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Subharmonic and Summed-and-Differential Harmonic

Oscillations of an Unsymmetrical Rotor*

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Subharmonic and summed-and-differential harmonic oscillations of a nonlinear rotating shaft system where an unsymmetrical rotor is mounted on a shaft with circular cross section are treated. It becomes clear theoretically that the resonance curves of subharmonic oscillations of order $1/2$ and summed-and-differential harmonic oscillations are classified into four types. Therefore, similarly to those of an unsymmetrical shaft system reported previously, stable oscillation, unstable oscillation, or no oscillation appears. In an experimental apparatus in which single-row deep groove ball bearings were used, we observed the same types of the resonance curves as the theoretical results by changing the assembly of the apparatus and the size and location of the existing unbalance.

1. Introduction

Let one of the three principal axes of a rotor coincide with a rotating shaft and the polar moment of inertia around this axis be denoted by I_p and the other two principal moments of inertia by I_1 and I_2 . An unsymmetrical rotor means a rigid body such that $I_1 \neq I_2$. It is mounted on a flexible shaft with a circular cross section (i. e. a round shaft). When the shaft is supported vertically by single-row deep groove ball bearings, the restoring force of the shaft has a nonlinear spring characteristic owing to angular clearances of the ball bearings. We have already reported on the subharmonic oscillations⁽¹⁾ and the summed-and-differential harmonic oscillations⁽²⁾ in the unsymmetrical shaft system carrying a disc. In this paper, we shall report on these nonlinear oscillations which occur in the unsymmetrical rotor system as mentioned above, and confirm that qualitatively the same phenomena as those in the unsymmetrical shaft system appear also in this unsymmetrical rotor system.

Experiments are performed in a four-degree-of-freedom system where the deflection and the inclination of a shaft couple with each other. In a theoretical analysis, for the sake of simplicity, we treat the inclination oscillation when they do not

couple. This becomes a two-degree-of-freedom system.

2. Equations of motion

The experimental apparatus is a four-degree-of-freedom system. The deflection r and the inclination θ of the shaft at the position where an unsymmetrical rotor is mounted couple with each other. But in the theoretical analysis, we consider only the inclination oscillation in the system where r and θ do not couple. The same procedure as that in the unsymmetrical shaft system⁽²⁾ can lead to the conclusion that the results for the four-degree-of-freedom system as in this experimental apparatus are qualitatively similar to those for the two-degree-of-freedom system.

Let $O-xyz$ be the stationary rectangular coordinate system where z -axis coincides with the center line of the upper and lower bearing pedestals, and θ_x and θ_y be the projectional angles of θ to xz - and yz -planes, respectively. We use the notations $I = (I_1 + I_2)/2$ and $\Delta I = (I_1 - I_2)/2$ instead of the principal moments of inertia I_1 and I_2 ($I_1 > I_2$). Let the dynamic unbalance be τ , the bending stiffness of the shaft be δ , the damping coefficient be c , the rotating speed of the shaft be ω , the time be t , and the coefficients of unsymmetrical and symmetrical nonlinear terms be ε_{ij} ($i+j=3$) and β_{ij} ($i+j=4$), respectively. Now we consider a stationary rectangular coordinate system consisting of θ_x - and θ_y -axes. In addition, let the rectangular coordinate system rotating with an angular velocity ω be $O-\theta'_x\theta'_y$, and let the θ'_x - and θ'_y -axes coincide with the principal axes of the rotating body. The time when θ'_x -axis coincides with the θ_x -axis is taken as $t=0$. We denote the angular position of τ measured from the positive direction of the θ'_x -axis by α . Introducing the dimen-

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sionless quantities

$$\left. \begin{aligned} i_p &= I_p/I, \quad t' = t\sqrt{\delta/I}, \quad \omega' = \omega\sqrt{I/\delta} \\ c' &= c/\sqrt{\delta I}, \quad \Delta = \Delta I/I, \quad N_{\theta x}' = N_{\theta x}/\delta \\ N_{\theta y}' &= N_{\theta y}/\delta, \quad \varepsilon_{ij}' = \varepsilon_{ij}/\delta, \quad \beta_{ij}' = \beta_{ij}/\delta \end{aligned} \right\} \dots\dots\dots (1)$$

we obtain the following equations of motion provided we omit the primes from the symbols⁽³⁾.

$$\left. \begin{aligned} \ddot{\theta}_x + i_p \omega \dot{\theta}_y + c \dot{\theta}_x + \theta_x - \Delta \frac{d}{dt} (\dot{\theta}_x \cos 2\omega t + \dot{\theta}_y \sin 2\omega t) \\ + N_{\theta x} = \tau \omega^2 \{ (1-i_p) \cos(\omega t + \alpha) + \Delta \cos(\omega t - \alpha) \} \\ \ddot{\theta}_y - i_p \omega \dot{\theta}_x + c \dot{\theta}_y + \theta_y - \Delta \frac{d}{dt} (\dot{\theta}_y \sin 2\omega t - \dot{\theta}_x \cos 2\omega t) \\ + N_{\theta y} = \tau \omega^2 \{ (1-i_p) \sin(\omega t + \alpha) + \Delta \sin(\omega t - \alpha) \} \end{aligned} \right\} \dots\dots\dots (2)$$

$N_{\theta x}$ and $N_{\theta y}$ are the nonlinear components of the restoring force, and we can write them in the following forms provided that we take

$$\begin{aligned} V = \frac{1}{2} \delta \theta^2 + (\varepsilon_{\varepsilon^{(1)}} \cos \varphi + \varepsilon_{\varepsilon^{(2)}} \sin \varphi + \varepsilon_{\varepsilon^{(3)}} \cos 3\varphi + \varepsilon_{\varepsilon^{(4)}} \sin 3\varphi) \theta^3 \\ + (\beta_{\varepsilon^{(0)}} + \beta_{\varepsilon^{(2)}} \cos 2\varphi + \beta_{\varepsilon^{(4)}} \sin 2\varphi + \beta_{\varepsilon^{(6)}} \cos 4\varphi + \beta_{\varepsilon^{(8)}} \sin 4\varphi) \theta^4 \dots\dots\dots (4) \end{aligned}$$

In Eq.(4), the magnitudes of the nonlinear terms change periodically during one cycle of φ (from 0 to 2π), and the numbers of times of these changes are represented by the numbers in the parentheses of the coefficients. Hereafter, we denote the nonlinear component, whose magnitude changes n times, by the symbol $N(n)$. The following relations hold between the coefficients in Eqs.(3) and (4)⁽⁴⁾.

$$\left. \begin{aligned} \varepsilon_{\varepsilon^{(1)}} &= (3\varepsilon_{30} + \varepsilon_{12})/4, \quad \varepsilon_{\varepsilon^{(2)}} = (\varepsilon_{21} + 3\varepsilon_{03})/4, \quad \varepsilon_{\varepsilon^{(3)}} = (\varepsilon_{30} - \varepsilon_{12})/4 \\ \varepsilon_{\varepsilon^{(4)}} &= (\varepsilon_{21} - \varepsilon_{03})/4, \quad \beta_{\varepsilon^{(2)}} = (3\beta_{40} + \beta_{22} + 3\beta_{04})/8, \quad \beta_{\varepsilon^{(4)}} = (\beta_{40} - \beta_{04})/2 \\ \beta_{\varepsilon^{(6)}} &= (\beta_{31} + \beta_{13})/4, \quad \beta_{\varepsilon^{(8)}} = (\beta_{40} - \beta_{22} + \beta_{04})/8, \quad \beta_{\varepsilon^{(10)}} = (\beta_{31} - \beta_{13})/8 \end{aligned} \right\} \dots\dots\dots (5)$$

We denote the natural frequency of the system by p , and put $2\omega - p = \bar{p}$. The frequency equation of Eq.(2) is obtained as follows⁽⁵⁾:

$$G(p)G(\bar{p}) - (\Delta \bar{p} \bar{p})^2 = (1 + i_p \omega p - p^2)(1 + i_p \omega \bar{p} - \bar{p}^2) - (\Delta \bar{p} \bar{p})^2 = 0 \dots\dots\dots (6)$$

Eq.(6) has four real roots at most against a certain value of ω . Among them, we designate two natural frequencies corresponding to those for the symmetrical system obtained when $\Delta \rightarrow 0$ (i.e. a rotating shaft system carrying a disc on a round shaft) by p_f and p_b . And the other roots are denoted by $\bar{p}_f (= 2\omega - p_f)$ and $\bar{p}_b (= 2\omega - p_b)$. As an example, a p - ω diagram is shown in Fig.1 for $i_p = 0.4$ and $\Delta = 0.18$. The broken lines represent the values of $2p_f$ and $p_f - p_b$.

3. Theoretical analyses of the subharmonic oscillation of order 1/2 and the summed-and-differential harmonic oscillation

By means of the harmonic balance method which was used in the theoretical analyses for the unsymmetrical shaft system^{(1),(2)}, approximate solutions of Eq.(2) for these two kinds of oscillations can be obtained.

3.1 The subharmonic oscillation of order 1/2 of the type $[2p_f]$ When the frequency of the external force ω becomes nearly equal to the natural frequency of this system $2p_f$ (in the vicinity of the point A in Fig.1), this force performs a work on the system. Consequently, the oscillation with the frequency $\omega_f = 1/2 \cdot \omega (= p_f)$ appears remarkably, and we can observe a subharmonic oscillation having the mode of forward precession. We designate this kind of oscillation by the symbol $[2p_f]$. As the unsymmetry Δ is the quantity of $O(\varepsilon^0)$, this oscillation component of frequency $1/2 \cdot \omega$ always yields a component of frequency $2\omega - 1/2 \cdot \omega = 3/2 \cdot \omega$ in the same magnitudes of amplitudes. Therefore, the approximate solution must be assumed in the following forms with the accuracy of $O(\varepsilon^1)$.

$$\left. \begin{aligned} \theta_x &= R \cos \left(\frac{1}{2} \omega t + \delta_1 \right) + \bar{R} \cos \left(\frac{3}{2} \omega t - \delta_1 \right) + P \cos(\omega t + \beta) + \varepsilon f \left(\frac{1}{2} \omega, \omega, \frac{3}{2} \omega, \dots \right) \\ \theta_y &= R \sin \left(\frac{1}{2} \omega t + \delta_1 \right) + \bar{R} \sin \left(\frac{3}{2} \omega t - \delta_1 \right) + P \sin(\omega t + \beta) + \varepsilon f' \left(\frac{1}{2} \omega, \omega, \frac{3}{2} \omega, \dots \right) \end{aligned} \right\} \dots\dots\dots (7)$$

where the amplitudes R and $\bar{R}$, and the phase angle δ_1 , are the quantities of $O(\varepsilon^0)$ varying slowly with time. The functions $\varepsilon \cdot f$ and $\varepsilon \cdot f'$ are the components of $O(\varepsilon)$ which appear owing to the nonlinear terms in Eq.(3), and their concrete forms are the same as those of the unsymmetrical shaft system⁽¹⁾. The amplitude P and the phase angle β of the harmonic oscillation can be determined with the accuracy of $O(\varepsilon^0)$ as follows:

$$P = \tau \omega^2 \sqrt{\left(\frac{1-i_p+\Delta}{G-\Delta \omega^2} \cos \alpha \right)^2 + \left(\frac{1-i_p-\Delta}{G+\Delta \omega^2} \sin \alpha \right)^2}, \quad \tan \beta = \frac{(G-\Delta \omega^2)(1-i_p-\Delta)}{(G+\Delta \omega^2)(1-i_p+\Delta)} \tan \alpha \dots\dots\dots (8)$$

where $G = G(\omega)$.

account of the terms up to the third power of coordinates θ_x and θ_y .

$$\left. \begin{aligned} N_{\theta x} &= (3\varepsilon_{30}\theta_x^2 + 2\varepsilon_{21}\theta_x\theta_y + \varepsilon_{12}\theta_y^2) \\ &\quad + (4\beta_{40}\theta_x^3 + 3\beta_{31}\theta_x^2\theta_y + 2\beta_{22}\theta_x\theta_y^2 + \beta_{13}\theta_y^3) \\ N_{\theta y} &= (\varepsilon_{21}\theta_x^2 + 2\varepsilon_{12}\theta_x\theta_y + 3\varepsilon_{03}\theta_y^2) \\ &\quad + (\beta_{31}\theta_x^3 + 2\beta_{22}\theta_x^2\theta_y + 3\beta_{13}\theta_x\theta_y^2 + 4\beta_{04}\theta_y^3) \end{aligned} \right\} \dots\dots\dots (3)$$

We denote the small quantity by ε , and assume in Eq.(2) that θ_x , θ_y , and Δ are magnitudes of the zeroeth order of ε (we will use the symbol $O(\varepsilon^0)$ later), and c , $N_{\theta x}$, and $N_{\theta y}$ are the magnitudes of $O(\varepsilon)$.

It is reasonable that the nonlinear spring characteristics in rotating shaft systems are expressed with the polar coordinates (θ, φ) given by $\theta_x = \theta \cos \varphi$ and $\theta_y = \theta \sin \varphi$. By these coordinates, the potential energy V are expressed in the following form.

As the next step, substituting Eq.(7) into Eq.(2) and setting the terms of the same frequencies to be zero separately with the accuracy of $O(\varepsilon)$, we obtain the simultaneous equations for the amplitude R and the phase angle δ_1 .

$$\left. \begin{aligned} A\omega R\dot{\delta}_1 &= [\sigma + \beta^{(1)}(gR^2 + kP^2) + 4\Delta Q(\beta_s^{(2)} \sin 4\delta_1 + \beta_e^{(2)} \cos 4\delta_1)R^2 \\ &\quad + 2\varepsilon^{(1)}P(\bar{G}_1 \cos(2\delta_1 - \beta) + 2\Delta Q \cos(2\delta_1 + \beta))]R \\ A\omega \dot{R} &= [\bar{c} + 4\Delta Q(\beta_e^{(2)} \sin 4\delta_1 - \beta_s^{(2)} \cos 4\delta_1)R^2 + 2\varepsilon^{(1)}P(\bar{G}_1 \sin(2\delta_1 - \beta) + 2\Delta Q \sin(2\delta_1 + \beta))]R \end{aligned} \right\} \dots\dots\dots(9)$$

where we put $\varepsilon_e^{(1)} = \varepsilon^{(1)} (>0)$ and $\varepsilon_s^{(1)} = 0$ by rotating the coordinate axes by some angles. The notations in Eq.(9) are as follows:

$$\left. \begin{aligned} A &= (1 - i_p)\bar{G}_1 + (i_p - 3)G_1 + 2\Delta^2 Q, & \bar{c} &= c\omega(3G - \bar{G}_1)/2, & G_1 &= G\left(\frac{1}{2}\omega\right) \\ \bar{G}_1 &= G\left(\frac{3}{2}\omega\right), & g &= 4(G_1^2 + 4G_1\bar{G}_1 + \bar{G}_1^2)/\bar{G}_1, & k &= 8(G_1 + \bar{G}_1 + \Delta Q \cos 2\beta) \\ \beta^{(2)} &= \sqrt{\beta_e^{(2)2} + \beta_s^{(2)2}}, & 2\varphi_2 &= \tan^{-1}(\beta_s^{(2)}/\beta_e^{(2)}), & \sigma &= G_1\bar{G}_1 - (\Delta Q)^2, & Q &= \frac{3}{4}\omega^2 \end{aligned} \right\} \dots\dots\dots(10)$$

By putting $\dot{\delta}_1 = 0$ and $\dot{R} = 0$ in Eq.(9), the stationary solution $R = R_0$ can be classified into two kinds of solutions, i.e. $R_0 \neq 0$ and

$R_0 = 0$. Especially, it is noticeable that the nonlinear component $N(2)$, which has no effects on the subharmonic oscillation of order $1/2$ in the symmetrical system, is contained in Eq.(9). This component $N(2)$ causes the theoretical resonance curves of the subharmonic oscillation $[2p_f]$ in the unsymmetrical rotor system to be complicated. We find that the right side of Eq.(9) coincides with the equation of the subharmonic oscillation $[2p_f]$ in the unsymmetrical shaft system if we replace $\Delta (= \delta\delta/\delta)$ with $\Delta\Omega (= \Delta I/I \cdot (3/4 \cdot \omega^2))^{(1)}$. Therefore, this suggests that the shapes of the theoretical resonance curves in the unsymmetrical rotor system are qualitatively the same as those in the unsymmetrical shaft system. Introducing the small deviations from the stationary solutions, we can examine the stability of these solutions by the same procedure as that in the unsymmetrical shaft system⁽¹⁾.

Figures 2~7 show the typical theoretical resonance curves obtained by numerical computation. The stable and unstable solutions are drawn with solid and broken lines, respectively. In order to avoid confusion, we do not draw the solutions $R_0 = 0$. As for the stability of the solution $R_0 = 0$, the points at which the resonance curves for the

Fig.1 A p - ω diagram in the two-degree-of-freedom system.

Fig.2 Theoretical resonance curves of the subharmonic oscillation $[2p_f]$ showing the influence of the unsymmetry Δ .

Fig.3 Theoretical resonance curves showing the influence of the dynamic unbalance τ .

solutions $R_0 \neq 0$ intersect the ω -axis give the boundaries between the stable parts and the unstable ones, and the solutions are unstable between such two intersecting points, and stable outside them. If there are no intersecting points, the solutions of $R_0 = 0$ are stable at any value of ω .

Figure 2 shows the influence of the unsymmetry Δ . By making the value of Δ larger, the stable resonance curve tends to bend toward the lower angular velocity, and at last, all resonance curves of $R_0 \neq 0$ become unstable. In this case, all resonance curves containing the solution $R_0 = 0$ are unstable between two points at which the curves for $R_0 \neq 0$ intersect the ω -axis. Therefore, in this region (i.e. unstable region) no stable stationary oscillations

are obtained, but an unstable vibration of frequency $+1/2 \cdot \omega$ whose amplitude increases with time occurs. This is a remarkable phenomenon which is not observed in the symmetrical system.

The influence of the dynamic unbalance τ is shown in Fig.3. This is such an example that all the solutions of $R_0 \neq 0$ are unstable. When the value of τ is larger, the curve for $R_0 \neq 0$ intersects the ω -axis, and then the unstable region appears. The value of τ has no direct effect on the stability of the resonance curve, but contributes to the difficulty of appearance of this vibration.

Figure 4 shows an example representing the influence of the damping coefficient c . When each parameter takes the value shown

Fig.4 Theoretical resonance curves showing the influence of the damping coefficient (when $\beta^{(0)}$ is comparatively small).

Fig.6 Theoretical resonance curves showing the influence of the nonlinear component $N(1)$ ($\epsilon^{(1)}$).

Fig.5 Theoretical resonance curves showing the influence of the nonlinear component $N(0)$ ($\beta^{(0)}$).

Fig.7 Theoretical resonance curves showing the influence of the nonlinear component $N(0)$ ($\beta^{(2)}$).

in this diagram, the unstable region appears for a small value of c , and it vanishes for a large value of it. When the stationary stable oscillation occurs for different values of these parameters, the oscillation is also hard to appear for a large value of c .

Figure 5 shows the influence of the nonlinear component $N(0)$ ($\beta^{(0)}$). The sign of the value of $\beta^{(0)}$ decides the type of the resonance curves; i.e. for the positive $\beta^{(0)}$ the curve is a hard spring type bending toward the higher angular velocity, and for the negative $\beta^{(0)}$ a soft spring type bending toward the lower angular velocity. Especially, when the absolute value of $\beta^{(0)}$ is comparatively small, the unstable region appears.

Figure 6 shows the influence of the nonlinear component $N(1)$ ($\epsilon^{(1)}$). The effect of $\epsilon^{(1)}$ is qualitatively the same as that of τ shown in Fig.3, because Eq.(9) includes $\epsilon^{(1)}$ and P in the product form.

Figure 7 shows the influence of the nonlinear component $N(2)$ ($\beta^{(2)}$). In this diagram where the curves for the comparatively large $\beta^{(0)}$ are shown, the value of $\beta^{(2)}$ gives a remarkable effect on the spread of the curves in the part of large amplitudes. Since Eq.(9) includes $\beta^{(2)}$ and Δ in the product form, Fig.7 is similar to Fig.2 showing the influence of Δ . The fact that the unstable region appears in the unsymmetrical shaft or rotor system is due to the influence of the nonlinear component $N(2)$. This component has no effects

on the symmetrical system, but has great influence through the unsymmetry of the shaft or the rotor.

There are qualitatively no big changes for various values of phase angle φ_2 of $\beta^{(2)}$ and the angular position α of τ .

3.2 The summed-and-differential harmonic oscillation of the type $[p_f - p_b]$ When the natural frequency $p_f - p_b$ is nearly equal to the angular velocity ω in the nonlinear multi-degree-of-freedom shaft system (in the vicinity of the point B in Fig.1), two oscillations with the frequencies ω_f and ω_b , which are nearly equal to p_f and p_b , respectively, appear simultaneously. In this case, the equations

$$\omega_f \approx p_f, \quad \omega_b \approx p_b, \quad \omega_f - \omega_b = \omega \quad \dots\dots\dots(11)$$

hold. We find that this kind of oscillation (which is designated by the symbol $[p_f - p_b]$) occurs also in the unsymmetrical rotor system, and that its vibratory characteristic is similar to that of the unsymmetrical shaft system^[2]. We try to perform the theoretical analysis by the same procedure as that for the unsymmetrical shaft system. First, we suppose an approximate solution with the accuracy of $O(\epsilon)$ in the following forms. It is assumed that the amplitudes R_f , R_b , $\bar{R}_f$, $\bar{R}_b$ and the phase angles δ_f , δ_b are the quantities of $O(\epsilon^0)$ varying slowly with time.

$$\begin{aligned} \theta_s = & R_f \cos(\omega_f t + \delta_f) + R_b \cos(\omega_b t + \delta_b) + \bar{R}_f \cos(\bar{\omega}_f t - \delta_f) + \bar{R}_b \cos(\bar{\omega}_b t - \delta_b) + P \cos(\omega t + \beta) \\ & + \epsilon \left(g(\omega_f, \omega_b, \bar{\omega}_f, \bar{\omega}_b, 2\omega + \omega_f, 2\omega + \omega_b, 2\omega + \bar{\omega}_f, 2\omega + \bar{\omega}_b) \right. \\ & \left. + g'(\omega_f, \omega_b, \bar{\omega}_f, \bar{\omega}_b, 2\omega + \omega_f, 2\omega + \omega_b, 2\omega + \bar{\omega}_f, 2\omega + \bar{\omega}_b) \right) \quad \dots\dots\dots(12) \end{aligned}$$

where $\bar{\omega}_f = 2\omega - \omega_f$ and $\bar{\omega}_b = 2\omega - \omega_b$. The terms $\epsilon \cdot g$ and $\epsilon \cdot g'$ represent the compensative terms with the magnitude of $O(\epsilon)$, and include the components with the frequencies shown in the parentheses. Their concrete forms are the same as those in the unsymmetrical shaft system^[2].

Substituting Eq.(12) into Eq.(2), we set the *sine* and *cosine* terms of the respective frequencies ω_f , $\bar{\omega}_f$, ω_b and $\bar{\omega}_b$ to be zero separately with the accuracy of $O(\epsilon)$. After arranging these simultaneous equations, we obtain the following equations.

$$\left. \begin{aligned} A_f R_f \dot{\delta}_f &= -\sigma_f R_f - 4\beta^{(0)}(g_f R_f^2 + 2\bar{g}_b R_b^2 + 2k_f P^2) R_f + 6m_b(\beta_e^{(2)} \cos 2\phi \\ &\quad - \beta_s^{(2)} \sin 2\phi) R_f R_b^2 - 2\epsilon^{(1)}(a_e \cos \phi + a_s \sin \phi) P R_b \\ A_f \dot{R}_f &= c_f R_f + 6m_b(\beta_e^{(2)} \cos 2\phi + \beta_s^{(2)} \sin 2\phi) R_f R_b^2 + 2\epsilon^{(1)}(a_e \cos \phi - a_s \sin \phi) P R_b \\ A_b R_b \dot{\delta}_b &= -\sigma_b R_b - 4\beta^{(0)}(2\bar{g}_f R_f^2 + g_b R_b^2 + 2k_b P^2) R_b + 6m_f(\beta_e^{(2)} \cos 2\phi \\ &\quad - \beta_s^{(2)} \sin 2\phi) R_f^2 R_b - 2\epsilon^{(1)}(b_e \cos \phi + b_s \sin \phi) P R_f \\ A_b \dot{R}_b &= c_b R_b - 6m_f(\beta_e^{(2)} \cos 2\phi + \beta_s^{(2)} \sin 2\phi) R_f^2 R_b - 2\epsilon^{(1)}(b_e \cos \phi - b_s \sin \phi) P R_f \end{aligned} \right\} \quad \dots\dots\dots(13)$$

where we put $\epsilon_c^{(1)} = \epsilon^{(1)}(\infty)$ and $\epsilon_s^{(1)} = 0$. The notations

$$\left. \begin{aligned} A_f &= (i_p \omega - 2\omega_f) \bar{G}_f - (i_p \omega - 2\bar{\omega}_f) G_f - 4\Delta^2 \Omega_f (\omega - \omega_f) \\ A_b &= (i_p \omega - 2\omega_b) \bar{G}_b - (i_p \omega - 2\bar{\omega}_b) G_b - 4\Delta^2 \Omega_b (\omega - \omega_b) \\ a_e &= \left(\bar{G}_f + \frac{\Omega_f}{\Omega_b} G_b + \Delta \Omega_f \right) \cos \beta, \quad a_s = \left(\bar{G}_f + \frac{\Omega_f}{\Omega_b} G_b - \Delta \Omega_f \right) \sin \beta \\ b_e &= \left(\bar{G}_b + \frac{\Omega_b}{\Omega_f} G_f + \Delta \Omega_b \frac{\bar{G}_b}{\bar{G}_f} \right) \cos \beta, \quad b_s = \left(\bar{G}_b + \frac{\Omega_b}{\Omega_f} G_f - \Delta \Omega_b \frac{\bar{G}_b}{\bar{G}_f} \right) \sin \beta \\ c_f &= c(\omega_f \bar{G}_f - \bar{\omega}_f G_f), \quad c_b = c(\omega_b \bar{G}_b - \bar{\omega}_b G_b), \quad G_f = G(\omega_f), \quad G_b = G(\omega_b) \\ \bar{G}_f &= G(\bar{\omega}_f), \quad \bar{G}_b = G(\bar{\omega}_b), \quad g_f = (G_f^2 + \bar{G}_f^2 + 4G_f \bar{G}_f) / \bar{G}_f \\ g_b &= (G_b^2 + \bar{G}_b^2 + 4G_b \bar{G}_b) / \bar{G}_b, \quad \bar{g}_f = ((G_f + \bar{G}_f)(G_b + \bar{G}_b) + 2\Delta^2 \Omega_f \Omega_b) / \bar{G}_f \\ \bar{g}_b &= ((G_f + \bar{G}_f)(G_b + \bar{G}_b) + 2\Delta^2 \Omega_f \Omega_b) / \bar{G}_b, \quad k_f = G_f + \bar{G}_f + \Delta \Omega_f \cos 2\beta \\ k_b &= G_b + \bar{G}_b + \Delta \Omega_b \cos 2\beta, \quad m_f = -\Delta(\Omega_f \bar{G}_b + \Omega_b G_f) / \bar{G}_f \\ m_b &= -\Delta(\Omega_f \bar{G}_b + \Omega_b G_f) / \bar{G}_b, \quad \sigma_f = G_f \bar{G}_f - (\Delta \Omega_f)^2 \\ \sigma_b &= G_b \bar{G}_b - (\Delta \Omega_b)^2, \quad \phi = \delta_f - \delta_b, \quad \Omega_f = \omega_f \bar{\omega}_f, \quad \Omega_b = \omega_b \bar{\omega}_b \end{aligned} \right\} \quad \dots\dots\dots(14)$$

are used in Eq.(13). The solution of the harmonic oscillation is given by Eq.(8). From Eq.(13), we find that the nonlinear components $N(0)$, $N(1)$, and $N(2)$ have effects

Fig.8 Theoretical resonance curves of the summed-and-differential harmonic oscillation $[p_f - p_b]$ showing the influence of the unsymmetry Δ .

Fig.9 Experimental resonance curves of the subharmonic oscillation of order 1/2 $[2p_2]$ (a hard spring type)

on the summed-and-differential harmonic oscillation of the type $[p_f - p_b]$. Similarly to the subharmonic oscillation of order 1/2 $[2p_f]$ in the previous section, the component $N(2)$, which has no influence on the symmetrical system, is related. In the same manner as that in the unsymmetrical shaft system⁽²⁾, we can get the stationary solutions $R_f = R_{f0}$ and $R_b = R_{b0}$, and can examine the stabilities of them. The influences of various parameters on the resonance curves obtained by numerical computations are nearly the same as those in the subharmonic oscillation of order 1/2 $[2p_f]$. For example, the influence of the unsymmetry Δ is shown in Fig.8. In this diagram, the ordinate represents the sum of the stationary amplitudes of the two oscillations with the frequencies ω_f and ω_b .

4. Other kinds of subharmonic oscillations and summed-and-differential harmonic oscillations

Up to the previous chapter, we have shown that the vibratory characteristics of the subharmonic oscillation of order 1/2 $[2p_f]$ and the summed-and-differential harmonic oscillation $[p_f - p_b]$ in the unsymmetrical rotor system are quite different from those in the symmetrical system. In addition to these oscillations, the subharmonic oscillation of order 1/2 of the type $[-2p_b]$, the subharmonic oscillations of order 1/3 of the types $[3p_f]$ and $[-3p_b]$, and summed-and-differential harmonic oscillations of the types $[2p_f - p_b]$ and $[p_f - 2p_b]$ may occur in the unsymmetrical rotor system

Fig.10 Experimental resonance curves of the subharmonic oscillation $[2p_2]$ where an unstable region existed.

Fig.11 Waves of the unstable vibration $[2p_2]$ (at $\omega = 2842.3\text{rpm}$)

with the nonlinear spring characteristics represented by Eq.(3). But, the theoretical calculations for these oscillations lead to

Fig.12 Experimental resonance curves where the subharmonic oscillation $[2p_2]$ disappeared and the harmonic oscillation remained.

Fig.13 Experimental resonance curves of the summed-and-differential harmonic oscillation $[p_2-p_3]$ (a hard spring type)

Fig.14 Experimental resonance curves of the summed-and-differential harmonic oscillation $[p_2-p_3]$ (a soft spring type)

the same results as those of the symmetrical system⁽⁴⁾. Their resonance curves are not influenced greatly by the unsymmetry Δ , and the nonlinear components related to each oscillation are the same as those in the symmetrical system, and the vibratory characteristics are qualitatively the same.

5. Experimental apparatus

An unsymmetrical rotor was mounted on a round shaft at the position of the ratio 1:4 from the lower end of the shaft. The shaft was supported vertically by a self-aligning double row ball bearing (#1200) at the upper end and by a single-row deep groove ball bearing (#6200) at the lower end. The upper end was under the simply supported condition. The shaft had a nonlinear spring characteristics owing to the angular clearance of the lower bearing. Depending on the degree of discrepancy between the center lines of the upper and the lower bearings, the type of the nonlinear spring characteristics changed. The unsymmetrical rotor was 8.79kg in mass, and $I_p =$

Fig.15 Experimental resonance curves of the summed-and-differential harmonic oscillation $[p_2-p_3]$ where an unstable region appeared.

Fig.16 Experimental resonance curves where no summed-and-differential harmonic oscillations occurred

Fig.17 Waves of the unstable vibration $[p_2-p_3]$ (at $\omega=1724\text{rpm}$)

$0.2041 \text{ kg}\cdot\text{m}^2$, $I_1=0.1594 \text{ kg}\cdot\text{m}^2$, and $I_2=0.0480 \text{ kg}\cdot\text{m}^2$. The diameter of the shaft was 12mm, and the length between the bearings was 700mm. The shaft deflections were measured optically in the two directions which were at right angles to each other.

6. Experimental results

The resonance curves were measured under various assemblies. The values of parameters were varied by changing the assembly of the apparatus and size and location of unbalance.

A p - ω diagram of the experimental apparatus with four degrees of freedom is similar to that of an unsymmetrical rotor system⁽²⁾. Natural frequencies p_i ($i=1,4$) and $\bar{p}_i$ ($=2\omega-p_i$) coexist due to the unsymmetry of the rotor.

The experimental results for the subharmonic oscillations of order $1/2$ $[2p_2]$ are shown in Figs.9~12. Figure 9 is a hard spring type, and the arrow represents a jump phenomenon. Figure 10 is an example where unstable vibration appeared under different assembly. The envelope of an unstable wave form can be expressed approximately by $A_0 e^{m\tau}$ (A_0 : an arbitrary constant). The length of each arrow represents the value of m . Figure 11 shows a wave recorded at $\omega=2842.3\text{rpm}$ in Fig.10. In this picture, vertical white lines represent one revolution of the shaft. Figure 12 has the same assembling condition as Fig.10, and shows the case of a small unbalance. An unstable vibration does not appear and harmonic oscillations remain. This corresponds to that of the theoretical resonance curve for a small τ in Fig.3.

The results for the summed-and-differential harmonic oscillation $[p_2-p_3]$ are shown in Figs.13~17. Amplitudes in these resonance curves represent a maximum value of the amplitude in the wave. Figure 13 is a hard spring type and Fig.14 is a soft spring type. In these figures, the resonant angular velocities are different from each other. Because the degree of the discrepancy between the center lines of the upper and lower bearings in Fig.13 was smaller than that in Fig.14, and the natural frequency changed. An unstable vibration may occur as shown in Fig.15 in a certain assembly. After making the unbalance smaller in the same assembly as Fig.15, Fig.16 was obtained. The oscillation $[p_2-p_3]$ does not occur unless a large disturbance is given. The small peak in the figure represents the other kind of forced oscillation. Figure 17 shows a wave of the unstable vibration recorded at $\omega=1724\text{rpm}$ in Fig.15.

As for the summed-and-differential harmonic oscillation of the type $[p_2-p_4]$, the resonance frequency was so high that we could not experiment on it sufficiently.

7. Conclusions

(1) In an unsymmetrical rotor system with a nonlinear spring characteristic, the nonlinear components $N(0)$, $N(1)$, and $N(2)$ are concerned with the subharmonic oscillation of order $1/2$ of the type $[2p_i]$ and the summed-and-differential harmonic oscillation of the type $[p_i-p_j]$. Especially, the component $N(2)$, which has no effect in a symmetrical system, plays an important part through the unsymmetry Δ in these oscillations. As the result, different phenomena from those in the symmetrical system appear.

(2) The theoretical resonance curves of these two kinds of oscillations are qualitatively the same. They can be classified into four typical groups as follows: the stable resonance curve of a hard spring type, that of a soft spring type, the resonance curve having an unstable region in which only the unstable solutions exist, and the case in which these oscillations do not occur unless a large disturbance gets into the system.

(3) In experiments with the unsymmetrical rotor system having a single-row deep groove ball bearing, we obtained the same resonance curves in form as those calculated in the theoretical analyses. The types of these resonance curves were changed depending on the degree of discrepancy between the center lines of the upper and lower bearings and the size and location of the existing unbalance.

(4) In experiments, we obtained three kinds of resonance curves for the subharmonic oscillation of order $1/2$ of the type $[2p_2]$, and four kinds of resonance curves for the summed-and-differential harmonic oscillation of the type $[p_2-p_3]$.

(5) The qualitative characteristics of the subharmonic oscillations and the summed-and-differential harmonic oscillations in the unsymmetrical rotor system are theoretically and experimentally the same as those in the unsymmetrical shaft system.

References

- (1) Yamamoto, T., et al., *Bull. JSME*, Vol. 22, No.164 (1979-2), 164.
- (2) Yamamoto, T., et al., *Bull. JSME*, Vol. 24, No.187 (1981-1), 1.
- (3) Yamamoto, T. and Ota, H., *Bull. JSME*, Vol.6, No.21 (1963-2), 29.
- (4) Yamamoto, T. and Ishida, Y., *Trans. Japan Soc. Mech. Engrs.* (in Japanese), 41-345 (1975-5), 1374.
Yamamoto, T. and Ishida, Y., *Ing. Arch.*, 46 (1977), 125.