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The stability of flows of two different fluids which are separated by a thin elastic plate*

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The stability of flows of two different fluids which are separated by a thin elastic plate is studied on the basis of hydrodynamical stability with respect to small disturbances. The governing equations are derived assuming that both fluids are inviscid and incompressible, and compressive or tensile in-plane forces act on the elastic plate. The growth rate as a function of the wave number, the critical wave number and the critical in-plane force are found with the aid of an eigen relation. For long waves the stability of the system depends only on the density difference of both fluids and does not depend on the properties of the elastic plate at all. In the case in which the lower fluid is heavier than the upper one, instability does not arise, unless magnitude of compressive in-plane force exceeds the critical value.

1. Introduction

One method for continuous heat exchange in various processes is to use the flow of two immiscible fluids. To do so, it is desirable that two fluids contact directly each other, for the efficiency of heat exchange. But we seldom have perfectly immiscible fluids, such as water and motor oil, in practice, and even if we have such fluids, a wave motion takes place on the interface between the two immiscible fluids. To remove these difficulties, it is conceivable that two fluid layers are separated by a thin plate. In this case, it is good for heat exchange that the separating plate is thin. But the plate is limited in thickness from a point of view of the structural safety.

Let the flows of two fluid layers be separated by a horizontal plane under the action of gravity. If the force due to a pressure difference between both sides of the plate is greater than one which is required to bend the plate, the system considered will become unstable. This instability is a fluttering phenomenon in aeroelasticity⁽¹⁾. On the other hand, an instability called Rayleigh-Taylor type⁽²⁾ appears in two fluid layers under the action of gravity, even if these fluids are at rest.

Investigation as to instability of the plate in the two fluid flows has been made only for the former case as mentioned in Bolotin's book⁽³⁾. The flows of two fluid layers with the interface replaced with a thin plate have instability for the latter case also. Then there is a possibility that instability due to the density difference between two fluid layers appears in this problem.

In this paper, instability of two fluid

layers, separated horizontally by a thin elastic plate, which flow along the surface of the plate in any direction with any velocity is studied. In case that the flow velocities of two fluids are small, instability can occur. Then fluids are assumed to be incompressible. The limit of the region in which instability appears and the growth rate of disturbances in its region are mainly discussed.

2. Governing Equations

Let two fluids which are incompressible and inviscid be separated horizontally by a thin elastic plate with the thickness h , the density ρ_0 and the bending stiffness D as shown in Fig.1. We take x and y axes on the neutral surface of the plate and z axis vertically upward from its surface. The upper fluid of the density ρ_1 flows with a constant velocity of which x and y components are U_1 and V_1 , respectively, and the lower fluid of the density ρ_2 flows with a constant velocity U_2 in the x direction. In this steady state the pressure in the upper fluid P_1 and that in the lower one P_2 are $-\rho_1 g x(x-h/2)$ and $-\rho_2 g(z+h/2) + \rho_2 g h$, respectively. Here, g is the gravitational acceleration. The transverse displacement of the elastic plate is zero. The instability of this steady state is examined by the linear theory of the hydrodynamical stability.

Fig.1. Sketch of configuration and coordinate system.

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Let this steady state receive a slight perturbation, in which the velocity components and the pressures in the upper fluid and the lower one are denoted by u_1, v_1, w_1, p_1 and u_2, v_2, w_2, p_2 , respectively. These quantities are functions of coordinates x, y, z and time t . The transverse displacement ζ of the plate due to the perturbation is a function of x, y and t . The differential equations governing the perturbation quantities are obtained from the equations of the fluid motion, the continuity equations and the equation of the thin elastic plate⁽⁴⁾ to the first order of the small quantities as follows:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \quad (1)$$

$$\frac{\partial u_1}{\partial t} + U_1 \frac{\partial u_1}{\partial x} + V_1 \frac{\partial u_1}{\partial y} = -\frac{1}{\rho_1} \frac{\partial p_1}{\partial x} \quad (2)$$

$$\frac{\partial v_1}{\partial t} + U_1 \frac{\partial v_1}{\partial x} + V_1 \frac{\partial v_1}{\partial y} = -\frac{1}{\rho_1} \frac{\partial p_1}{\partial y} \quad (3)$$

$$\frac{\partial w_1}{\partial t} + U_1 \frac{\partial w_1}{\partial x} + V_1 \frac{\partial w_1}{\partial y} = -\frac{1}{\rho_1} \frac{\partial p_1}{\partial z} \quad (4)$$

$$\rho_0 h \frac{\partial^2 \zeta}{\partial t^2} + D \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \zeta - \left(N_x \frac{\partial^2}{\partial x^2} + 2N_{xy} \frac{\partial^2}{\partial x \partial y} + N_y \frac{\partial^2}{\partial y^2} \right) \zeta = f_1 - f_2 \quad (5)$$

Here, N_x, N_y are the x and y components of the normal in-plane force and N_{xy} is a shearing in-plane force. These forces are measured per unit length. f_1 and f_2 are forces on unit area which act on the plate from the upper fluid and the lower one, respectively: $f_1 = p_1 - \rho_1 g \zeta$ and $f_2 = p_2 - \rho_2 g \zeta$. The suffix i takes 1 for the upper fluid and 2 for the lower one. It is noticed that V_x is zero as mentioned above.

Let us now consider the boundary conditions for the above equations. The kinematic conditions, that the velocity of the transverse motion of the plate is equal to that of the fluid on both sides of the plate, are written as

$$z = h/2 : \frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x} + V_1 \frac{\partial \zeta}{\partial y} = w_1 \quad (6)$$

$$z = -h/2 : \frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} = w_2 \quad (7)$$

Since the perturbation must vanish at infinity, the following conditions are imposed on the above equations:

$$z \rightarrow \infty : u_1, v_1, w_1 \rightarrow 0 \quad (8)$$

$$z \rightarrow -\infty : u_2, v_2, w_2 \rightarrow 0 \quad (9)$$

These conditions are also expressed to the first order of the small quantities.

Eliminating u_i, v_i and p_i from Eqs. (1)-(9), we obtain the following differential system for w_i and ζ :

$$\Delta w_1 = 0 \quad (10)$$

$$\Delta w_2 = 0 \quad (11)$$

$$D \Delta^2 \zeta - \Delta \left(N_x \frac{\partial^2}{\partial x^2} + 2N_{xy} \frac{\partial^2}{\partial x \partial y} + N_y \frac{\partial^2}{\partial y^2} \right) \zeta$$

$$+ \rho_1 \left(\frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x} + V_1 \frac{\partial}{\partial y} \right) \frac{\partial w_1}{\partial z} \Big|_{z=h/2}$$

$$- \rho_2 \left(\frac{\partial}{\partial t} + U_2 \frac{\partial}{\partial x} \right) \frac{\partial w_2}{\partial z} \Big|_{z=-h/2}$$

$$+ g(\rho_1 - \rho_2) \Delta \zeta + \rho_0 h \Delta \frac{\partial^2 \zeta}{\partial t^2} = 0 \quad (12)$$

$$z = h/2 : \frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x} + V_1 \frac{\partial \zeta}{\partial y} = w_1 \quad (13)$$

$$z = -h/2 : \frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} = w_2 \quad (14)$$

$$z \rightarrow \infty : w_1 \rightarrow 0 \quad (15)$$

$$z \rightarrow -\infty : w_2 \rightarrow 0 \quad (16)$$

where, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. This differential system is the equations governing our problem.

3. Solution

Since our problem is uniform along the plate, perturbation quantities w_i and ζ can be expressed in the form of Fourier's component as follows:

$$w_i(t, x, y, z) = W_i(z) e^{i(k_x x + k_y y) + \sigma t} \quad (17)$$

$$\zeta(t, x, y) = \eta e^{i(k_x x + k_y y) + \sigma t} \quad (18)$$

where, k_x and k_y are the x and y components of the wave number vector k of which the magnitude k is given by $k^2 = k_x^2 + k_y^2$. σ is the complex growth rate. Substituting Eqs. (17), (18) into Eqs. (10), (11), (13)-(16), we have

$$W_1 = (\sigma + ik_x U_1 + ik_y V_1) \eta e^{-k(z-h/2)} \quad (19)$$

$$W_2 = (\sigma + ik_x U_2) \eta e^{k(z+h/2)} \quad (20)$$

The differential operations $\partial/\partial t$, $\partial/\partial x$ and $\partial/\partial y$ correspond to multiplying by σ , ik_x and ik_y , respectively. The derivatives $\partial w_1/\partial z$ and $\partial w_2/\partial z$ correspond to $-k w_1$ and $k w_2$. $(U_1 k_x + V_1 k_y)/k$ and $U_2 k_x/k$ equal to the projective components q_1 and q_2 in the direction of the traveling waves due to disturbances. Taking these facts into account and substituting Eqs. (17)-(20) into Eq. (12), we obtain the following relation from the compatibility condition:

$$(\rho_1 + \rho_2 + \rho_0 h k) \sigma^2 + 2ik(\rho_1 q_1 + \rho_2 q_2) \sigma + D k^4 + N k^4 - (\rho_1 q_1^2 + \rho_2 q_2^2) k^2 + k g(\rho_2 - \rho_1) = 0 \quad (21)$$

where, $N_x k_x^2 + 2N_{xy} k_x k_y + N_y k_y^2 = N$. If the real part of σ given by this equation is positive, our system becomes unstable. The eigen relation obtained above agrees formally with that of the stability problem of two fluid layers, which flow with velocities q_1 and q_2 in the x direction, separated by a thin plate.

The coefficient N contained in Eq. (21) corresponds to the coefficient of surface tension acting on the interface between two fluid layers⁽⁵⁾. The former has a positive value for the tensile in-plane force and a negative for the compressive one. But the latter has a positive value only. This is an important difference between the two. In the case of the heat exchanger considered in Chap. 1, the plate may be subjected to the compressive in-plane force due to thermal expansion.

4. Results and Discussion

The eigen relation (21), derived from the condition that our differential system has a non-trivial solution, can be rewritten by using the following non-dimensional variables:

$$\left. \begin{aligned} \rho &= \rho_1 / \rho_2, & q &= q_1 / q_2, & \gamma &= \rho_0 / \rho_2 \\ D^* &= D / \rho_2 g h^4, & N^* &= N / \rho_2 g h^2 \\ \sigma^* &= \sigma / \sqrt{g / h}, & k^* &= k h, & F &= q^2 / g h \end{aligned} \right\} \quad (22)$$

And we have

$$(\rho + 1 + \gamma k) \sigma^2 + 2i\sqrt{F}k(\rho q + 1)\sigma + Dk^2 + Nk^3 - F(\rho q^2 + 1)k^2 + k(1 - \rho) = 0 \quad (23)$$

For simplicity, we neglect symbol * denoting the non-dimensional variables after here. The discriminant of the quadratic equation of the complex growth rate σ ,

$$J = Fk^2(\rho q + 1)^2 + k(\rho + 1 + \gamma k)(Dk^2 + Nk^3 - F(\rho q^2 + 1)k + 1 - \rho) \quad (24)$$

is positive or negative. Depending on whether this discriminant is positive or negative, the real part of the complex growth rate σ_r and its imaginary part σ_i are given as follows: in case that $J > 0$,

$$\sigma_r = 0 \quad (25)$$

$$\sigma_i = (-\sqrt{F}k(\rho q + 1) \pm \sqrt{J}) / (\rho + 1 + \gamma k) \quad (26)$$

and in case that $J < 0$,

$$\sigma_r = \pm \sqrt{-J} / (\rho + 1 + \gamma k) \quad (27)$$

$$\sigma_i = -\sqrt{F}k(\rho q + 1) / (\rho + 1 + \gamma k) \quad (28)$$

The stability of our system is determined by the sign of the quantity σ_r . The system is stable or unstable according as it is negative or positive. When it is zero, the system becomes neutral. It is found from Eqs. (25)-(28) that there is not a stable state in our system. In case that $J \geq 0$, it is neutrally stable and in case that $J < 0$, it is unstable. The limit of the unstable region is given by

$$J = 0 \quad (29)$$

If Eq. (24) is divided by a positive value $(\rho + 1 + \gamma k)k$, the discriminant can be written in the following form,

$$J' = J / (\rho + 1 + \gamma k)k = Dk^2 + Nk^2 - (\rho - 1) - Fk \frac{\rho(q-1)^2 + \gamma k(\rho q^2 + 1)}{\rho + 1 + \gamma k} \quad (24)'$$

Then in case that J' is negative, the system is unstable and the limit is given by

$$J' = 0 \quad (29)'$$

4.1 Instability and critical conditions in case of no flowing system

The stability characteristics in case of no flowing system are determined by the discriminant when zero is substituted into F in Eq. (24)'

$$J_0 = Dk^2 + Nk^2 - (\rho - 1) \quad (24)''$$

N , denoting the in-plane force, can be positive or negative, and the density ratio ρ can be larger or smaller than unity. Then the value J_0 can be positive or negative. It is found from Eq. (24)'' that the system becomes unstable in the following two cases:

(i) When the upper fluid is heavier than the lower one, the system becomes unstable irrespective of the value of N . Then J_0 is negative in the region of wave numbers smaller than a critical value. The system becomes unstable in this region. That is, the system is unstable in the wave number region

$$0 < k < k_{cr} \quad (30)$$

where the critical wave number k_{cr} is given by

$$k_{cr}^2 = \frac{-N + \sqrt{N^2 + 4D(\rho - 1)}}{2D} \quad (31)$$

(ii) When the upper fluid is lighter than the lower one, J_0 can be positive or negative according as whether the in-plane force N is larger or smaller than the critical in-plane force

$$N_{cr} = -\sqrt{4D(1 - \rho)} \quad (32)$$

In case of $N < N_{cr}$, the system becomes unstable in the following region of the wave numbers

$$k_{cr}' < k < k_{cr} \quad (34)$$

where two critical wave numbers k_{cr} and k_{cr}' are given by

$$k_{cr}^2 = \frac{-N + \sqrt{N^2 - 4D(1 - \rho)}}{2D} \quad (31)$$

$$k_{cr}'^2 = \frac{-N - \sqrt{N^2 - 4D(1 - \rho)}}{2D} \quad (33)$$

When the upper fluid is lighter than the lower one, there is a possibility that even if the compressive in-plane force acts on the plate, the system is stable. N_{cr} corresponds to a buckling load of the plate in our system.

4.2 The effect of the flow velocities on the critical conditions

Irrespective of the sign of the velocity ratio q in the flows, the coefficient F in the last term of Eq. (24)' certainly become positive. When F is large, J' certainly becomes negative. Therefore, our system is surely unstable in case of large flow velocity. This corresponds to the fluttering phenomenon in aeroelasticity.⁽¹⁾ Since the instability appears in the stationary state in our system, the effect of the flows of two fluid layers on the stability is only examined in case of small flow velocity.

When the flow velocity is small, the critical wave numbers k_{cr} and k_{cr}' obtained in section 4.1 can be expanded to the first order of F as follows:

$$k_{cr} \quad \text{and} \quad k_{cr}' + F \frac{\rho(q-1)^2 + \gamma k(\rho q^2 + 1)}{2(\rho + 1 + \gamma k)(2Dk^2 + N)} \quad (35)$$

The second term in this expression means the effect of the flows on the critical wave numbers. The factor $(2Dk^2 + N)$ in this term may have a positive or negative value. This factor is positive near the critical wave number k_{cr} given by Eq. (31). Then the critical value, Eq. (31), increases as the flow velocity increases. On the other hand, this factor is negative near the critical wave number k_{cr}' given by Eq. (33). Then the critical value, Eq. (33), decreases as the flow velocity increases. Therefore, if there is a flow, the system destabilizes and the wave number region in which disturbances are unstable enlarges.

When F is small, the critical in-plane force N_{cr} given by Eq. (32) is written to the first order of F as follows:

$$N_{cr} + Fk\sqrt{D} \frac{\rho(q-1)^2 + \gamma k(\rho q^2 + 1)}{\sqrt{1 - \rho(\rho + 1 + \gamma k)}} \quad (36)$$

Since the second term in this expression is always positive, the critical in-plane force decreases as the flow velocities increase. That is, the critical load decreases.

4.3 The growth rate

When the system is unstable, the growth rate σ_r is given by Eq. (27), that is,

$$\sigma_r^2 = \frac{k}{\rho+1+\gamma k} \left\{ \rho-1-Nk^2-Dk^4 + Fk \frac{\rho(q-1)^2+\gamma k(\rho q^2+1)}{\rho+1+\gamma k} \right\} \dots\dots\dots(27)'$$

It is found from this equation that an increase in compressive in-plane force and bending stiffness causes a decrease in the growth rate.

Meanwhile, even if the flow velocities are zero, instabilities appear in cases of a compressive in-plane force being larger than the critical load and of the upper fluid heavier than the lower fluid. In the latter case, if fluid layers are separated by a plate with large bending stiffness, instability diminishes. But in the former case, since Eq. (27)' can be written in the following expanded form

$$\sigma_r = \sqrt{\frac{\rho-1}{\rho+1} k \left(1 - \frac{\gamma}{2(\rho+1)} k - \frac{N}{2(\rho-1)} k^2 - \frac{D}{2(\rho-1)} k^4 + O(k^6) \right)} \dots\dots\dots(37)$$

the system becomes unstable even if the fluids are separated by an elastic plate. In the case of the small wave number the growth rate depends on the density difference and does not depend on the properties of the plate. Eq. (37) is derived to the first order of k assuming that γ , Nk and Dk^3 are of the order of unity.

When there is a flow, the growth rate is given by Eq. (27)'. Since the term containing F is positive, the system is destabilized by the flow. The flow destabilizes the system as the result of the flow causing an increase in the unstable wave region and in the growth rate.

4.4 The wave number at the maximum growth rate

A disturbance with the maximum growth rate is most dangerous in the unstable region. Then let us find the wave number at the maximum growth rate.

If there is no fluid flow, the wave number corresponding to the maximum growth rate $k_{\max}$ is given from Eq. (27)' as follows:

Fig.2. The effect of the tensile in-plane force on the instability when the upper fluid is heavier than the lower one.

$$\begin{aligned} -2(\rho+1+\gamma k)^2 \sigma_r \frac{d\sigma_r}{dk} &= 4D\gamma k^3 + 5D(\rho+1)k^4 \\ &+ 2N\gamma k^3 + 3N(\rho+1)k^2 + 1 - \rho^2 = 0 \dots\dots(38) \end{aligned}$$

This wave number decreases as the bending stiffness and the tensile in-plane force increase, and increases as the compressive in-plane force increases.

Fig.3. The effect of the compressive in-plane force on the instability when the upper fluid is lighter than the lower one.

Fig.4. The effect of the flows on the instability when the upper fluid is heavier than the lower one.

Fig.5. The effect of the bending stiffness on the instability when the upper fluid is heavier than the lower one.

4.5 The numerical examples

The growth rates in the unstable re-

gion are found numerically as a function of the wave number for the practical materials, for example, metal, plastic and rubber.

In case that the upper fluid is heavier than the lower one as mentioned in section 4.2, the growth rate is as shown in Fig.2. In case that the upper fluid is lighter than the lower one and the compressive in-plane force acts on the plate, the growth rate is as shown in Fig.3. Fig.4 shows the destabilizing effect of the flows. Fig.5 shows the stabilizing effect of the bending stiffness. These figures have characteristics as explained in Chap.3.

5. Conclusion

A flow of two fluid layers separated horizontally by a thin elastic plate is unstable or neutrally stable. Instabilities appear in case that the upper fluid is

heavier than the lower fluid or a compressive in-plane force exceeding the critical value acts on the plate. The critical wave numbers and the critically compressive in-plane force are obtained. The effect of the flow velocities and the properties of the plate on the growth rate is examined.

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