Vibrations of Mindlin's Circular Plates with Thickness

Varied along Radius*

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The vibrations of circular plates with variable thickness are studied by the method of Mindlin's thick plate theory. From the Lagrangian of vibration of the circular plate with thickness varied along radius, the equations of vibration and the boundary conditions are deduced. The vibration problems of circular plates with two or more nodal diameters are solved in the case that the thickness is varied exponentially, the effects of some parameters on frequencies are discussed and some frequencies are compared with those from the classical theory.

1. Introduction

Reissner⁽¹⁾⁽²⁾ and Mindlin⁽³⁾⁽⁴⁾ have researched the vibrations of plates with consideration of the rotatory inertia and the shearing deformation and ever since, many papers have been published as seen in researches of Timoshenko beam. Some of them are researches on vibrations of elliptic plates by Callahan⁽⁵⁾, annular plates by Rao⁽⁶⁾ and of forced vibrations by Reismann⁽⁷⁾.

On the other hand, there are many researches on free and forced vibrations of circular plates with variable thickness. Confining ourselves to free vibrations, we find papers by Conway⁽⁸⁾, Raju and others⁽⁹⁾, Soni and others⁽¹⁰⁾, Ramaiah and others⁽¹¹⁾ and Juarez⁽¹²⁾ on axisymmetric vibrations and those by Harris⁽¹³⁾ and Takahashi and others⁽¹⁴⁾ on vibrations with nodal diameters.

However, researches on vibrations of plates with variable thickness considering both rotatory inertia and shearing deformation are quite limited and we can not find any except for one by Soni and Rao⁽¹⁵⁾, who have studied axisymmetric vibrations.

Based on Mindlin's theory, in this paper, we have studied the bending vibrations of a circular plate with two or more nodal diameters of which thickness is varied exponentially along the radius. Introducing the Lagrangian of vibration of the plate in a period, we have obtained equations of vibration and boundary conditions in a general form. We have calculated the frequencies in the case that thickness is

varied exponentially. After numerical calculations, we have compared Mindlin's theory with the classical one and showed frequencies corresponding to eight boundary conditions. We have also studied how frequencies are varied by changes of several parameters, which are ratio of radius to thickness, number of nodal diameters, rate of thickness variation, shear coefficient and Poisson's ratio.

2. Lagrangian and equations of vibration

The lateral vibrations are studied of a circular plate, which is symmetric about the middle surface in profile. Let R denote the radius of the circular plate; Rr and θ , the polar coordinates on the middle surface, where $r=0-1$, and we take z -axis perpendicular to this surface. Let also h_0 and $h(r)$ denote the thicknesses at origin and at any point, respectively. The time, the circular frequency and the period are denoted by t , p and $\tau'(=2\pi/p)$, and the density, Young's modulus and Poisson's ratio by γ/g , E and ν respectively.

We assume that the middle surface is inextensible and that the variations of displacements and strains along z -axis are so small that the terms higher than the second order of Z can be neglected.

Let the displacements along r , θ and z axes be $\bar{u}$, $\bar{v}$ and $\bar{w}$, respectively, and the nondimensional ones be u , v and w which are the functions of r and θ . We put as follows:

$$\bar{u} = zu \sin pt, \quad \bar{v} = zv \sin pt, \quad \bar{w} = R w \sin pt \quad \dots\dots\dots (1)$$

The normal and tangential strains ϵ , γ' are

$$\left. \begin{aligned} \epsilon_r &= \partial \bar{u} / R \partial r, & \epsilon_\theta &= \partial \bar{v} / R r \partial \theta + \bar{u} / R r, & \epsilon_z &= \partial \bar{w} / \partial z = 0, & \gamma_{r,\theta}' &= \partial \bar{u} / R r \partial \theta + \partial \bar{v} / R \partial r - \bar{v} / R r \\ \gamma_{\theta,z}' &= \partial \bar{v} / \partial z + \partial \bar{w} / R r \partial \theta, & \gamma_{r,z}' &= \partial \bar{w} / R \partial r + \partial \bar{u} / \partial z \end{aligned} \right\} \quad \dots\dots\dots (2)$$

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From Hooke's law, we obtain the following normal and tangential stresses,

$$\left. \begin{aligned} \sigma_r &= E(\varepsilon_r + \nu \varepsilon_\theta) / (1 - \nu^2), & \sigma_\theta &= E(\nu \varepsilon_r + \varepsilon_\theta) / (1 - \nu^2), & \tau_{r\theta} &= E\gamma_{r\theta}' / 2(1 + \nu) \\ \tau_{\theta z} &= k'E\gamma_{\theta z}' / 2(1 + \nu), & \tau_{rz} &= k'E\gamma_{rz}' / 2(1 + \nu) \end{aligned} \right\} \dots\dots\dots (3)$$

where k' is the shear coefficient. Moreover, we assume that the integration of σ_z along z axis is zero.

The above procedure is similar to Mindlin's method.

The Lagrangian L of vibration of the plate is

$$L = \frac{1}{2} \frac{2}{\tau'} \int \int \int \int \left[\frac{\gamma}{g} \left\{ \left(\frac{\partial \bar{u}}{\partial t} \right)^2 + \left(\frac{\partial \bar{v}}{\partial t} \right)^2 + \left(\frac{\partial \bar{w}}{\partial t} \right)^2 \right\} - (\sigma_r \varepsilon_r + \sigma_\theta \varepsilon_\theta + \tau_{r\theta} \gamma_{r\theta}' + \tau_{\theta z} \gamma_{\theta z}' + \tau_{rz} \gamma_{rz}') \right] R^2 r dr d\theta dz dt$$

Integrating the above equation about t and z , of which both limits are $(0 \sim \tau')$ and $(-h/2 \sim h/2)$, we obtain the Lagrangian in a period as follows:

$$L = (1/2) \int \int [D_1(u^2 + v^2) + D_2 w^2 - D\{(\partial u / \partial r)^2 + (\partial v / r \partial \theta + u/r)^2 + 2\nu(\partial u / \partial r)(\partial v / r \partial \theta + u/r) + (1 - \nu)(\partial u / r \partial \theta + \partial v / \partial r - v/r)^2 / 2\} - D_3\{(v + \partial w / r \partial \theta)^2 + (u + \partial w / \partial r)^2\}] r dr d\theta \dots\dots\dots (4)$$

We put the nondimensional frequencies α and α' , the parameter β depending on the ratio of radius and thickness and the representative bending rigidity D_0 as follows,

$$\alpha^4 = 2(1 + \nu)\gamma p^2 R^2 / gE, \quad \alpha'^4 = \alpha^4 / k', \quad \beta = 12k'R^2 / h_0^2, \quad D_0 = Eh_0^3 / 12(1 - \nu^2)$$

Then, D_1 etc. in eq. (4) become

$$D = D_0(h/h_0)^3, \quad D_1 = D_0(h/h_0)^3 \alpha^4 (1 - \nu) / 2, \quad D_2 = D_0(h/h_0) \beta \alpha'^4 (1 - \nu) / 2 \\ D_3 = D_0(h/h_0) \beta (1 - \nu) / 2$$

Their unit is kg-cm and they are functions of thickness h only. We denote the numerical results by α_0 which is defined as

$$\alpha_0^4 = \gamma h_0 p^2 R^4 / gD_0 [= \alpha^4 \beta (1 - \nu) / 2k']$$

For the sake of convenience, we put

$$\left. \begin{aligned} M_r &= Dr(\partial u / \partial r + \nu(\partial v / r \partial \theta + u/r)), & M_\theta &= Dr(\nu \partial u / \partial r + \partial v / r \partial \theta + u/r) \\ M_{r\theta} &= (1 - \nu)Dr(\partial u / r \partial \theta + \partial v / \partial r - v/r) / 2, & Q_r &= D_3 r(u + \partial w / \partial r) \\ Q_\theta &= D_3 r(v + \partial w / r \partial \theta) \end{aligned} \right\} \dots\dots\dots (5)$$

We denote the inner radius of a plate by r_1 and the integral limits of θ by θ_1 and θ_2 . After calculating the first variation of eq. (4), we integrate it by parts and obtain

$$\delta L = \int_{\theta_1}^{\theta_2} \int_{r_1}^1 (E_1 \delta u + E_2 \delta v + E_3 \delta w) r dr d\theta - \int_{\theta_1}^{\theta_2} [M_r \delta u + M_{r\theta} \delta v + Q_r \delta w] \Big|_{r_1}^1 d\theta \\ - \int_{r_1}^1 \frac{1}{r} [M_{r\theta} \delta u + M_\theta \delta v + Q_\theta \delta w] \Big|_{\theta_1}^{\theta_2} dr = 0 \dots\dots\dots (6)$$

$$\left. \begin{aligned} rE_1 &= \partial M_r / \partial r + \partial M_{r\theta} / r \partial \theta - M_\theta / r - Q_r + D_1 r u \\ rE_2 &= \partial M_\theta / r \partial \theta + \partial M_{r\theta} / \partial r + M_{r\theta} / r - Q_\theta + D_1 r v, & rE_3 &= \partial Q_r / \partial r + \partial Q_\theta / r \partial \theta + D_2 r w \end{aligned} \right\} \dots\dots\dots (7)$$

In the above equations, M_r and M_θ are the quantities proportional to bending moment, $M_{r\theta}$ to twisting moment and Q_r and Q_θ to shearing force, respectively. From eq. (6), equations $E_1 = E_2 = E_3 = 0$ (equations of vibration) hold at any point in a circular plate and $M_r u = M_{r\theta} v = Q_r w = 0$ (boundary conditions) hold at inner and outer peripheries. The last term in eq. (6) represents the boundary conditions along the radius for a sector plate etc. In the study of a circular plate, $\theta_1 = 0$ and $\theta_2 = 2\pi$ and so the last term becomes zero. We must consider eight combinations of boundary conditions at a periphery of a circular plate.

3. Solution

We treat a circular plate without a hole, of which the thickness h/h_0 is $\exp(-kr)$. We put the displacements as follows:

$$u = \sum_{n=0}^{\infty} u_n(r) \cos n\theta, \quad v = \sum_{n=0}^{\infty} v_n(r) \sin n\theta, \quad w = \sum_{n=0}^{\infty} w_n(r) \cos n\theta \quad \dots\dots\dots (8)$$

We restrict ourselves to the vibrations with two or more nodal diameters ($n \geq 2$). In the case that $n=0$ (axisymmetric vibration) and $n=1$ (vibration with one nodal diameter), the solutions of vibration equations take particular forms. Substituting eq.(8) into $E_3=0$, we obtain

$$-nv_n/r = (rd/dr + 2 - kr)(u_n/r) + \{d^2/dr^2 + (1/r - k)d/dr + \alpha'^4 - n^2/r^2\} w_n \quad \dots\dots\dots (9)$$

Substituting eq.(8) into $E_1=E_2=0$, and using eq.(9), we obtain the simultaneous differential equations of u_n/r and w_n , as follows:

$$\left. \begin{aligned} (rd^2/dr^2 + P_1 d/dr + P_2)(u_n/r) + [- \{ (1+\nu)/(1-\nu) \} d^3/dr^3 + P_3 d^2/dr^2 + P_4 d/dr + P_5] w_n &= 0 \\ (rd^3/dr^3 + P_6 d^2/dr^2 + P_7 d/dr + P_8)(u_n/r) + (d^4/dr^4 + P_9 d^3/dr^3 + P_{10} d^2/dr^2 + P_{11} d/dr + P_{12}) w_n &= 0 \end{aligned} \right\} \quad \dots\dots\dots (10)$$

$$\begin{aligned} P_1 &= 5 - \frac{5-7\nu}{1-\nu} kr, & P_2 &= \frac{4-n^2}{r} - \frac{7-9\nu}{1-\nu} k - \left(\frac{6\nu k^2}{1-\nu} - \alpha^4 \right) r - \beta r e^{2kr} \\ P_3 &= \frac{1-3\nu}{1-\nu} \frac{1}{r} + \frac{1+7\nu}{1-\nu} k, & P_4 &= \frac{n^2(1+\nu)+3-\nu}{1-\nu} \frac{1}{r^2} - \frac{2-8\nu}{1-\nu} \frac{k}{r} - \frac{6\nu k^2}{1-\nu} - \frac{1+\nu}{1-\nu} \alpha'^4 - \beta e^{2kr} \\ P_5 &= \frac{-4n^2}{r^3(1-\nu)} - \frac{6n^2\nu k}{r^2(1-\nu)} + \frac{2\alpha'^4}{r} + \frac{6\nu k\alpha'^4}{1-\nu}, & P_6 &= 7-4kr \\ P_7 &= \frac{9-n^2}{r} - 14k + (3k^2 + \alpha^4 - \beta e^{2kr})r \\ P_8 &= \left(\frac{-1+3\nu}{1-\nu} n^2 - 3 \right) \frac{k}{r} + 3k^2 + (\alpha^4 - \beta e^{2kr})(2-kr), & P_9 &= \frac{4}{r} - 4k \\ P_{10} &= \left(1 - \frac{3-\nu}{1-\nu} n^2 \right) \frac{1}{r^2} - \frac{6k}{r} + 3k^2 + \alpha'^4 + \alpha^4 - \beta e^{2kr} \\ P_{11} &= - \left(1 + \frac{1+\nu}{1-\nu} n^2 \right) \frac{1}{r^3} + \left(3 + \frac{5-3\nu}{1-\nu} n^2 \right) \frac{k}{r^2} + (3\alpha'^4 + \alpha^4 - \beta e^{2kr}) \left(\frac{1}{r} - k \right) \\ P_{12} &= \frac{2n^4}{r^4(1-\nu)} - \frac{6n^2k}{r^3} - \frac{n^2}{r^2} \left(\frac{2\alpha'^4}{1-\nu} + \alpha^4 \right) + \alpha'^4 (\alpha^4 - \beta e^{2kr}) \end{aligned}$$

We assume the solutions of eq.(10) as follows,

$$u_n/r = (\log r) \sum_{m=0}^{\infty} a_m r^{p+m} + \sum_{m=0}^{\infty} a'_m r^{p+m-2}, \quad w_n = (\log r) \sum_{m=0}^{\infty} b_m r^{p+m+2} + \sum_{m=0}^{\infty} b'_m r^{p+m} \quad \dots\dots\dots (11)$$

Substituting eq.(11) into eq.(10), we obtain

$$\left. \begin{aligned} (\log r) \sum_{m=0}^{\infty} [\{ (p+m+2)^2 - n^2 \} [a_m - \{ (p+m+2)(1+\nu) - 4 \} b_m / (1-\nu)] + f_m] r^{p+m-1} \\ + \sum_{m=0}^{\infty} [\{ (p+m)^2 - n^2 \} [a'_m - \{ (p+m)(1+\nu) - 4 \} b'_m / (1-\nu)] + F_m] r^{p+m-3} = 0 \\ (\log r) \sum_{m=0}^{\infty} [\{ (p+m+2)^2 - n^2 \} [(p+m)a_m + \{ (p+m+2)(p+m) - 2n^2 / (1-\nu) \} b_m] + g_m] r^{p+m-2} \\ + \sum_{m=0}^{\infty} [\{ (p+m)^2 - n^2 \} [(p+m-2)a'_m + \{ (p+m)(p+m-2) - 2n^2 / (1-\nu) \} b'_m] + G_m] r^{p+m-4} = 0 \end{aligned} \right\} \quad \dots\dots\dots (12)$$

where $\rho=k(1-\nu)$, and F_m and G_m are

$$\begin{aligned}
 F_m = & \rho \{ (-5+7\nu)(p+m) + 8-12\nu \} a_{m-1}' + \left(\alpha^4 - \frac{6\nu k^2}{1-\nu} \right) a_{m-2}' \\
 & - \beta \sum_{q=0}^{m-2} \frac{(2k)^{m-q-2}}{(m-q-2)!} \{ a_q' + (p+q)b_q' \} + 2(p+m)a_{m-2} + \rho(-5+7\nu)a_{m-3} \\
 & + \rho \{ (p+m)^2(1+7\nu) - (p+m)(5+13\nu) + 4+6\nu(1-\nu^2) \} b_{m-1}' \\
 & - \left\{ \left(\alpha^4 \frac{1+\nu}{1-\nu} + \frac{6\nu k^2}{1-\nu} \right) (p+m-2) - 2\alpha^4 \right\} b_{m-2}' + 6\rho\nu\alpha^4 b_{m-3}' \\
 & - \frac{1}{1-\nu} [\{ 3(p+m)^2 - n^2 \} (1+\nu) - 8(p+m)] b_{m-2} + \rho \{ 2(1+7\nu)(p+m) - 5-13\nu \} b_{m-3} \\
 & - \left(\alpha^4 \frac{1+\nu}{1-\nu} + \frac{6\nu k^2}{1-\nu} \right) b_{m-4} - \beta \sum_{q=0}^{m-4} \frac{(2k)^{m-q-4}}{(m-q-4)!} b_q \\
 G_m = & -k \left\{ 4(p+m)^2 - 14(p+m) + 9 + \frac{1-3\nu}{1-\nu} n^2 \right\} a_{m-1}' + \{ 3k^2(p+m-3) + \alpha^4(p+m-2) \} a_{m-2}' \\
 & - k\alpha^4 a_{m-3}' - \beta \sum_{q=0}^{m-2} \frac{(2k)^{m-q-2}}{(m-q-2)!} \{ (p+q)a_q' + (p+q)^2 b_q' \} + k\beta \sum_{q=0}^{m-3} \frac{(2k)^{m-q-3}}{(m-q-3)!} \{ a_q' + (p+q)b_q' \} \\
 & - \alpha^4 \beta \sum_{q=0}^{m-4} \frac{(2k)^{m-q-4}}{(m-q-4)!} b_q' - k \left[(p+m-1) \{ 4(p+m)^2 - 14(p+m) + 9 \} \right. \\
 & \left. - \frac{n^2}{1-\nu} \{ (5-3\nu)(p+m) - 11+9\nu \} \right] b_{m-1}' + \left[3k^2(p+m-2)(p+m-3) \right. \\
 & \left. + \alpha^4 \left\{ (p+m)(p+m-2) - \frac{2n^2}{1-\nu} \right\} + \alpha^4 \{ (p+m-2)^2 - n^2 \} \right] b_{m-2}' - k(3\alpha^4 + \alpha^4)(p+m-3)b_{m-3}' \\
 & + \alpha^4 \alpha^4 b_{m-4}' + \{ 3(p+m)^2 - 4(p+m) - n^2 \} a_{m-2} - k \{ 8(p+m) - 14 \} a_{m-3} + (3k^2 + \alpha^4) a_{m-4} \\
 & + 2 \left[(p+m)^2(2p+2m-3) + n^2 \left\{ 1 - \frac{(p+m)(3-\nu)}{1-\nu} \right\} \right] b_{m-2} - k \left\{ 12(p+m)^2 - 36(p+m) + 23 \right. \\
 & \left. - \frac{5-3\nu}{1-\nu} n^2 \right\} b_{m-3} + \{ 3k^2(2p+2m-5) + 2\alpha^4(p+m-1) + 2\alpha^4(p+m-2) \} b_{m-4} \\
 & - k(3\alpha^4 + \alpha^4) b_{m-5} - \beta \sum_{q=0}^{m-4} \frac{(2k)^{m-q-4}}{(m-q-4)!} \{ a_q + 2(p+q+2)b_q \} + k\beta \sum_{q=0}^{m-5} \frac{(2k)^{m-q-5}}{(m-q-5)!} b_q
 \end{aligned} \tag{13}$$

And we can obtain f_m and g_m in eq.(12), respectively, by putting $a_i=b_i=0$, ($i=0 \sim m-1$) in F_m and G_m , then by substituting a_i , b_i and $(p+2)$ for a_i' , b_i' and p into F_m and G_m . From eq.(12), it is seen that four expressions in double brackets $\{\{\}$ must be zero, and we obtain

$$\begin{aligned}
 G'a_m' = & \{ 2n^2 - (p+m)(p+m-2)(1-\nu) \} F_m - \{ (p+m)(1+\nu) - 4 \} G_m \\
 G'b_m' = & (1-\nu) \{ (p+m-2)F_m - G_m \}, \quad G' = 2 \{ (p+m)^2 - n^2 \} \{ (p+m-2)^2 - n^2 \}
 \end{aligned} \tag{14}$$

We also obtain a_m and b_m by substitution of f_m , g_m and $(p+2)$ for F_m , G_m and p into eq.(14). There are six pairs of solutions of equations of vibration and three pairs of them are finite at $r=0$. The vanishing of the coefficient of the lowest power of r gives $p=\pm n$ or $p-2=\pm n$ for the two pairs of solutions without $(\log r)$. Taking $(-n)$ in the above equations of p yields the same solutions as when taking $(+n)$, since the coefficients of the first few powers vanish inevitably. We now call these two solutions the first solution and the second one, respectively. And the solution including the term with $(\log r)$ will be called the third solution. Three pairs of solutions of a plate without a hole are as follows:

where, $G_0 = \{ (n+2)(1+\nu) - 4 \} / (1-\nu)$

(i) The first solution: $a_i=b_i=0$ ($i=0, 1, 2, \dots$). $a_0'=a_1'=a_2'=0$,

$$b_1' = nk b_0' / (2n+1), \quad b_2' = -b_0' [\alpha^4 - n(n+1)k^2 / (2n+1) + n\beta / G_0] / 4(n+1).$$

We can give an arbitrary value to b_0' and obtain a_m' and b_m' from eq.(14), when $m \geq 3$.

(ii) The second solution: $a_i=b_i=0$. $a_0'=G_0 b_0'$. As the first solution, b_0' is arbitrary and we can obtain a_m' and b_m' from eq.(14), when $m \geq 1$.

- (iii) The third solution: In this solution, a and b are the same as a_c' and b_c' in the second solution.

$$a_0' = -8(4n^2 - 1)b_0/9k^2(n-1)(1-\nu^2),$$

$$b_0' = 0, \quad a_1' = 3k(n-1)\{n(3-\nu)+1-\nu\}a_0'/(4n^2-1), \quad b_1' = k\{-n(1-3\nu)+2-3\nu\}a_0'/(4n^2-1), \quad a_2' = 0.$$

We can give an arbitrary value to b_0 and obtain a_m' ($m \geq 3$) and b_m' ($m \geq 2$) from eq.(14).

4. Comparison with Mindlin's solution

Mindlin⁽³⁾ has constructed an excellent theory of vibrations of a circular plate with uniform thickness. Rewriting eq.(37) in his paper with our notations we obtain

$$\left. \begin{aligned} d^4 w_n / dr^4 + (2/r) d^3 w_n / dr^3 + \{K - (2n^2 + 1)/r^2\} d^2 w_n / dr^2 \\ + \{K/r + (2n^2 + 1)/r^3\} dw_n / dr + \{K' - n^2 K / r^2 + (n^4 - 4n^2)/r^4\} w_n = 0 \end{aligned} \right\} \dots\dots\dots (15)$$

$$K = \alpha'^4 + \alpha'^4(1-\nu)/2, \quad K' = \alpha'^4(1-\nu)(\alpha'^4 - \beta)/2$$

Substituting w_n in eq.(11) into the above equation, we obtain

$$\begin{aligned} (\log r) \sum_{m=0}^{\infty} \{[(p+m+2)^2 - n^2]\{(p+m)^2 - n^2\} b_m + K[(p+m)^2 - n^2] b_{m-2} + K' b_{m-4}\} r^{p+m-2} \\ + \sum_{m=0}^{\infty} \{[(p+m)^2 - n^2]\{(p+m-2)^2 - n^2\} b_m' + K[(p+m-2)^2 - n^2] b_{m-2}' + K' b_{m-4}' \\ + 4(p+m-1)\{(p+m)(p+m-2) - n^2\} b_{m-2} + 2K(p+m-2) b_{m-4}\} r^{p+m-4} = 0 \end{aligned} \dots\dots\dots (16)$$

Putting $k=0$ in equations shown in chapter.2 and the expressions with $\log r$ in the first equation of eq.(12) equal to zero, we obtain the following relations,

$$\begin{aligned} \{(p+m)^2 - n^2\} a_{m-2} + (\alpha'^4 - \beta) a_{m-4} - \{(p+m)^2 - n^2\}\{(p+m)(1+\nu) - 4\} b_{m-2}/(1-\nu) \\ - \{\alpha'^4(p+m-2)(1+\nu)/(1-\nu) - 2\alpha'^4 + \beta(p+m-2)\} b_{m-4} = 0 \end{aligned}$$

Substituting the above relation into the right side of eq.(14), we obtain an equation with a 's eliminated. It is easy to show that b_m and b_m' obtained thus agree with those obtained from eq.(16). That is, we can obtain Mindlin's solutions by putting $k=0$ in our solutions.

5. Numerical calculation

We have carried out numerical calculations to clarify the effects of parameters on the nondimensional frequency α_0 . Restricted in the capacity of the computer used, we took 100 terms for a_m and b_m etc., respectively (50 terms when $k=0.4$).

If β is larger than 400 or α is larger than 4, the moments and the shearing forces obtained (with 100 terms of a_m etc.) are accurate to 4~5 digits, while the frequencies are not so accurate, because of dropped digits. So we have had to calculate in the range where β is smaller than or equal to 200 and α smaller than 3. In this case, the moments etc. are accurate to 8~9 digits and the values shown in tables have slight errors at the last digit. We obtain α from the frequency equation (the third order determinant) and reduce it to α_0 . The material constants used are $\nu=0.3$ and $k'=0.822$ (the value used by Mindlin), unless we specify them particularly.

The boundary conditions at outer periphery are eight kinds. "H, T, S, U, V and W" stand for the conditions that the bending and twisting moments, the shearing force, and the displacements along r , θ and z axes are zero, respectively. (e.g. "HTS" means "free at outer periphery".)

In Table 1, the variations of α_0 due to β ($n=2, k=0.4$) are shown. The upper 4

lines are values of α_0 in the first mode and the lower 4 lines are ones in the second mode, respectively. We have placed boundary conditions in the order of the magnitudes of α_0 in the first mode, but in the vibrations of higher modes, α_0 of VHS is always smaller than that of UTS. In HTS, the classical theory gives $\alpha_0=2.122$ (first mode) and 5.303 (second mode) and these values are obtained from a rigorous solution by interpolation. It seems that the values in the column of HTS come close to the classical ones if β becomes large. In general, the following tendencies appear between α_0 and β , and the same is true for other values of n and k . As β increases, α_0 becomes large, and the variation of α_0 due to β is large in the higher modes. The frequencies in the case that the shearing force is zero, are always smaller than those for $w=0$. In the vibrations of the second and the higher modes, generally, there are 4 groups of frequencies, i.e., (HTS, VHS), (UTS, UVS), (WHT, VWH), (UWT, UVW), and as the mode is higher, the frequencies in parentheses become nearly equal (and the differences among these groups become close with each other).

In Fig.(1), the relation between α_0 and (number of nodal diameters) is shown, where $\beta=100$ and $k=0.03$ (nearly uniform thickness) ~ 1 . The relation $\alpha_0 \sim n$ of a circular plate with free and built-in peripheries obtained from the improved theory is shown by solid and dotted lines, respectively. Broken

Table 1 Nondimensional frequency α_0 ($n=2, k=0.4$)

β	HTS	UTS	VHS	UVS	WHT	VWH	UWT	UVW
20	1.91	2.24	2.16	2.41	3.26	3.30	3.36	3.38
80	2.048 0	2.481 6	2.604 7	2.825 7	3.909 4	3.933 9	4.226 1	4.238 2
100	2.060 6	2.505 0	2.677 9	2.883 7	3.988 3	4.011 6	4.348 2	4.358 1
130	2.072 9	2.528 0	2.764 4	2.949 8	4.070 3	4.092 0	4.479 6	4.487 3
160	2.080 8	2.543 1	2.833 2	3.001 0	4.127 0	4.147 5	4.573 5	4.579 7
200	2.088 0	2.556 8	2.907 5	3.055 4	4.180 4	4.199 5	4.664 2	4.669 0
20	3.66	3.93	3.66	3.93	4.43	4.43	4.55	4.55
80	4.510 7	4.908 0	4.522 2	4.924 6	5.774 5	5.784 8	5.920 0	5.929 0
100	4.618 5	5.040 6	4.634 4	5.060 5	5.964 8	5.974 4	6.146 3	6.153 9
130	4.732 3	5.183 2	4.754 9	5.207 5	6.175 2	6.184 5	6.402 9	6.409 2
160	4.812 3	5.285 3	4.841 6	5.313 6	6.330 1	6.339 1	6.596 5	6.602 0
200	4.888 8	5.383 9	4.926 7	5.417 0	6.484 0	6.492 6	6.793 1	6.797 8

Fig.1 α_0 vs number of nodal diameters, n

lines are for α of a uniform plate with free periphery obtained from the classical theory. The value of α_0 increases with n and α_0 becomes large as k becomes small. The values of α_0 for $k=1$ obtained from the classical theory are also indicated by marks $\circ$ (free) and $\triangle$ (built-in). From positions of these marks and broken lines, it is obvious that the difference of α_0 between the classical and improved theories becomes large as n or mode number increases when $\beta=100$.

In Fig.2, the relation between α_0 and k (rate of variation of thickness along the radius) is shown, when $n=4$ and $\beta=100$. As mentioned in Fig.1, α_0 decreases with an increasing k . To avoid complexity, α_0 for only 4 boundary conditions is shown and other 4 curves come among these. The tendency

Fig.2 α_0 vs thickness parameter, k

that curves come down as k increases is the same for other values of n and β . This tendency, though values for higher modes are omitted in figure, diminishes and curves are levelled for higher modes, and especially they are almost horizontal for small k .

As mentioned above, the shear coefficient k' is 0.822. Reissner and others have used 0.833 and for a bar with rectangular cross section Cooper has used 0.850. The effect of a large change of k' on α_0 is shown in Table 2. It is a case that $n=2$, $k=0.4$ and $\nu=0.3$ and for other values of n , k and ν , the tendency is similar. As seen from the table, α_0 is influenced little by a large change of k' in lower modes.

The variation of α_0 due to Poisson's ratio ν is shown in Table 3, when $n=2$,

Table 2 Change of α_0 due to shear coefficient, k'

β	k'	HTS, I	WHT, I	UVW, I	HTS, II	WHT, II	UVW, II	HTS, III	WHT, III	UVW, III	HTS, IV
100	0.78	2.0618	3.9914	4.3611	4.6254	5.9708	6.1593	6.4155	7.4829	7.5764	7.8233
	0.86	2.0595	3.9854	4.3554	4.6122	5.9592	6.1491	6.3942	7.4695	7.5630	7.7958
200	0.78	2.0886	4.1827	4.6714	4.8938	6.4897	6.8030	7.0246	8.3365	8.5307	8.7729
	0.86	2.0874	4.1784	4.6669	4.8842	6.4787	6.7929	7.0060	8.3206	8.5161	8.7478

Table 3 Change of α_0 due to Poisson's ratio, ν

β	ν	HTS, I	WHT, I	UVW, I	HTS, II	WHT, II	UVW, II	HTS, III	WHT, III	UVW, III	HTS, IV
100	0.25	2.0937	4.0001	4.3896	4.6384	6.0100	6.2142	6.4524	7.5529	7.6566	7.8864
	0.35	2.0254	3.9738	4.3230	4.5954	5.9139	6.0876	6.3503	7.3910	7.4741	7.7226
200	0.25	2.1213	4.1860	4.6914	4.9022	6.5173	6.8477	7.0520	8.3930	8.6022	8.8264
	0.35	2.0526	4.1730	4.6438	4.8731	6.4458	6.7421	6.9724	8.2559	8.4357	8.6851

$k=0.4$ and $k'=0.822$. In lower modes, like the effect of k' , α_0 is influenced little but it is influenced largely in higher modes. The value of k' changes with ν and so, a study of the effect of one under the condition that the other is constant, will not be rigorous, but we can have a rough estimation of such effect.

6. Conclusions

Based on Mindlin's theory, we have studied the vibrations of a circular plate of which the thickness is variable along the radius and obtained equations of vibration and boundary conditions. As an example, we solved such a case that the thickness changes exponentially. We have shown that the results obtained from this paper come close to those from the classical theory as the ratio of radius to thickness becomes large, and that the frequencies of both theories are considerably different if the ratio is not so large or the plate vibrates in higher modes.

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