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Coupled Torsional-flexural Vibration of a Shaft in a Geared System of Rotors (1st Report) *

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Coupled torsional-flexural vibration of a shaft in a spur geared system is investigated. Equation of motion which contain the terms of geometrical eccentricity of gears and mass unbalance are introduced. The coupling between torsional and flexural displacements is considered in the equations which assume that surfaces of tooth of meshed gears are always in contact during the rotation. By solving the equation, it is possible to get natural frequencies, mode shapes, and frequency response of the system. It is usual to ignore the effect of the coupling between torsional and flexural vibrations. But comparing the result with them, some new phenomena were revealed. They are: (1) shifts of the critical speeds (2) transformations of the mode shapes (3) a forced vibration which is caused by geometrical eccentricity of a gear.

1. Introduction

This investigation is concerned with a coupled torsional-flexural vibration of a shaft in a spur geared system. It is well known that in a geared system the noise and vibration often become a serious problem. Though the problems have been investigated⁽¹⁾⁻⁽³⁾, in these investigations the shafts are considered rigid for bending or at most torsional only. But it seems that the flexure of a shaft becomes more important when it is rotating at high speed. So it is necessary to consider the vibration of a shaft in a geared system as a coupled one of torsion and flexure.

Fig.1 Analytical model

In the present paper, we discuss the free and forced vibrations of a shaft in a spur geared system from this point of view, especially the effects of geometrical eccentricity and unbalance of a gear on the forced vibration.

2. Equation of motion of a geared system

The analytical model is shown in Fig. 1, where the suffixes 1,2 indicate respectively driving and driven parts of the model. J_i is the moment of inertia of the motor ($i=1$) and dynamo ($i=2$), I_i is the moment of inertia of the gear. T_i is the output torque absorbed by the dynamo. ϕ_i is the rotational angle of the shaft, ϕ_1 being that of the motor and ϕ_2 that of the dynamo. θ_i is the rotational angle of the gear, and K_i is the torsional rigidity. In this model, shaft II is considered flexible, while the shaft I is rigid so that a flexure appears only in shaft II. Torsion is possible in shaft II and the flexible coupling which is in driving side. So this model is a 3-degree-of-freedom system. We consider that the power is transmitted from gear I to gear II in a plane O_3-xy which is perpendicular to both shafts as shown in Fig.2. In this figure the symbols O_2 , O_3 are positions of the center of shaft II where O_2 is when it is rotating and O_3 is not, and O_2' is the center of gear II, so that e_2 is the geometrical eccentricity. The angle α is the pressure angle of the gear, and r_i is the radius of base circle. The stiffness, damping coefficient to flexure of shaft II are denoted by k and c respectively, and the mass, unbalance of gear II are denoted by m and U . Moreover denoting the force and

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damping coefficient to torsion by F and C_i respectively, we have following equations of motion.

$$m\ddot{x} + c\dot{x} + kx = U\dot{\theta}_2^2 \cos(\theta_2 + \theta_f) \quad \dots\dots\dots (1)$$

$$m\ddot{y} + c\dot{y} + ky = F + U\dot{\theta}_2^2 \sin(\theta_2 + \theta_f) \quad \dots\dots\dots (2)$$

$$J_1\ddot{\phi}_1 = T_1 - K_1(\phi_1 - \theta_1) \quad \dots\dots\dots (3)$$

$$J_2\ddot{\phi}_2 = K_2(\theta_2 - \phi_2) - T_2 \quad \dots\dots\dots (4)$$

$$I_1\ddot{\theta}_1 = K_1(\phi_1 - \theta_1) - F(r_1 + \varepsilon_1 \cos \theta_1) - c_1\dot{\theta}_1 \quad \dots\dots\dots (5)$$

$$I_2\ddot{\theta}_2 = F(r_2 + \varepsilon_2 \cos \theta_2) - K_2(\theta_2 - \phi_2) - c_2\dot{\theta}_2 \quad \dots\dots\dots (6)$$

Additionally, the following equation (Eq.7) is derived from the condition that the tooth contact is maintained during the rotation.

$$\varepsilon_2 \sin \theta_2 + r_2 \theta_2 + y = r_1(\theta_1 + \beta) + \varepsilon_1 \sin \theta_1 \quad \dots\dots\dots (7)$$

where β is the initial angle of gear I.

Fig.2 Coordinate system

3. Theoretical analysis

Although the flexural displacement occurs in both directions (x, y) , only the one in y direction concerns the coupled vibration. So, we discuss the flexural displacement only in y direction unless otherwise mentioned.

3.1 Undamped natural frequency

Undamped natural frequency can be obtained from Eq.s (2)-(7) putting the terms which relate to force, torque, and damping equal to zero respectively. At first we introduce the frequency equation as follows.

$$\begin{vmatrix} k - m\omega^2 & 0 & 0 & 0 & 0 & -1 \\ 0 & K_1 - J_1\omega^2 & -K_1 & 0 & 0 & 0 \\ 0 & -K_1 & K_1 - I_1\omega^2 & 0 & 0 & r_1 \\ 0 & 0 & 0 & K_2 - I_2\omega^2 & -K_2 & -r_2 \\ 0 & 0 & 0 & -K_2 & K_2 - J_2\omega^2 & 0 \\ -1 & 0 & r_1 & -r_2 & 0 & 0 \end{vmatrix} = 0 \quad \dots\dots\dots (8)$$

Expansion of Eq. (8) yields Eq. (9).

$$A\omega^6 + B\omega^4 + C\omega^2 + D = 0$$

$$\left. \begin{aligned} \text{where } A &= J_1 J_2 (I_1 I_2 + m(r_1^2 I_2 + r_2^2 I_1)) \\ B &= -(r_1^2 I_2 J_2 (k J_1 + m K_1) + J_1 K_2 (I_2 + J_2) (I_1 + m r_1^2) + r_2^2 I_1 J_1 (k J_2 + m K_2) + J_2 K_1 (I_1 + J_1) (I_2 + m r_2^2)) \\ C &= k(r_1^2 K_1 I_2 J_2 + r_2^2 K_2 I_1 J_1) + r_1^2 K_2 (I_2 + J_2) (k J_1 + m K_1) + K_1 K_2 (I_1 + J_1) (I_2 + J_2) + r_2^2 K_1 (I_1 + J_1) (k J_2 + m K_2) \\ D &= -k K_1 K_2 (r_1^2 (I_2 + J_2) + r_2^2 (I_1 + J_1)) \end{aligned} \right\} \quad \dots\dots\dots (9)$$

The solutions of Eq. (9) give the undamped natural frequencies of a torsion-flexure coupled system. We can also obtain the natural mode of the system from Eq. (8) as follows.

$$\left. \begin{aligned} \frac{y}{\theta_1} &= \frac{(K_1(I_1 + J_1) - I_1 J_1 \omega^2) \omega^2}{r_1(k - m\omega^2)(K_1 - J_1 \omega^2)}, & \frac{y}{\theta_2} &= \frac{(I_2 J_2 \omega^2 - K_2(I_2 + J_2)) \omega^2}{r_2(k - m\omega^2)(K_2 - J_2 \omega^2)} \\ \theta_1/\phi_1 &= 1 - J_1 \omega^2 / K_1, & \theta_2/\phi_2 &= 1 - J_2 \omega^2 / K_2 \end{aligned} \right\} \quad \dots\dots\dots (10)$$

3.2 Steady-state response of vibration

In Eq.s (2)-(7) we put $T_i = T_{i0} - q_i \Delta \phi_i$, $y = y_0 + \Delta y$, $\theta_i = \theta_{i0} + \omega_i t + \Delta \theta_i$, $\phi_i = \phi_{i0} + \omega_i t + \Delta \phi_i$, $F = F_0 + \Delta F$ where suffix 0 indicates the constant part and ω_i denotes the constant rotational speed. With this substitution, six equations are derived for the constant part and the variable part respectively. The solution for the constant part can be easily obtained. The equations of variable part are shown as follows.

$$\left. \begin{aligned} m\Delta \ddot{y} + c\Delta \dot{y} + k\Delta y - \Delta F - U\omega_2^2 \sin(\omega_2 t + \Delta \theta_2 + \theta_{f0}) &= 0 \\ J_1 \Delta \ddot{\phi}_1 + q_1 \Delta \phi_1 + K_1(\Delta \phi_1 - \Delta \theta_1) &= 0, & J_2 \Delta \ddot{\phi}_2 - q_2 \Delta \phi_2 - K_2(\Delta \theta_2 - \Delta \phi_2) &= 0 \\ I_1 \Delta \ddot{\theta}_1 - K_1(\Delta \phi_1 - \Delta \theta_1) + \Delta F r_1 + c_1 \Delta \dot{\theta}_1 &= 0 \\ I_2 \Delta \ddot{\theta}_2 - F_0 \varepsilon \cos(\omega_2 t + \Delta \theta_2 + \theta_{20}) - \Delta F(r_2 + \varepsilon \cos(\omega_2 t + \Delta \theta_2 + \theta_{20})) + K_2(\Delta \theta_2 - \Delta \phi_2) + c_2 \Delta \dot{\theta}_2 &= 0 \\ r_1 \Delta \theta_1 - \varepsilon \sin(\omega_2 t + \Delta \theta_2 + \theta_{20}) - \Delta y - r_2 \Delta \theta_2 &= 0 \end{aligned} \right\} \quad \dots\dots\dots (11)$$

where we put $\varepsilon_1 = 0$ in order to simplify the vibrational phenomena, and we put ε instead of ε_2 .

From Eq. (11), we eliminate F and y , and put the solution of $\Delta \theta_i$ and $\Delta \phi_i$ as follows:

$$\Delta \theta_i = \theta_{i1} \cos \omega_2 t + \theta_{i2} \sin \omega_2 t, \quad \Delta \phi_i = \phi_{i1} \cos \omega_2 t + \phi_{i2} \sin \omega_2 t \quad (i=1, 2)$$

Neglecting the terms of higher order, we obtain the following equation in a matrix form.

$$\begin{bmatrix}
 -J_1\omega_2^2 + K_1 & q_1\omega_2 & -K_1 & 0 \\
 -q_1\omega_2 & -J_1\omega_2^2 + K_1 & 0 & -K_1 \\
 -K_1 & 0 & -I_1\omega_2^2 + K_1 + r_1^2(k - m\omega_2^2) & c_1\omega_2 + c\omega_2 r_1^2 \\
 0 & -K_1 & -c_1\omega_2 - c\omega_2 r_1^2 & -I_1\omega_2^2 + K_1 + r_1^2(k - m\omega_2^2) \\
 0 & 0 & -r_1 r_2 (k - m\omega_2^2) & -r_1 r_2 c\omega_2 \\
 0 & 0 & r_1 r_2 c\omega_2 & -r_1 r_2 (k - m\omega_2^2) \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 -r_1 r_2 (k - m\omega_2^2) & -r_1 r_2 c\omega_2 & 0 & 0 \\
 r_1 r_2 c\omega_2 & -r_1 r_2 (k - m\omega_2^2) & 0 & 0 \\
 -I_2\omega_2^2 + K_2 + r_2^2(k - m\omega_2^2) & c_2\omega_2 + r_2^2 c\omega_2 & -K_2 & 0 \\
 -c_2\omega_2 - r_2^2 c\omega_2 & -I_2\omega_2^2 + K_2 + r_2^2(k - m\omega_2^2) & 0 & -K_2 \\
 -K_2 & 0 & -J_2\omega_2^2 + K_2 & -q_2\omega_2 \\
 0 & -K_2 & q_2\omega_2 & -J_2\omega_2^2 + K_2
 \end{bmatrix}
 \begin{Bmatrix}
 \varphi_{11} \\
 \varphi_{12} \\
 \theta_{11} \\
 \theta_{12} \\
 \theta_{21} \\
 \theta_{22} \\
 \varphi_{21} \\
 \varphi_{22}
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 0 \\
 0 \\
 r_1 [\varepsilon \{ (k - m\omega_2^2) \sin \theta_{20} + c\omega_2 \cos \theta_{20} \} + U\omega_2^2 \sin(\theta_f + \theta_{20})] \\
 r_1 [\varepsilon \{ (k - m\omega_2^2) \cos \theta_{20} - c\omega_2 \sin \theta_{20} \} + U\omega_2^2 \cos(\theta_f + \theta_{20})] \\
 \varepsilon F_0 \cos \theta_{20} - r_2 [\varepsilon \{ (k - m\omega_2^2) \sin \theta_{20} + c\omega_2 \cos \theta_{20} \} + U\omega_2^2 \sin(\theta_f + \theta_{20})] \\
 -\varepsilon F_0 \sin \theta_{20} - r_2 [\varepsilon \{ (k - m\omega_2^2) \cos \theta_{20} - c\omega_2 \sin \theta_{20} \} + U\omega_2^2 \cos(\theta_f + \theta_{20})] \\
 0 \\
 0
 \end{Bmatrix} \quad \dots\dots\dots(12)$$

where θ_{20} is the torsional angle of gear II due to the mean transmitting torque (constant). The solution of Eq. (12) gives the steady-state vibration at the rotational speed ω_2 .

Since the term $\varepsilon(k - m\omega_2^2)$ which concerns the geometrical eccentricity of the gear has the same amount of value as that of unbalance force $U\omega_2^2$ it is clear that not only the unbalance but also the geometrical eccentricity acts on the system as the cause of vibration.

3.3 Example of numerical analysis

We will analyze numerically the torsional-flexural vibration in a geared system with the quantities shown in Table 1.

(1) Natural mode of the vibration

Fig. 3 shows the natural modes of the vibration. Where flexural displacement y is expressed with the value y/r_2 to match the dimension with the torsion. Though the torsional displacement is rather emphasized in the first mode, it is comparable to the flexural one in 2nd and 3rd modes.

(2) Undamped natural frequency

Figs. 4-6 show the relations between the undamped natural frequency and some variables. The broken lines indicate the undamped natural frequencies obtained from the condition that the torsion and flexure do not couple.

Fig. 4 shows the relation between the natural frequency and the moment of inertia of gear I (J_1). It is clear that there are great differences between the coupled ones and the uncoupled ones.

Fig. 5 shows the relation between the natural frequency and the torsional rigidity K_1 . There are also found great differences between the coupled and the uncoupled ones.

Fig. 6 shows the relation between the natural frequency and the flexural stiffness k . A little difference is found between the coupled and the uncoupled ones in the 1st mode, but the 2nd and 3rd natural frequencies of torsion-flexure mode differ greatly from the uncoupled ones. The 2nd coupled one comes close to the value of higher one of the uncoupled

Fig. 3 Natural modes of vibration

Fig.4 Relation between the natural frequency and the moment of inertia

Fig.5 Relation between the natural frequency and the torsional stiffness

Fig.7 Steady-state response (effect of geometrical eccentricity and unbalance)

Fig.6 Relation between the natural frequency and the flexural stiffness

torsional vibration with an increase of stiffness. This is because the effect of flexure decreases with an increase of stiffness.

(3) Steady-state response of vibration

Fig. 7 shows the steady-state flexure vibration response obtained from the solution of Eq.(12) under the condition that gear II has both geometrical eccentricity and some amount of unbalance. In this figure, the angle in the parenthesis means the one between the directions of the eccentricity and the unbalance. Line ① shows the response caused by geometrical eccentricity, and line ② shows the

effect of the unbalance on the geometrical eccentricity in this mode. It is clear that the response can be reduced by unbalance.

4. Experimental analysis

4.1 Experiment

An outline of the experimental equipment is shown in Fig.8, and its details are listed in Table 1.

The power is transmitted from motor to shaft I by flexible coupling and it makes gear I rotate. Gear I drives gear II which is mounted on the middle of the flexible shaft II, and the power is absorbed by the dynamo. Gear I has 54 teeth and gear II has 30 teeth. Both gears are spur ones with module 4, and their pressure angle is 20°. The gears are lubricated using oil bath.

The flexural displacement of shaft II is measured near the gear by gap pick-ups from x -direction (power transmitting direction which is normal to the gear surface) and y -direction (tooth sliding direction which is perpendicular to x). The span of shaft II and the flexible coupling are changeable respectively, so that the values K_1 and k are variable. At the same time the value of I_1 is varied by adding a flywheel to shaft I.

4.2 Effect of the moment of inertia

Fig.9 shows the relation between steady-state response in the y -direction and the moment of inertia under the condition that the value $k=23.6$ Kg/mm. It is found that the critical speed in the case the gears are meshed differs from that in the case of unmeshed, and in the meshed case, the critical speed decreases with an increase of the moment of inertia I_1 . It is clear that in the meshed case the coupled vibrations of torsion and flexure occur. The experimental result coincide with the theoretical ones shown in Fig. 4.

The peaks in Fig.9 are the responses at the 2nd critical speeds, and the peaks at the other critical speeds do not appear although the degree of freedom is 3. This is because: (1) in the 1st mode, the flexural vibration is not emphasized compared with the torsional one; and (2) the 3rd critical speed is out of the measuring range.

In the x -direction, it seems that there is no effect of the moment of inertia on the response. This is because the vibration is not a coupled one, but the flexure only.

4.3 Effect of the torsional stiffness

Fig. 10 shows a relation between the response in y -direction and the torsional rigidity K_1 under the condition that the value $k=23.6$ kgf/mm. It is found that the 2nd critical speed increases with an increase of the torsional rigidity. This is because the vibration is a coupled one, and the natural frequencies coincide with the theoretical result shown in Fig.5. In the response in x -direction, the torsional rigidity seemed to have no effect on it.

- 1. Motor
- 2. Coupling
- 3. Ball Bearing
- 4. Shaft I (rigid)
- 5. Shaft II (flexible)
- 6. Ball Bearing
- 7. Gear II ($m=4, z_2=30$)
- 8. Rubber Coupling
- 9. Dynamo
- 10. Gear I ($m=4, z_1=54$)
- 11. Flywheel

Fig.8 Experimental equipment

Table 1 Details of the experimental equipment

Symbol	Value
J_1	46.8 kgfmm ²
J_2	56.0 kgfmm ²
I_1	(Variable) standard: 3.06 kgfmm ²
I_2	0.64 kgfmm ²
K_1	(Variable) standard: 1.17×10^4 kgfmm/rad
K_2	2.82×10^5 kgfmm/rad
k	(Variable) standard: 20.0 kgfmm/rad
r_1	101.5 mm
r_2	56.4 mm
m	5.77×10^{-4} kgs ² /mm

Fig.9 Steady-state response (effect of moment of inertia)

4.4 Effect of the geometrical eccentricity and unbalance of the gear

Fig.11 shows the response when gear II has geometrical eccentricity. In this fig-

ure, we call response curve i case i . Case 1 shows the response in which the gear is unmeshed, case 2 shows the one meshed, and case 3 shows the one meshed with unbalance whose direction is opposite to that of eccentricity. Fig. 11 consists of two figures, figure (a) being the response in y -direction (power transmitting direction) and (b) the response in x -direction (tooth sliding direction which is perpendicular to x). The geometrical eccentricity is $100\text{ }\mu\text{m}$ and the unbalance added in case 3 is $1.11 \times 10^{-4}\text{ kgm}$.

At first we discuss Fig. (a). Considering that the system has little damping, the rotor seems to be well balanced in case 1. In case 2, the geometrical eccentricity acts as an exciting force, the peak amplitude is increased over that of case 1. In case 3 the peak amplitude is decreased under that of case 2 because the unbalance reduces the effect of geometrical eccentricity.

Next, we discuss Fig. (b). Only case 3 has an unnatural peak of response. This is considered thus: although the addition of unbalance indeed reduces the effect of geometrical eccentricity in y -direction, in x -direction it simply means an addition of unbalance. But the maximum amplitude in x direction is limited to a small value because of a backlash of the gear. As a result, the response curve becomes like case 3. Although the maximum amplitude in y -direction is limited to a certain value by the backlash, it is not so small as that in x -direction.

Small peaks appear in Fig. (a) at the speed 28 rps., and in Fig. (b) at 25 rps. Seemingly it is because of the vibration being picked up in another direction.

4.5 Effect of the transmitting torque

In a system in which gears are mounted on a flexible shaft, the increase of transmitting torque increases the static deflection of shaft. In addition, it may increase a fluctuating torque which is caused by the geometrical eccentricity of the gear, and may make the vibration much more severe. However such phenomenon was not observed experimentally because the transmitting torque cannot be varied over a wide range. But in the case of a small backlash, the amplitude becomes greater with an increase of transmitting torque as shown in Fig. 12. The reason seems to be as follows: the maximum value of the vibration amplitude is restricted to a certain value by the backlash when it is small, and the backlash is a factor of distance between both shafts on which the gear is mounted. As the static deflection widens the distance, an increase of the transmitting torque make the value of backlash greater. As a result, it seems, the vibration amplitude is increased. In the case of sufficient backlash, such phenomenon was not observed.

Fig. 10 Steady-state response (effect of torsional stiffness)

(a) Vibration in the direction of power transmission

(b) Vibration in the direction of tooth sliding

Fig. 11 Steady-state response (effect of geometrical eccentricity and unbalance)

5. Conclusions

We have analyzed a coupled torsional-flexural vibration of a shaft in a spur geared system theoretically and experimentally, and obtained the following con-

clusions.

(1) In a system in which gears are mounted on a flexible shaft, the torsional vibration is coupled with the flexural vibration, and the natural frequencies of the system differ from the ones obtained without considering the effect of coupling.

(2) The geometrical eccentricity of gear acts on the system as a cause of the vibration in the direction of power transmission, and the steady-state response at one of the critical speeds can be reduced by adding an unbalance.

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Fig.12 Steady-state response (effect of transmitting torque when the system has small amount of backlash)