532.135:532.542

Paper No. 187-14

Flow of a Viscoelastic Fluid in an Inlet Region*

(Numerical Solutions and Discussion)

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The behaviour of the Maxwell fluid in the inlet region of a channel composed of flat plates is revealed by solving the governing equations numerically. Two cases are studied, one with the boundary condition of a uniform flow far upstream of the entrance and the other with the condition of a uniform flow at the entrance.

It is shown that a negative vorticity and a distinct flow pattern appear near the leading edge of the plate caused by a steep gradient of a normal stress for the former case. The numerical results also indicate that the profiles of kinematic variables in cross sections have inflection points and extreme values, that the region where the internal energy of fluid elements decreases exists near the leading edge and that the vorticity distribution along the wall has extreme values.

1. Introduction

It is well known that one of the flow fields where the behaviour of viscoelastic fluids is markedly different from that of Newtonian fluids is a field where the rate of deformation and stress of fluid particles change rapidly. It is very difficult to analyse such complex flows theoretically even if an appropriate constitutive equation can be assumed. Therefore, if a numerical analysis can yield the characteristic patterns of velocity and stress fields, it will give effective information for theoretical and experimental investigations. This study deals with the flow of a viscoelastic fluid in an inlet region using a numerical method. The Maxwell model is adopted as the constitutive equation viewing it as one of the simplest models which exhibit characteristic behaviour of viscoelastic fluids in flow fields where the rate of deformation changes rapidly as well as in elongational flow fields.

For such flow fields as considered here, a constitutive equation of a differential type should include the term of derivative of stress tensor and the stress tensor cannot be substituted directly into the equation of motion. The constitutive equation, therefore, cannot be removed by substitution but it remains to be solved simultaneously with other governing equations. As for the existing numerical analysis under such a circumstance, Gatski and Lumley⁽¹⁾⁽²⁾ dealt with a flow of the Oldroyd-type model in a channel with a contraction whose boundaries were rectangu-

lar hyperbolae. They considered the flow intending to contribute to an understanding of the Toms effect and obtained information on the extension and alignment of macromolecules.

Newtonian flows in inlet regions have long been studied mostly by the boundary layer theory and rather recently some attempts were made to get directly numerical solutions of the full Navier-Stokes equations.^{(3) (4)} Wang and Longwell⁽³⁾ obtained numerical solutions of a two-dimensional flow for two cases: one with the boundary condition at the entrance and the other with the condition far upstream of the entrance, and discussed the applicability of boundary layer approximation. The present study is also concerned with these two cases.

As for the inlet flows of viscoelastic fluids, there exist studies by Metzner and White⁽⁵⁾ and by Kobayashi and Tomita.⁽⁶⁾ These papers begin with the analysis using the boundary layer approximation and the constitutive relations which are valid where the rate of deformation does not change rapidly. For the region where the effect of elasticity is pronounced, Metzner and White⁽⁵⁾ expected the appearance of solid-like behaviour and Kobayashi and Tomita⁽⁶⁾ tried an analysis by introducing an approximate constitutive relation.

Although this study deals with inlet flows of a viscoelastic fluid, its main purpose is not to reveal quantitatively the entire flow field of real fluids but to show the characteristics of the behaviour of viscoelastic fluids in such fields in view of the restrictive nature of the fluid model employed.

2. Nomenclature

The symbols other than listed here will be described when they appear.

e_{xx} , e_{xy} , e_{yy} : physical components of rate of deformation tensor
 $= ae_{xx}^*/U$, ae_{xy}^*/U , ae_{yy}^*/U

* Received 10th April, 1979.

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p : pressure = $p^*/(1/2)\rho U^2$
 Re : Reynolds number = $2\rho Ua/\mu$
 T_{xx}, T_{xy}, T_{yy} : physical components of deviatoric stress tensor
 $= T_{xx}^*/(1/2)\rho U^2, T_{xy}^*/(1/2)\rho U^2, T_{yy}^*/(1/2)\rho U^2$
 u, v : x, y components of velocity = $u^*/U, v^*/U$
 W : Weissenberg number = $\tau U/2a$
 x, y : coordinates = $x^*/a, y^*/a$
 ζ : vorticity = $a\zeta^*/U$
 ψ : stream function = ψ^*/aU
 The above-listed symbols are dimensionless and the symbols with * are the corresponding dimensional ones.
 a : half of distance between two plates
 e^{ij} : rate of deformation tensor
 T^{ij} : deviatoric stress tensor
 t : time
 U : velocity of uniform flow
 v^i : velocity vector
 μ : viscosity
 ρ : density
 τ : relaxation time of fluid
 $,$: covariant differentiation

3. Governing equations and boundary conditions

The present study deals with two cases of inlet flows shown in Fig.1: (Case 1) the flow through a row of an infinite number of flat plates of semi-infinite length separated from one another by distances $2a$ where the velocity distribution infinitely upstream is uniform and (Case 2) the flow through a channel composed of two flat plates of semi-infinite length separated by a distance $2a$ where the velocity distribution at the entrance is uniform. The flow is assumed to be steady and incompressible. Only the region: $0 \leq y^* \leq a$

Fig.1 Flow field and coordinate system

is considered for both cases due to the symmetry of the flow fields.

The convective Maxwell model with contravariant components of tensors:

$$\left(1 + \tau \frac{\partial}{\partial t}\right) T^{ij} = 2\mu e^{ij} \quad (1)$$

is adopted as the constitutive equation of a viscoelastic fluid, where $\delta T^{ij}/\delta t$ is the convective derivative of T^{ij} given by

$$\frac{\delta T^{ij}}{\delta t} = \frac{\partial T^{ij}}{\partial t} + v^k T^{ij}_{,k} - v^i_{,k} T^{kj} - v^j_{,k} T^{ik}.$$

The Cartesian coordinate system shown in Fig.1 is employed. Nondimensionalizing each variable by U, a , etc., introducing the stream function ψ by

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (2)$$

and defining the vorticity ζ by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (3)$$

we write the governing equations, i.e., the equation of motion, the equation for the definition of vorticity and the constitutive equations as follows:

$$2\left(\frac{\partial \psi}{\partial x} \frac{\partial \zeta}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \zeta}{\partial x}\right) = \frac{\partial^2 T_{xy}}{\partial y^2} - \frac{\partial^2 T_{xy}}{\partial x^2} + \frac{\partial^2}{\partial x \partial y} (T_{xx} - T_{yy}) \quad (4)$$

$$\zeta = -\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^2 \partial x}\right) \quad (5)$$

$$T_{xx} + 2W\left(\frac{\partial \psi}{\partial y} \frac{\partial T_{xx}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T_{xx}}{\partial y} - 2T_{xx} \frac{\partial^2 \psi}{\partial x \partial y} - 2T_{xy} \frac{\partial^2 \psi}{\partial y^2}\right) = \frac{8}{Re} \frac{\partial^2 \psi}{\partial x \partial y} \quad (6)$$

$$T_{yy} + 2W\left(\frac{\partial \psi}{\partial y} \frac{\partial T_{yy}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T_{yy}}{\partial y} + 2T_{yy} \frac{\partial^2 \psi}{\partial x \partial y} + 2T_{xy} \frac{\partial^2 \psi}{\partial x^2}\right) = -\frac{8}{Re} \frac{\partial^2 \psi}{\partial x \partial y} \quad (7)$$

$$T_{xy} + 2W\left(\frac{\partial \psi}{\partial y} \frac{\partial T_{xy}}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T_{xy}}{\partial y} + T_{xx} \frac{\partial^2 \psi}{\partial x^2} - T_{yy} \frac{\partial^2 \psi}{\partial y^2}\right) = \frac{4}{Re} \left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right) \quad (8)$$

These five equations constitute a system to be solved for five unknowns, i.e., the stream function ψ , the vorticity ζ and the deviatoric stress components T_{xx}, T_{yy}, T_{xy} .

We next give the boundary conditions. Infinitely upstream for Case 1 and at the entrance for Case 2, ψ is determined by the uniformity of u . Furthermore, the requirement is adopted that the vorticity and the deviatoric stress components be zero there. Thus we have there

$$\psi = y, \quad \zeta = T_{xx} = T_{yy} = T_{xy} = 0 \quad (9)$$

On the wall, from $u=v=0$ we have

$$\psi = 1, \quad \zeta = -\frac{\partial^2 \psi}{\partial y^2}, \quad T_{xx} = 16 \frac{W}{Re} \zeta^2, \quad T_{yy} = 0, \quad T_{xy} = -\frac{4}{Re} \zeta \quad (10)$$

Infinitely downstream, assuming each variable depends only on y we have

$$\psi = \frac{3}{2}y - \frac{1}{2}y^3, \quad \zeta = 3y, \quad T_{xx} = 144 \frac{W}{Re} y^2, \quad T_{yy} = 0, \quad T_{xy} = -\frac{12}{Re} y \quad (11)$$

On the center line ($y=0$), the following equations hold owing to the symmetry:

$$\left. \begin{aligned} \phi = \zeta = T_{xy} = 0, \quad T_{xx} + 2W \left(\frac{\partial \phi}{\partial y} \frac{\partial T_{xx}}{\partial x} - 2T_{xx} \frac{\partial^2 \phi}{\partial x \partial y} \right) &= \frac{8}{R_e} \frac{\partial^2 \phi}{\partial x \partial y} \\ T_{yy} + 2W \left(\frac{\partial \phi}{\partial y} \frac{\partial T_{yy}}{\partial x} + 2T_{yy} \frac{\partial^2 \phi}{\partial x \partial y} \right) &= -\frac{8}{R_e} \frac{\partial^2 \phi}{\partial x \partial y} \end{aligned} \right\} \dots\dots\dots (12)$$

On the streamline upstream of the leading edge of the plate ($y=1, x<0$) for Case 1, Eqs.(12) also hold except that $\psi=0$ is replaced by $\psi=1$. At the leading edge, Eqs.(10) are applied for Case 1 and Eqs.(9) for Case 2.

For the convenience of numerical calculation, α and β are introduced by

$$\left. \begin{aligned} \alpha &= \frac{16}{R_e} \frac{\partial^2 \phi}{\partial x \partial y} - (T_{xx} - T_{yy}) \\ \beta &= \frac{4}{R_e} \left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial x^2} \right) - T_{xy} \end{aligned} \right\} \dots\dots\dots (13)$$

and with them Eq.(4) is rewritten as

$$\begin{aligned} 2 \left(\frac{\partial \phi}{\partial x} \frac{\partial \zeta}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial \zeta}{\partial x} \right) \\ = -\frac{4}{R_e} \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) - \frac{\partial^2 \alpha}{\partial x \partial y} + \frac{\partial^2 \beta}{\partial x^2} - \frac{\partial^2 \beta}{\partial y^2} \end{aligned} \dots\dots\dots (14)$$

Then boundary conditions for α and β are added to the above conditions, that is, $\alpha=\beta=0$ to Eqs.(9), $\alpha=-T_{xx}$ and $\beta=0$ to Eqs.(10) and (11) and

$$\alpha = \frac{16}{R_e} \frac{\partial^2 \phi}{\partial x \partial y} - (T_{xx} - T_{yy}), \quad \beta = 0$$

to Eqs.(12). The transformation:

$$\xi = 1 - e^{-cx} \dots\dots\dots (15)$$

is also introduced for the convenience of numerical calculation. Thus, employing a grid of equal spacing on the ξ - y plane, we can obtain on the x - y plane a grid which is finer in the neighborhood of $x=0$ where gradients of variables are steep. The region: $0 \leq x \leq \infty, 0 \leq y \leq 1$ is transformed into the region: $0 \leq \xi, y \leq 1$ by assigning a positive value to c . The upstream region: $-\infty \leq x \leq 0, 0 \leq y \leq 1$ for Case 1 is also transformed into the region: $0 \leq \xi, y \leq 1$ by assigning to c a negative value whose absolute value is equal to the value for the downstream region. We can obtain final equations by transforming Eqs.(14), (5)~(8), (13) and the boundary conditions by means of Eq.(15). (Since the replacements of $\partial/\partial x$ by $c(1-\xi)\partial/\partial \xi$ and of $\partial^2/\partial x^2$ by

Fig.2 Grid

$c^2(1-\xi)^2(\partial^2/\partial \xi^2) - c^2(1-\xi)\partial/\partial \xi$ easily yield the final equations, they are not described here.)

4. Method of numerical solution

Since the present problem involves many unknowns and needs relatively much computation for solution, the numerical method employed should be one that can give rather rapid convergence. Thus we have decided to use the so-called up-wind method.⁽⁷⁾ The up-wind difference is applied to the material derivative term in the constitutive equation as well as to the convective term in the equation of motion. Figure 2 shows the grid used for the finite difference scheme, where the region: $0 \leq \xi \leq 1, 0 \leq y \leq 1$ is divided to yield $m \times m$ mesh points. Expressing the differential equations in the preceding section in finite difference form, we obtain the expressions for iterative calculation as follows.

$$\phi_{ij} = A_1 \phi_{i+1,j} + A_2 \phi_{i-1,j} + A_3 \phi_{i,j+1} + A_4 \phi_{i,j-1} + A_5 \dots\dots\dots (16)$$

where

$$A_1 = \left(f_2 - \frac{1}{2} h f_3 \right) / A, \quad A_2 = \left(f_2 + \frac{1}{2} h f_3 \right) / A, \quad A_3 = A_4 = 1/A, \quad A_5 = h^2 \zeta_{ij} / A, \quad A = 2(f_2 + 1)$$

and f 's are given by

$$f_1 = c(1-\xi_i), \quad f_2 = c^2(1-\xi_i)^2, \quad f_3 = c^2(1-\xi_i).$$

For the vorticity and the deviatoric stress components, the expressions are shown only for the case: $\psi_{i+1,j} - \psi_{i-1,j} \geq 0, \psi_{i,j+1} - \psi_{i,j-1} \geq 0$ and $c > 0$, although the expressions are different for other cases. That is,

$$\zeta_{ij} = B_1 \zeta_{i+1,j} + B_2 \zeta_{i-1,j} + B_3 \zeta_{i,j+1} + B_4 \zeta_{i,j-1} + B_5 \dots\dots\dots (17)$$

where

$$B_1 = (4f_2 - 2hf_3)/B, \quad B_2 = (R_e f_1 A_2 + 4f_2 + 2hf_3)/B, \quad B_3 = (R_e f_1 A_1 + 4)/B, \quad B_4 = 4/B$$

$$B_3 = R_e \left\{ \frac{1}{4} f_1 (\alpha_{i+1,j+1} - \alpha_{i+1,j-1} - \alpha_{i-1,j+1} + \alpha_{i-1,j-1}) - f_2 (\beta_{i+1,j} - 2\beta_{ij} + \beta_{i-1,j}) \right. \\ \left. + \frac{1}{2} f_3 h (\beta_{i+1,j} - \beta_{i-1,j}) + (\beta_{i,j+1} - 2\beta_{ij} + \beta_{i,j-1}) \right\} / B \\ B = 8(f_2 + 1) + R_e f_1 (\Delta_1 + \Delta_2), \\ T_{xxij} = D_1 T_{xxi+1,j} + D_2 T_{xxi-1,j} + D_3 T_{xxi,j+1} + D_4 T_{xxi,j-1} + D_5 \quad (18)$$

where

$$D_1 = 0, \quad D_2 = W f_1 \Delta_2 / D, \quad D_3 = W f_1 \Delta_1 / D, \quad D_4 = 0 \\ D_5 = \left(4W \Delta_1 T_{xyij} + \frac{2}{R_e} f_1 \Delta_5 \right) / D, \quad D = h^2 + W f_1 (\Delta_1 + \Delta_2 - \Delta_5), \\ T_{yyij} = E_1 T_{yyi+1,j} + E_2 T_{yyi-1,j} + E_3 T_{yyi,j+1} + E_4 T_{yyi,j-1} + E_5 \quad (19)$$

where

$$E_1 = 0, \quad E_2 = W f_1 \Delta_2 / E, \quad E_3 = W f_1 \Delta_1 / E, \quad E_4 = 0 \\ E_5 = \left\{ -2W (2f_2 \Delta_3 - h f_3 \Delta_1) T_{xyij} - \frac{2}{R_e} f_1 \Delta_5 \right\} / E, \quad E = h^2 + W f_1 (\Delta_1 + \Delta_2 + \Delta_5), \\ T_{xyij} = F_1 T_{xyi+1,j} + F_2 T_{xyi-1,j} + F_3 T_{xyi,j+1} + F_4 T_{xyi,j-1} + F_5 \quad (20)$$

where

$$F_1 = 0, \quad F_2 = W f_1 \Delta_2 / F, \quad F_3 = W f_1 \Delta_1 / F, \quad F_4 = 0 \\ F_5 = \left[-W \{ -2\Delta_4 T_{xyij} + (2f_2 \Delta_3 - h f_3 \Delta_1) T_{xxij} \} + \frac{2}{R_e} (2\Delta_4 - 2f_2 \Delta_3 + h f_3 \Delta_1) \right] / F, \quad F = h^2 + W f_1 (\Delta_1 + \Delta_2).$$

In these expressions,

$$\Delta_1 = \phi_{i+1,j} - \phi_{i-1,j}, \quad \Delta_2 = \phi_{i,j+1} - \phi_{i,j-1}, \quad \Delta_3 = \phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j} \\ \Delta_4 = \phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}, \quad \Delta_5 = \phi_{i+1,j+1} - \phi_{i+1,j-1} - \phi_{i-1,j+1} + \phi_{i-1,j-1}.$$

In addition, the finite difference expressions for α and β become

$$\alpha_{ij} = \frac{4f_1 \Delta_5}{R_e h^2} - (T_{xxij} - T_{yyij}), \quad \beta_{ij} = \frac{4}{R_e} \left(\frac{\Delta_4}{h^2} - \frac{f_2 \Delta_3}{h^2} + \frac{f_3 \Delta_1}{2h} \right) - T_{xyij} \quad (21)$$

Since the expressions for boundary conditions can be derived straightforward by applying the conditions in Section 3 at $i=1, m; j=1, m$, only a few points are mentioned. The second expression in Eqs.(10) for the vorticity is replaced by the usual expression:

$$\zeta_{im} = \frac{2}{h^2} (1 - \phi_{i,m-1}), \quad (22)$$

T_{xx} , T_{yy} and α on the symmetry lines are expressed in terms of the values at interior points of the region owing to the symmetry of the flow. At $x=0$ for Case 1 we use the finite difference expressions derived from the differential equations in x, y coordinates with the spacings $-(1/c)\ln(1-h)$ and h in the directions of x and y respectively. In the procedure of iterative calculation the relaxation factors A_ψ and A_ζ are used for the stream function and the vorticity respectively. That is,

$$\psi_{ij}^l = \psi_{ij}^{l-1} + A_\psi (\psi_{ij}^{l*} - \psi_{ij}^{l-1}) \\ \zeta_{ij}^l = \zeta_{ij}^{l-1} + A_\zeta (\zeta_{ij}^{l*} - \zeta_{ij}^{l-1}) \quad (23)$$

where the superscript denotes the step number of iteration and * indicates the value calculated from the expression for iteration. The results in this paper have been obtained by applying $A_\psi=1.0$, $0 < A_\zeta < 1.0$.

The calculation is advanced from the upstream side to the downstream side and in this procedure the latest values available are used at neighboring points. ψ_{ij} is first calculated over the whole region, next ζ_{ij} , T_{xxij} , T_{yyij} and T_{xyij} are similarly treated in order and lastly α_{ij} and β_{ij} are calculated. Then, for each variable (written representatively as ϕ_{ij}), ϵ_ϕ^l and $\bar{\epsilon}_\phi^l$ (defined as the maximum values of $|\phi_{ij}^l - \phi_{ij}^{l-1}| / \phi_{ij}^{l-1}$ and $|\phi_{ij}^l - \phi_{ij}^{l-1}|$ over the whole region respectively) are calculated for the judgement of convergence. This procedure is repeated until the calculation is judged to have converged. When the convergence is reached, we calculate the velocity, the pressure and the rate of deformation components using the converged values. Table 1 shows the conditions for the present calculation and the values of ϵ_ϕ and $\bar{\epsilon}_\phi$ at the time when the calculation

Table 1 Values of flow and computational parameters and values of ϵ and $\bar{\epsilon}$

Case	W	ϵ_ψ	ϵ_ζ	ϵT_{xx}	ϵT_{yy}	ϵT_{xy}	$\bar{\epsilon}_\psi$	$\bar{\epsilon}_\zeta$	$\bar{\epsilon} T_{xx}$	$\bar{\epsilon} T_{yy}$	$\bar{\epsilon} T_{xy}$	notes
1	0	1.5×10^{-7}	6.8×10^{-6}									$\zeta = 0$ at the leading edge $c = 0.65$ $c = 0.3$
	0	6.2×10^{-8}	8.8×10^{-6}									
	0	6.2×10^{-8}	1.5×10^{-5}									
	0	6.3×10^{-8}	1.6×10^{-5}									
	0.1	6.1×10^{-8}	2.4×10^{-5}	6.9×10^{-5}	2.3×10^{-4}	5.3×10^{-5}	6.0×10^{-8}	8.5×10^{-5}	9.9×10^{-6}	9.4×10^{-7}	5.3×10^{-6}	
2	0	6.2×10^{-8}	1.4×10^{-5}									
	0.001	1.3×10^{-7}	2.0×10^{-5}	4.0×10^{-4}	9.9×10^{-5}	4.4×10^{-5}						
	0.01	1.3×10^{-7}	1.5×10^{-5}	3.0×10^{-5}	3.9×10^{-4}	1.6×10^{-5}			1.5×10^{-6}	2.4×10^{-7}	1.4×10^{-6}	
	0.05	6.8×10^{-8}	1.4×10^{-5}	8.2×10^{-5}	6.5×10^{-5}	3.3×10^{-5}	6.0×10^{-8}	4.8×10^{-5}	5.1×10^{-6}	1.5×10^{-6}	3.7×10^{-6}	
	0.1	6.0×10^{-8}	8.1×10^{-6}	6.5×10^{-6}	2.1×10^{-3}	6.1×10^{-6}	6.0×10^{-8}	1.6×10^{-5}	7.8×10^{-7}	3.0×10^{-7}	4.8×10^{-7}	
	$R_e = 150, m = 21, c = 1.0$ (except for the cases where the value is given in the notes)											

tion is judged to have converged, where ε_ϕ and $\bar{\varepsilon}_\phi$ are the maximum values of ε_ϕ and $\bar{\varepsilon}_\phi$ over the range of $\ell : n-29 \leq \ell \leq n$ respectively (n : the total number of iteration).

5. Results and discussion for a Newtonian fluid

Profiles of the velocity component u and lines of constant vorticity are shown for Case 1 in Figs.3 and 4 respectively and for Case 2 in Figs.5 and 6 respectively. Table 2 gives the location of cross sections in figures of velocity profiles. These results essentially correspond to those by Wang and Longwell.⁽³⁾ It is seen for Case 1 that the velocity component near $y=1$ becomes smaller as the entrance is approached in the upstream region and a "boundary layer" of a finite thickness appears at the entrance. For Case 2, it is noted that the region extends significantly where the maximum of u in a cross section occurs near the wall and where velocity profiles are concave and have an inflection point. As for the vorticity, the diffusion in the upstream direction is appreciable for Case 1 and there exists a region for Case 2 where the maximum in a cross section does not occur on the wall.

Figure 7 shows distributions of u in the direction of x for Case 1 for several computational conditions together with the values by Wang and Longwell.⁽³⁾ Since the region of higher accuracy shifts downstream as c decreases, for $x \geq 2$ the results for $c=0.65, 0.3$ which are seen to agree well with those by Wang and Longwell⁽³⁾ are considered to be more accurate. The principal calculations,

Fig.3 Velocity profiles (Newtonian, Case 1)

Fig.4 Lines of constant vorticity (Newtonian, Case 1)

Fig.5 Velocity profiles (Newtonian, Case 2)

Fig.6 Lines of constant vorticity (Newtonian, Case 2)

Fig.7 Comparison of velocity distributions for several computational conditions (Newtonian, Case 1)

however, employ the value: $c=1$ because this study is mainly concerned with the region near the entrance. (All the results except in Fig.7 have been obtained for $c=1$.) The figure shows also that some difference

Table 2 Values of $x_{-5} \sim x_{10}$

x_{-5}	-0.693 1	x_{-1}	-0.105 4	x_3	0.356 7	x_7	1.204 0
x_{-4}	-0.510 8	x_0	0	x_4	0.510 8	x_8	1.609 4
x_{-3}	-0.356 7	x_1	0.105 4	x_5	0.693 1	x_9	2.302 6
x_{-2}	-0.223 1	x_2	0.223 1	x_6	0.916 3	x_{10}	∞

Fig.8 Lines of constant value of deviatoric normal stress T_{xx} (Newtonian, Case 1)

Fig.9 Lines of constant value of shear stress T_{xy} (Newtonian, Case 1)

Fig.10 Velocity profiles ($W=0.1$, Case 1)

is produced according to whether Eq.(22) is applied or zero is assumed for the vorticity at the leading edge and for the former each variable shifts upstream to some extent.

Figures 8 and 9 show lines of constant value of the deviatoric stress components T_{xx} and T_{xy} respectively, which are compared with the results for the viscoelastic fluid later. They of course correspond to the components of the rate of deformation. The whole field is divided into the regions of acceleration and deceleration in u by the curve $T_{xx}=0$ which is seen to lie transversely. It is also noted that $|T_{xy}|$ is generally larger than $|T_{xx}|$ near the wall and the maximum of $|T_{xy}|$ in each cross section for $x>0$ occurs on the wall.

6. Results and discussion for a viscoelastic fluid

6.1 Case of boundary conditions far upstream (Case 1)

All the results for this case have been obtained for $W=0.1$. Figure 10 shows profiles of the velocity component u and Fig.11 lines of constant value of the rate of deformation component e_{xx} in terms of $(8/R_e)e_{xx}$. $((8/R_e)e_{xx}$ corresponds to the "stress T_{xx} " which a Newtonian fluid would exhibit with the same kinematic field.) From Fig.10 it is seen that the minimum of u in a cross section does not occur at $y=1$ in the upstream region near the entrance. As can be seen from Fig.11, in the neighborhood of the streamline upstream of the leading edge u decreases little or conversely increases a little as the leading edge is approached in contrast with the Newtonian case, which brings about the characteristic velocity profiles described above. As the entrance is approached further, rapid deceleration appears near the wall and the corresponding acceleration occurs mainly in $0.8 \leq y \leq 0.9$ (see Fig.11), so that the characteristic velocity profile appears at the entrance: $x=0$ as shown in Fig.10. This profile has inflection points,

which move away from the wall in the downstream direction. In addition, it is noted that the maximum of u occurs on the center line in the neighborhood of the entrance but in a little downstream region it is near the wall.

To clarify the aspects of flow behaviour further, the streamlines near the entrance are shown together with those for the Newtonian case in Fig.12. As can be seen, for the viscoelastic case fluid particles move nearly in parallel with the x -axis upstream of the entrance and change their paths

abruptly in the immediate vicinity of the

Fig.11 Lines of constant value of rate of deformation component e_{xx} ($W=0.1$, Case 1)

Fig.12 Streamlines

Fig.13 Lines of constant vorticity ($W=0.1$, Case 1)

Fig.14 Lines of constant value of deviatoric normal stress T_{xx} ($W=0.1$, Case 1)

entrance in contrast with the Newtonian curves which are rather slowly varying.

Figure 13 shows lines of constant vorticity. A remarkable difference from the Newtonian case is that the negative vorticity whose absolute value is fairly large arises in the neighborhood of a little upstream region of the leading edge. Although the magnitude level of vorticity is not very different from that for a Newtonian fluid for $x > 0$, in the region near the entrance wall the distribution of vorticity is not monotonic in cross sections and the maximum does not occur on the wall. (Upstream of the broken line $\zeta=0$ in Fig.13, the regions of positive or negative vorticity whose absolute value was very small were recognized, which may be due to a residual computational error. The maximum value of $|\zeta|$ in this region was 8.1×10^{-4} .)

The distributions of kinematic variables have been seen so far. Among the shown aspects the appearance of negative vorticity near the leading edge is especially remarkable, which corresponds to the characteristic velocity profiles and streamlines near the leading edge.

We next show distributions of the stresses and examine their relation to the kinematic variables. Figures 14 and 15 show lines of constant T_{xx} and T_{xy} respectively. As is expected from the constitutive equation (6) and the boundary condition on the wall: $T_{xx}=16(W/Re)\zeta^2$ derived from this equation, the distribution of T_{xx} near the wall is seen to correspond qualitatively not to that of e_{xx} but rather to those of e_{xy} and ζ . On the other hand, the distribution of T_{xx} near the center line is analogous to that of e_{xx} although the former shifts downstream to some extent. Thus, downstream of the entrance there exist no regions where T_{xx} is negative. The distribution of T_{xx} shown is most different from the corresponding Newtonian distribution among the distributions of three components of the deviatoric stress. It should be noted that a region where the gradient of T_{xx} is steep appears in the vicinity of the leading edge due to the above-mentioned boundary conditions at the wall. The appearance of negative vorticity can be interpreted in terms of this steep gradient of T_{xx} . That is, such a steep gradient of T_{xx} must give a large positive value to the term $(\partial^2/\partial x \partial y)T_{xx}$ in Eq.(4) and consequently yield a negative $D\zeta/Dt$ whose absolute value is large. As for T_{xy} , it is seen from Fig.15 that the distribution is non-monotonic near $y=1$ just like that of vorticity. The magnitude level of T_{xy} is essentially the same as that for the Newtonian case and the region where the magnitude level of T_{xx} is higher than that of T_{xy} extends significantly near the wall unlike the Newtonian case.

For a Newtonian fluid, $T_{xx}e_{xx} + T_{yy}e_{yy} + 2T_{xy}e_{xy}$ is a (dimensionless) dissipation function which is of course always positive. For the present viscoelastic flow, it becomes negative near the leading edge due to the negative $T_{xx}e_{xx}$ whose absolute value is very large (see Figs.11 and 14). This means a decrease in the internal energy of fluid

Fig.15 Lines of constant value of shear stress T_{xy} ($W=0.1$, Case 1)

Fig.16 Deviatoric stress distribution along wall (Case 1)

Fig.17 Distributions of deviatoric stress and rate of deformation along center line (Case 1)

elements in that region. Accordingly it is suggested that when fluid elements pass near the leading edge a rapid storage and release of elastic energy occur in them although this is not clear from the figures.

Figure 16 shows distributions of the deviatoric stress on the wall together with the distribution for a Newtonian fluid. It should be noted that both the distributions of T_{xx} and $-T_{xy}$ for the viscoelastic fluid

have extreme values in contrast with the monotonic distribution of T_{xy} for a Newtonian fluid. Furthermore, it is inferred from the figure that the value of $|T_{xy}|$ in the immediate vicinity of the leading edge for the viscoelastic fluid is larger than that for a Newtonian fluid. Hermes and Fredrickson⁽⁸⁾ measured the drag on a flat plate in a uniform flow of a viscoelastic liquid and found a very large drag. Although the present calculation does not treat the same flow condition as that of their measurement, the behaviours in the vicinity of the leading edge are considered to correspond locally. Thus, it can be suggested that the large drag found by them is due to such a large shear stress in the vicinity of the leading edge as described above.

Figure 17 shows distributions of the deviatoric stress and the rate of deformation on the center line together with the Newtonian T_{xx} distribution. The typical aspect that the distribution of the stress shifts downstream from the corresponding distribution of the rate of deformation is seen to appear for the viscoelastic fluid. The extreme values of T_{xx} and T_{yy} are derived from Eq.(12) as $8e_{xx}/Re(1-4We_{xx})$ and $8e_{yy}/Re(1-4We_{yy})$ respectively, which agree well with the values found in the figure.

Figure 18 shows distributions of the pressure. Since the value near the wall or near the leading edge where gradients of variables are steep is not expected to be very accurate, the results for $y=1$ should be considered as semi-quantitative. It is seen that near the center line the pressure drop for the viscoelastic fluid is smaller than that for a Newtonian fluid, but near the wall the trend is inferred to be converse. It is suggested that non-uniformity of the pressure in cross sections in the channel is significant near the entrance for the viscoelastic fluid. As is expected from the values of velocity near the center line in Figs.3 and 10, the pressure drop for the viscoelastic fluid approaches rapidly to that for a Newtonian fluid in the downstream direction.

6.2 Case of boundary conditions at the entrance (Case 2)

The results for $W=0.1$ are mainly shown also for this case and only the

7. Concluding remarks

Characteristic aspects of the behaviour of the Maxwell fluid in inlet regions have been revealed with numerical solutions of the governing equations. Two cases have been studied: one with boundary conditions far upstream of the entrance (Case 1) and the other with those at the entrance (Case 2). It has been shown for Case 1 that a negative vorticity and a distinct flow pattern appear near the leading edge caused by a steep gradient in the distribution of a deviatoric normal stress T_{xx} . The numerical results have also indicated that profiles of kinematic variables in cross sections involve inflection points and extreme values for both cases. Furthermore, the results have shown that there exists a region where internal energy of fluid elements decreases and that the vorticity distribution along the wall has extreme values.

Although the numerical accuracy can be improved further since an up-wind difference scheme has been employed and a finer grid would be desirable, we can consider that characteristic behaviours of a viscoelastic fluid in inlet regions have been

revealed adequately.

Finally it is acknowledged that numerical computations in this study were carried out on HITAC 8800/8700 System at the Computer Center of the University of Tokyo.

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