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An oil sheet on the wall rising
to a heavy fluid under gravity*

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When a heavy fluid is superposed over a light oil sheet adhering to a horizontal solid wall, the gravity will cause the lower fluid to rise to the upper fluid. In such a case, whether or not there is a stable stationary state is studied theoretically. It is assumed that the flow is two-dimensional, the fluids are incompressible, the upper fluid is ideal and the lower one is viscous. To the fourth order for wave number and wave amplitude, the height equation of the interface is derived from the long wave expansion. It is found from this equation that nonlinearities of the governing equation have a stabilizing effect and there is a stable equilibrium state with amplitude of about 0.4 times the height of the lower liquid layer.

1. Introduction

A vessel is often used for storage of various liquids over and over again. In such a case, a part of a liquid, which has been stored, tends to adhere to the inner wall of the vessel. A highly viscous liquid easily adheres to the wall. When a new liquid is poured into the vessel in which an old liquid remains on its wall, these liquids will mix with each other. This fact sometimes poses various engineering problems. One of the methods to settle this problem is perfect washing of the vessel. But even if an adhering liquid on the wall does not come off, we shall have substantially no difficulty in storage of a liquid in the used vessel. Then the question is under what condition the adhering liquid rises or falls away from the wall when a new liquid is poured into the vessel.

The problem considered is closely related to a pendent drop on the lower surface of the plate. Instability of the pendent drop is recently examined using the variational principle with respect to energy of the system.^{(1),(2)} In the investigation about the pendent drop of this kind, the contact area between the drop and the plate is assumed to be finite. But a highly viscous liquid adhering to the inner wall of the vessel has a form similar to a sheet. This is an important difference between our problem and the pendent drop. Our problem, under what condition the thin liquid layer on the lower or upper surface of the horizontal plate falls or rises in the other liquid, is one of the stability of the interface between the two fluids of different densities superposed one over the other under the action of gravity. Since instability in the latter case is caused by a density difference between the two fluids, we can

examine it without loss of essence, under the assumption that fluids are ideal. In fact, the critical condition under which instability occurs does not change even if we take the viscosity of the fluids into account⁽³⁾. We now wish to study a deformation of the free surface of a thin liquid film on the plane surface. In this case the Reynolds number based on the thickness of the liquid film is very small. Then there is a possibility that the fluid viscosity plays an important role. However, the viscosity of the fluid with which a thin liquid film is surrounded is not important.

The motion of an oil film rising into the heavier fluid largely differs from that of a pendent drop and of the interface due to instability of the Rayleigh-Taylor type. Under the assumption that the liquid film on the plane is viscous and the surrounding fluid is ideal, we study theoretically the motion of a horizontal interface between two fluids under the action of gravity taking the nonlinearity in the governing equation into account. We can suggest that an adhering liquid film lighter than the surrounding fluid will not rise if the film is very thin.

2. The governing equations

Two kinds of fluids fill the space between two horizontal flat plates set with a distance H as shown in Fig.1. The lower thin layer is a viscous liquid of density ρ and viscosity μ and average thickness h , and the upper layer is an ideal fluid of density ρ_* and velocity potential ϕ . The flow is assumed to be two-dimensional and incompressible. We take the x -axis along the lower plate and the y -axis vertically upward from it. The upper fluid satisfies the Laplace equation

$$\phi_{xx} + \phi_{yy} = 0 \quad \dots\dots\dots (1)$$

because the flow is irrotational. On the other hand the lower fluid satisfies the Navier-Stokes equation because it is a viscous, incompressible fluid. Making use of

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Fig.1 System and coordinate.

the stream function ψ , we have the following equation governing the flow of the lower fluid:

$$(\psi_{xx} + \psi_{yy})_t + [\psi_y(\psi_{xx} + \psi_{yy})]_x - [\psi_x(\psi_{xx} + \psi_{yy})]_y = \frac{\mu}{\rho}(\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) \dots (2)$$

where, t is the time. The boundary conditions for this equation are as follows: the condition that the normal velocity component vanishes at the upper plane, i.e.,

$$y=H: \psi_y=0 \dots (3)$$

the no slip condition at the lower plane, i.e.,

$$y=0: \psi_x=\psi_y=0 \dots (4)$$

The conditions that the normal and tangential forces are continuous at the interface $y=h(t, x)$ between the upper fluid and the lower fluid, are

$$\begin{aligned} y=h: (\rho_a - \rho)g(h - \bar{h}) + p - p_a \\ = -\frac{2\mu}{1+h_s^2} \left\{ (1-h_s^2)\psi_{yy} \right. \\ \left. + \frac{4h_s^2}{1-h_s^2} \psi_{xy} \right\} - \frac{\sigma h_{ss}}{(1+h_s^2)^{3/2}} \dots (5) \\ (1-h_s^2)(\psi_{yy} - \psi_{ss}) = 4h_s \psi_{sy} \dots (6) \end{aligned}$$

The kinematic condition that the velocity of the interface is equal to one of the fluids, is

$$y=h: h_t + \psi_x h_s = -\psi_s \dots (7)$$

and the condition that the normal velocities of both fluids are continuous at the interface, is

$$y=h: (\psi_s - \phi_s) h_s = -\phi_s - \phi_y \dots (8)$$

Here, g is the acceleration of gravity and subscripts t, x, y denote the differentiation with respect to the time t , coordinates x and y . The above equations (1)-(8) containing the unknown variables ϕ, ψ and h are the equations governing a flow of two fluids that consists of the upper layer with thickness $H - \bar{h}$ and the lower layer with thickness $\bar{h}$ at rest. In the equations derived above, p_a and p are the pressures of the upper fluid and the lower one, respectively. They are related by the equation of motion to ϕ and ψ :

$$p_a = -\rho_a \phi_t - \rho_a \frac{1}{2}(\phi_s^2 + \phi_y^2) \dots (9)$$

$$p = \mu(\psi_{xxx} + \psi_{yyy}) - \rho(\psi_{xt} + \psi_y \psi_{xy} - \phi_s \psi_{yy}) \dots (10)$$

We discuss in the following chapter how these equations can be solved.

3. Non-dimensionalized equations

As mentioned in the introduction, one of the remarkable features of this system is that the lower fluid layer is very thin in comparison with the upper one. We derive the non-dimensionalized equations taking account of this feature.

Since the upper and lower walls are infinitely long in the x direction, the disturbances appearing on the interface between the two fluid layers are periodic in the x coordinate. The characteristic wave length of these periodic disturbances is assumed to be L . Since the lower layer is very thin, the variables with respect to the lower fluid are non-dimensionalized as follows:

$$\left. \begin{aligned} x &= Lx^*, & y &= \bar{h}y^*, & h &= \bar{h}h^* \\ t &= \frac{L}{\sqrt{g\bar{h}}} t^*, & \phi &= \sqrt{g\bar{h}} \bar{h} \phi^* \\ p &= \rho g \bar{h} p^* \end{aligned} \right\} \dots (11)$$

On the other hand, since the upper layer is thick, the variables associated with the upper fluid are non-dimensionalized as follows:

$$\left. \begin{aligned} x &= Lx^*, & y &= Ly^*, & h &= \bar{h}h^*, & H &= LH^* \\ t &= \frac{L}{\sqrt{g\bar{h}}} t^*, & \phi &= \sqrt{g\bar{h}} L \phi^*, & p_a &= \rho g \bar{h} p^* \end{aligned} \right\} \dots (12)$$

In these expressions, the symbol $*$ denotes a non-dimensional quantity. Expressing Eqs. (1)-(8), (9) and (10) by these new variables, the equations governing the upper layer and the lower one reduce to

$$\phi_{xx} + \phi_{yy} = 0 \dots (13)$$

$$y=H: \phi_y = 0 \dots (14)$$

$$y=\alpha h: \phi_y = \alpha [(\phi_s - \psi_y) h_s - \phi_s] \dots (15)$$

$$\begin{aligned} \psi_{yyy} &= \alpha R(\psi_{yy} + \phi_s \psi_{yy} - \phi_s \psi_{yy}) - 2\alpha^2 \psi_{yyy} \\ &+ \alpha^2 R(\phi_{ss} + \phi_s \phi_{ss} - \phi_s \phi_{ss}) - \alpha^4 \psi_{yyy} \dots (16) \end{aligned}$$

$$y=h: (\gamma - 1)(h - 1) + p - p_a$$

$$= -\alpha \frac{2}{R} \frac{1}{1 + \alpha^2 h_s^2} \psi_{yy} - \alpha^2 \frac{W h_{ss}}{(1 + \alpha^2 h_s^2)^{3/2}} + \alpha^2 \frac{2}{R} \frac{h_s^2}{1 + \alpha^2 h_s^2} \left(1 - \frac{4}{1 - \alpha^2 h_s^2} \right) \psi_{xy} \dots (17)$$

$$\psi_{yy} = \alpha^2 (\phi_{ss} + h_s^2 \phi_{yy} + 4h_s \phi_{sy}) - \alpha^4 h_s^2 \phi_{ss} \dots (18)$$

$$h_t + \psi_x h_s + \phi_s = 0 \dots (19)$$

$$y=0: \psi_x = \psi_y = 0 \dots (20)$$

The equations governing the pressures p and p_a reduce to

$$p_a = -\gamma \left\{ \phi_t + \frac{1}{2}(\phi_s^2 + \phi_y^2) \right\} \dots (21)$$

$$\begin{aligned} p_s &= \frac{1}{\alpha R} \psi_{yyy} - (\psi_{xt} + \psi_y \psi_{xy} - \phi_s \psi_{yy}) \\ &+ \alpha \frac{1}{R} \phi_{yyy} \dots (22) \end{aligned}$$

where,

$$\gamma = \frac{\rho_a}{\rho}, \quad R = \frac{\sqrt{g\bar{h}} \bar{h}}{\mu/\sigma}, \quad \alpha = \frac{\bar{h}}{L}, \quad W = \frac{\sigma}{\rho g \bar{h}^2} \dots (23)$$

It should be noticed in the above equations that the position of the interface between the two layers is $y = \alpha h$ for the upper flu-

id and $y=h$ for the lower one. Assuming that the lower layer is thin compared with the upper one, that is, α is small, we solve the above equations. For simplicity, we neglect the symbol $*$ in Eqs.(13)-(22).

4. Solution when the thickness of the lower layer is small compared with the wave length of the capillary wave

The non-dimensional quantity α introduced in the above chapter is the ratio of thickness of the lower layer to the wave length of disturbance. The surface tension will play an important role in this case as inferred from example of the pendent drop mentioned in the introduction. There is a strong possibility that the disturbances caused by the surface tension appear on the interface. Then the wave length of disturbances will be of the order of 1 cm corresponding to the wave length of the capillary wave ⁽⁴⁾. While the thickness of the viscous liquid adhering to the wall is at most of the order of 1 mm in case of a highly viscous liquid. Then the non-dimensional variable α is small. The governing equations can be solved in the following form by expanding the velocity potential ϕ and the stream function ψ with respect to the small parameter α :

$$\phi = \phi_0 + \alpha\phi_1 + \alpha^2\phi_2 + \dots \quad (24)$$

$$\psi = \psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + \alpha^3\psi_3 + \dots \quad (25)$$

From substitution of these expanded forms into the equations in the above chapter, it follows that the zeroth order solution ϕ_0 is zero as the pressure is finite. Then the velocity potential ϕ of the upper fluid is at most of the order of α^2 . We start in the expanding procedure from the first order approximation for ϕ and the second order one for ψ .

4.1 The first approximation

The equations for the first approximation with respect to α , which are independent of p_* , are obtained from substitution of Eq.(25) into Eqs.(16)-(20) as follows:

$$\psi_{1,yyy} = 0 \quad (26)$$

$$y=h: (\gamma-1)(h-1) + p_0 = -W\alpha^2 h_{,ss} \quad (27)$$

$$\psi_{1,yy} = 0 \quad (28)$$

$$y=0: \psi_{1,s} = \psi_{1,y} = 0 \quad (29)$$

where,

$$p_0 = \frac{1}{R} \phi_{1,yyy} \quad (30)$$

We have

$$\phi_1 = A_2 y^2 + A_3 y^3 \quad (31)$$

where

$$A_2 = \frac{1}{2} R h \zeta \quad (32)$$

$$A_3 = -\frac{1}{6} R \zeta \quad (33)$$

$$\zeta = (\gamma-1)h_s + \alpha^2 W h_{,ss} \quad (34)$$

Since non-dimensional quantity W , which means the magnitude of the surface tension, becomes very large as h is very small, $\alpha^2 W$ is assumed to be of the order of unity.

4.2 The second approximation

ϕ is noticed above to be at most of the order of α^2 . Meanwhile, as ϕ_1 is of the order of α , h_s is from Eq.(19) of the order of α . We examine from Eq.(19) the order of p_* taking the above results into account. The virtually lowest solution of p_* with respect to α is derived from ϕ_1 in Eq.(21). As ϕ_1 is related to h_s , ϕ_1 is of the order of α^2 which is higher order than α^2 . Then p_* in Eq.(17) is of the order of α^3 . It is found from this consideration that the stream function ψ is independent of the motion of the upper fluid with a precision of the order of α^4 . Taking this result into account, we obtain an expanded solution in the following.

Substituting Eq.(25) into Eqs.(16)-(20), the equations governing ψ_1 is derived from the terms of the order of α^2 as follows:

$$\psi_{1,yyy} = R\psi_{1,yy} \quad (35)$$

$$y=h: p_1 = 0 \quad (36)$$

$$y=h: \psi_{1,yy} = 0 \quad (37)$$

$$y=0: \psi_{1,s} = \psi_{1,y} = 0 \quad (38)$$

where,

$$p_{1,s} = \frac{1}{R} \psi_{1,yyy} - \psi_{1,yy} \quad (39)$$

Then we have

$$\begin{aligned} \psi_1 = & -\frac{R}{2} (A_2 h^2 + A_3 h^3) y^2 + R \left(\frac{1}{12} A_2 y^4 \right. \\ & \left. + \frac{1}{20} A_3 y^5 \right) \quad (40) \end{aligned}$$

4.3 The third approximation

As noticed later in the determination of the shape of the interface, the amplitude of a variation in the position of the interface from average one, $h-1$, is assumed to be of the same order as the non-dimensional parameter α . We carry out this analysis to the fourth order of α under this condition. The equation to determine the height of the interface is given from Eq. (19) to the order of α^4 by

$$h_s + [\alpha\dot{\psi}_1(t, h) + \alpha^2\dot{\psi}_2(t, h) + \alpha^3\dot{\psi}_3(t, h)]_s = 0 \quad (41)$$

Then if we obtain ψ_1 , the third approximation of solution ψ , up to the first order of the amplitude of disturbances on the interface, the position of the interface h can be determined with a desired accuracy. The equations of ψ_1 are from Eq. (25) and Eqs. (16)-(20)

$$\begin{aligned} \psi_{1,yyy} = & R(\psi_{1,yy} + \psi_{1,y}\psi_{1,yy} - \psi_{1,s}\psi_{1,yy}) \\ & - 2\psi_{1,yyy} \quad (42) \end{aligned}$$

$$y=h: p_2 = -\frac{2}{R} \psi_{1,yy} + W\alpha^2 \frac{3}{2} h_s^2 h_{,ss} \quad (43)$$

$$y=h: \psi_{1,yy} = \psi_{1,ss} + h_s^2 \psi_{1,yy} + 4h_s \psi_{1,ys} \quad (44)$$

$$y=0: \psi_{1,s} = \psi_{1,y} = 0 \quad (45)$$

where,

$$p_{12} = \frac{1}{R} \dot{\psi}_{1222} - (\dot{\psi}_{121} + \dot{\psi}_{112} \dot{\psi}_{122} - \dot{\psi}_{12} \dot{\psi}_{122}) + \frac{1}{R} \dot{\psi}_{1222} \dots (46)$$

Substituting the first and second approximate solutions into the right hand side of Eq. (42) yields

$$\dot{\psi}_{1222} = -\frac{1}{3} R^2 \zeta_{11} - 2R^2 \zeta_{22} + 2R^2 \zeta_{33} y + \frac{1}{2} R^2 \zeta_{11} y^2 - \frac{1}{6} R^2 \zeta_{11} y^3 + (\text{nonlinear terms in } h) \dots (47)$$

A solution of the third order approximation corresponding to the linear part with respect to h is obtained as

$$\dot{\psi}_1 = \left(\frac{1}{15} R^2 \zeta_{11} + \frac{3}{4} R^2 \zeta_{22} \right) y^2 - \frac{1}{12} R^2 \zeta_{33} y^3 + \dot{\psi}_1 \dots (48)$$

where $\dot{\psi}_1$ is a special solution of $\dot{\psi}_1$ which is given by

$$\dot{\psi}_1 = -\frac{1}{72} R^2 \zeta_{11} y^4 - \frac{1}{12} R^2 \zeta_{22} y^4 + \frac{1}{60} R^2 \zeta_{33} y^5 + \frac{1}{720} R^2 \zeta_{11} y^6 - \frac{1}{5040} R^2 \zeta_{11} y^7 \dots (49)$$

5. The equation governing the interface height and its solution

Substituting the first, second and third approximate solutions of $\dot{\psi}$, i.e. Eqs. (31), (40), (48) into Eq. (41), an equation governing the interface height is obtained. If the interface height h is re-written in this equation as

$$h = 1 + \epsilon \zeta(t, x) \dots (50)$$

using the amplitude ϵ of disturbance on the interface, the equation governing the interface height up to the fourth order of small quantities, is

$$\eta_1 + \frac{1}{3} \alpha R^2 \zeta_{11} - \frac{2}{15} \alpha^2 R^2 \zeta_{11} + \frac{3}{5} \alpha^3 R^2 \zeta_{11} + \alpha \epsilon R (\eta_1 \zeta_{11} + \eta_1 \zeta_{22}) + \alpha \epsilon^2 R \eta_1 (2\eta_1 \zeta_{11} + \eta_1 \zeta_{22}) = 0 \dots (51)$$

where, $\zeta = (\tau - 1)\eta_1 + W\alpha^2 \eta_{11}$. With the exception that we take the weight of the upper fluid into account, the equation of h obtained here is the same one as derived by Nayfeh-Saric in order to discuss the instability of Kelvin-Helmholtz type appearing in the thin viscous layer⁽³⁾.

Let us now solve Eq. (51). To do so, $\eta(t, x)$ is expanded in the amplitude ϵ of disturbances as follows:

$$\eta(t, x) = \eta_1(t) e^{i\theta} + \eta_{-1}(t) e^{-i\theta} + \epsilon \{ \eta_2(t) e^{2i\theta} + \eta_{-2}(t) e^{-2i\theta} \} + \dots (52)$$

Substituting Eq. (52) into Eq. (51), we have the following equations for η_1 and η_{-1} :

$$\eta_{11} - \mu_1 \eta_1 = -\alpha R \epsilon^2 [(\beta - \beta') \eta_{-1} \eta_1 - \beta \eta_1^2 \beta_{-1}] \dots (53)$$

$$\eta_{21} - \frac{2}{3} \alpha R \beta' \eta_1 = 2\alpha R \beta \eta_1^2 \dots (54)$$

where,

$$\beta = \tau - 1 - \alpha^2 W \dots (55)$$

$$\beta' = 2(\tau - 1) - 8\alpha^2 W \dots (56)$$

$$\mu_1 = \frac{1}{3} \alpha R \beta \left(1 - \frac{9}{5} \alpha^2 \left(1 + \frac{2}{27} R^2 \beta \right) \right) \dots (57)$$

If the right hand side of Eq. (53) is equal to zero, the equation governing the linear stability is obtained. μ_1 is the growth rate based on the linear theory. Then the critical condition for stability of the system is given by $\mu_1 = 0$. The system becomes unstable or stable corresponding to $\beta > 0$ or $\beta < 0$. Therefore, in case where the upper fluid is heavier than the lower one, instability always takes place. This result is the same as that of Rayleigh-Taylor instability in case that the upper and lower fluids are infinite in thickness⁽³⁾. In our problem, the viscosity of the fluid is inferred to be important because one fluid layer is thin compared with the other. However, as μ_1 is proportional to Reynolds number R , the growth rate merely decreases with the viscosity of the fluid. There is not the effect of the viscosity of the fluid on the critical condition. We know from experience that when once one plate polished perfectly is superposed over another, the former is not easily torn from the latter. This phenomenon is based on the action of the pressure. The fact that the stability characteristics in our problem is independent of the viscosity is similar to the example mentioned above. Therefore, it is inferred that the pressure plays an important role in our case also.

Let us now consider the effect of the nonlinearity of the equation. Substituting the above linear solution

$$\eta_1 = \exp(\alpha R \beta t / 3) \dots (58)$$

into the right hand side of Eq. (54), we obtain η_2 as follows:

$$\eta_2 = \frac{3\beta}{\beta - \beta'} \eta_1^2 \dots (59)$$

The equation governing η_1 is derived as

$$\eta_{11} = [\mu_1 - 2\alpha R \beta (\epsilon \eta_1 \epsilon \eta_{-1})] \eta_1 \dots (60)$$

The quantity contained in the bracket on the right hand side of Eq. (60) is the growth rate which takes the nonlinearity into account. In case that the system is unstable in the linear theory, the quantity in the bracket is always positive. The growth rate decreases through nonlinearity of the equation of motion. Then the nonlinearity has a stabilizing effect. The value of the right hand side of Eq. (60) decreases as the amplitude of disturbance grows in unstable region. The system assumes a stationary state when the amplitude of disturbances is equal to

$$\sqrt{(\epsilon \eta_1 \epsilon \eta_{-1})} = \frac{1}{\sqrt{6}} \sqrt{1 - \frac{9}{5} \alpha^2 \left(1 + \frac{2}{27} R^2 \beta \right)} \dots (61)$$

That is, the equilibrium amplitude δ is given by Eq. (61). For two systems, i.e., ben-

benzen-water : $\gamma=1.14$, $R=153$, $W=3.57$
 n-octane-water: $\gamma=1.42$, $R=182$, $W=5.18$
 Fig.2 The relation between the equilibrium amplitude and the wave number

zene-water ($\rho=0.879 \text{ g/cm}^3$, $\sigma=35.0 \text{ dyn/cm}$, $\mu=0.647 \text{ cP}$)⁽⁶⁾ and n-octane-water ($\rho=0.703 \text{ g/cm}^3$, $\sigma=50.8 \text{ dyn/cm}$, $\mu=0.545 \text{ cP}$)⁽⁶⁾, which are a couple of immiscible fluids, the equilibrium amplitudes are shown in Fig.2. In these examples the thickness of the viscous layer is assumed to be 0.1 cm. It is found from Fig.2 that at a small wave number the equilibrium amplitude is about 0.4 and it decreases as the wave number increases. The magnitude of δ at a small wave number is not very small, but it is sufficiently small compared with unity. Therefore, our analysis holds for the small wave number. On the other hand, in case that the wave length of disturbances is short, the right hand side of Eq.(61) becomes imaginary. Then the equilibrium amplitude can not be determined from Eq.(61). However, in such cases the system will be stabilized by the surface tension. There is no inconsistency in this theory. The equilibrium amplitude certainly exists in our system. This result explains the fact that if the thickness of paint adhering to ceiling is very thin, paint does not fall off.

6. Conclusion

In case that a heavier liquid layer is superposed over a sheet of a viscous liquid layer, the stability of the system is studied theoretically taking account of nonlinearity. The upper ideal fluid and the lower viscous one are assumed to be incompressible. If the upper fluid is heavier than the lower one, the system is always unstable in the linear theory irrespective of the thickness and the viscosity of the lower liquid layer. However, taking the nonlinearity into account, there exists an equilibrium state in the unstable region of the linear theory.

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