, \alpha_0=20^\circ, z=28, x=0, P_n/b=10 \text{ kgf/mm}$$

Fig. 6 Relation between total tooth deflection and meshing position

point s , and here s is defined as the distance from the interference point to the load point on the line of action. The interference point is a point of contact of the line of action and the base circle. In order to compare with our calculation result, in Fig. 6, the value obtained from Ishikawa's formula⁽²⁾ which is derived by approximating the gear tooth with both a rectangle and a trapezoid, the value from his new formula derived by using three trapezoids⁽⁶⁾ and the value measured by him⁽⁶⁾ also are shown. Further, since Ishikawa's formula has the assumption that there are no displacement and an appropriate angle of rotation at the intersecting point of the center line of tooth and the root circle, in order to compare with this formula we calculated the deflection under the above assumption, and the result is also shown in Fig. 6 (calculation B). It is seen from Fig. 6 that our calculation result is in a good agreement in tendency with the measured value and the value obtained from the new formula derived by using three trapezoids, but our total tooth deflection is rather smaller than the others.

The reason can be discussed as follows. In Fig. 5, the deflection along the center line of tooth is already shown. From this figure, it seems that there is little influence of the Hertzian contact deformation on the deflection on the center line of tooth. Therefore, when the deflection (vector $\overline{FF'}$ in Fig. 5) at the intersecting point F of the line of action and the center line of tooth is known, and when the component of this deflection in the direction of the line of action is calculated, we go on to obtaining the deflection by subtracting the Hertzian contact deformation from the total deflection at the load point w . The resulting deflection (calculation A) at different meshing positions (roll distances at mesh point s) is shown in Fig. 7. Also this figure shows the values obtained by

$$m=4, \alpha_0=20^\circ, z=28, x=0, P_n/b=10 \text{ kgf/mm}$$

Fig. 7 Deflection obtained by subtracting Hertzian contact deformation

$$m=4, \alpha_0=20^\circ, z=28, x=0, P_n/b=10 \text{ kgf/mm}$$

Fig. 8 Hertzian contact deformation

subtracting the Hertzian contact deformation from both the values by Ishikawa's formula and his new formula using three trapezoids. Here, putting the deflections by the bending, the shearing and the inclination of tooth foundation as δ_B , δ_S and δ_G respectively, the resulting value is equal to $\delta_B + \delta_S + \delta_G$. Furthermore, in a similar manner, the resulting deflection for calculation B was evaluated from Fig. 6, and it is also shown in Fig. 7. It is seen from Fig. 7 that our resulting deflection is in a good agreement with the value resulting from the new formula using three trapezoids. From this fact it seems that it is on account of the smaller Hertzian contact deformation that our total deflection shown in Fig. 6 is smaller than the others. Therefore, the Hertzian contact deformation will be discussed in the following section.

3.3 Hertzian contact deformation

Figure 8 shows the Hertzian contact deformation δ_P (calculation A) at different

meshing positions. This deformation was obtained by subtracting the deflection at the point F (Fig.5) from the total deflection at the load point along the direction of the line of action. For comparison with our Hertzian contact deformation, the values (Lundberg A, Lundberg B) obtained from Lundberg's formula⁽⁵⁾ which is used in Ishikawa's formula, the value (Hertz A) from Hertzian formula⁽⁷⁾ and the deformation (Hertz B) calculated by taking account of only the height of a circular arc that Hertzian contact width holds⁽⁸⁾ also are shown in Fig.8. It is seen from Fig.8 that the closer a meshing position gets to the tooth tip, i.e., the longer the roll distance at mesh point becomes, the larger the value of Hertz A becomes. However, the value of Hertz B is constant regardless of the meshing position. Our Hertzian contact deformation is the amount of hollow of one of a pair of meshing teeth, while the deformation obtained from Lundberg's formula contains also the amount of hollow of the contacting opposite body in addition to tooth. Therefore, Lundberg A is shown in Fig.8, which is half of the deformation which is obtained when two cylinders having the same radius equal to the roll distance at mesh point S contact each other. When our Hertzian contact deformation is compared with Lundberg A, it can be seen that both are in good agreement in the tendency to decrease as the meshing position approaches the tooth tip, but our Hertzian contact deformation has a smaller value than Lundberg A, and it shows an intermediate value between Hertz B and Lundberg A. Lundberg B in Fig.8 shows the amount of the contact deformation used in both Ishikawa's formula and his new formula. Since the value of Lundberg B contains also the amount of hollow of the opposite plate contacting with the tooth, it ought to become larger than our contact deformation. From the above consideration, one of the reasons that our total deflection shows a smaller value than the others in Fig.6 may lie in the difference of the condition that our contact deformation is that of one of a pair of meshing teeth, while other contact deformation contains also that of the opposite body in addition to tooth. However,

Fig.9 Relation between tooth deflection and tooth load

even if this difference is taken into consideration, our contact deformation still shows a small value.

3.4 Relation between deflection and tooth load

Figure 9 shows the calculated deflection for the gear tooth of $m=4$, $\alpha=20^\circ$ and $x=0$ at different tooth loads. Figures affixed to the deflection curves stand for the roll distance at mesh point S, i.e., the meshing position. $S=3.28\text{mm}$ shows the starting position in the engagement of a pair of same gears having $z=18$ and $x=0$, and $S=21.34\text{mm}$ shows the meshing position at the tooth tip. In Fig.9, the tooth load is limited to its maximum under which the contact stress amounts to 200kgf/mm^2 at the meshing position of $S=3.28\text{mm}$. It is seen from Fig.9 that when the meshing position comes nearly at the tooth root, the tooth deflection increases nearly proportionally with the increase in the tooth load, but as the meshing position approaches the tooth tip, the relation between them deviates slightly from the proportional expression. However, we can consider that the tooth deflection increases in proportion to the tooth load. Therefore, in a calculation of vibration of spur gears, the tooth stiffness can be treated as a constant independent of the tooth load.

4. Conclusions

The results obtained in the present paper are summarized as follows:

(1) A calculation method for the deflection of the spur gear teeth is shown, which was developed by using the two-dimensional elastic theory and the conformal mapping function. By using this method, the tooth deflection based on the true stress arising in the tooth under load can be calculated without approximating the tooth profile with another contour, and the Hertzian contact deformation can also be obtained at the same time.

(2) The mapping function with seven fractional terms shown in Eq.(14) can satisfactorily express various gear tooth profiles.

(3) The tooth deflection calculated by using the present method is in good agreement in tendency with the experimental value, but the total tooth deflection has a smaller value than the latter. From comparison with the well-known tooth deflection formulas, it can be said that the cause for the above result lies in the smaller Hertzian contact deformation.

(4) The Hertzian contact deformation calculated by using the present method is larger than that obtained by taking account of only the height of a circular arc that Hertzian contact width holds, and has a smaller value than the contact deformation of Lundberg's formula, and it has an intermediate value of these two contact deformations.

(5) The tooth deflection is nearly proportional to the tooth load. Therefore,

in a calculation of vibration of spur gears, the tooth stiffness can be treated as a constant independent of the tooth load.

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