Transient Responses of a Cantilever Cylindrical Shell
Subjected to Impulsive Loads on Its Free Edge

: The Influences of Rotatory Inertia
and Transverse Shear Deformation *

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In this paper, the transient responses of a cantilever circular cylindrical shell subjected to impulsive loads on its free edge are analyzed on the basis of an improved dynamic shell theory, namely, Mirsky-Herrmann's theory, considering the influences of the rotatory inertia and the transverse shear deformation by the use of Laplace transformation. The numerical results are compared with those previously obtained by the author via Flügge's classical dynamic shell theory. It is shown that the results predicted by the classical and the improved dynamic shell theories are in good agreement about the displacement components and the in-plane stress resultants, but are considerably different from each other about the out-of-plane stress resultants, namely, the bending moments and the transverse shearing forces.

1. Introduction

In dealing with the dynamic boundary-value problems of circular cylindrical shells, there exist various shell theories which differ in the degree of approximation and in consideration of the in-plane inertia, the rotatory inertia and the transverse shear deformation in the constitutive relations. The accuracy of the shell theories, therefore, has been examined mainly by the studies on the natural vibrations or the propagation of elastic waves [1]-[6]. Studies on the transient and forced responses of the circular cylindrical shells under some specific edge conditions have been made in detail by Reissmann, et al. by the use of Flügge's classical and Mirsky-Herrmann's improved dynamic shell theories in comparison with the three-dimensional dynamic theory of elasticity.

These studies are conducted to examine the influences of the rotatory inertia and the transverse shear deformation on the transient behavior of cylindrical shells. Then, it is found that the improved shell theory yields quite accurate results compared to the exact three-dimensional theory [7]-[10]. Huang also obtained the same conclusions as those predicted by Reissmann about the influences of the rotatory inertia and the transverse shear deformation as the result of an analysis of the transient responses of a simply-supported shell subjected to longitudinal tensile step forces at both ends [11].

On the other hand, Liao investigated the dynamic responses of a simply-supported cylindrical shell under a concentrated load by the use of two classical shell theories considering the influences of the axial, circumferential and radial inertia and showed the relative errors between Donnell's theory and Flügge's theory [12]. More recently, Shirakawa also presented an analytical study on a simply-supported cylindrical shell subjected to concentrated impulsive loads by the use of a classical theory and elucidated the influences of the in-plane inertia in the axial and the circumferential directions [13].

These analyses published hitherto, however, being restricted to the problems on the cases of axially-symmetric loads or locally distributed loads, the dynamic problems of a thin-walled beam under impulsive bending loads remain unsolved.

Previously, the authors analyzed the dynamic responses of a cantilever circular cylindrical shell suddenly loaded by shearing forces on its free edge by the use of Donnell’s and Flügge's dynamic shell theories [14]. In this paper, the same problem is analyzed by using a Mirsky-Herrmann type theory considering the influences of the rotatory inertia and the transverse shear deformation. Then, the differences in the constitutive relations are revealed by comparing the results of this theory with those of the classical theory previously given. Meanwhile, the errors of the elementary beam theory applied to the bending problems those of the shell theory are pointed out by comparing them.

2. Basic Equations

A circular cylindrical shell of thickness \( t \), mean radius \( R \), Young's modulus \( E \), shear modulus \( G \), Poisson's ratio \( \nu \), and weight per unit volume \( \gamma \), is considered. The coordinate system, \((x, \theta, z)\), referred to the mid-surface of the shell and the components of the
stress resultants are shown in Fig. 1.

The components of displacement on the mid-surface along this coordinate system are designated by \((u,v,w)\) are the angles of rotation of a normal to the mid-surface in \((x,z)\) and \((\theta,z)\) planes are denoted by \((\psi_x, \psi_\theta)\). Then, the basic equations employed in this paper may be given in the following form.

(I) Mîrsky-Herrmann Type Shell Theory \([2]\)

The basic equations of motion derived by Mîrsky and Herrmann, considering the influences of the rotatory inertia and the transverse shear deformation, are written in the form of a set of five coupled differential equations in terms of \((u,v,w)\) and \((\psi_x, \psi_\theta)\):

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} + (1 + \kappa^2) \rho_1 & \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \theta^2} \rho_1 \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial z^2} \rho_1 \frac{\partial^2}{\partial z^2} \left( \rho_1 \frac{\partial^2}{\partial \theta^2} - \rho_1 \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2}{\partial \theta^2} + \rho_1 \frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2} \end{align*}
\]

\[
\begin{align*}
\kappa^2 \left( \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2}{\partial \theta^2} + \kappa^2 \left( \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2}{\partial \theta^2} \rho_1 \frac{\partial^2}{\partial \theta^2} - \kappa^2 \frac{\partial^2}{\partial \theta^2} \rho_1 \frac{\partial^2}{\partial \theta^2} = \rho_1 \frac{\partial^2}{\partial \theta^2} \\
\rho_2 \frac{\partial^2}{\partial \theta^2} - \rho_3 \frac{\partial^2}{\partial \theta^2} - \rho_4 \frac{\partial^2}{\partial \theta^2} \rho_2 \frac{\partial^2}{\partial \theta^2} + \rho_3 \frac{\partial^2}{\partial \theta^2} - \rho_4 \frac{\partial^2}{\partial \theta^2} \rho_2 \frac{\partial^2}{\partial \theta^2} = \rho_2 \frac{\partial^2}{\partial \theta^2} \\
\rho_3 \frac{\partial^2}{\partial \theta^2} + \rho_4 \frac{\partial^2}{\partial \theta^2} + \rho_5 \frac{\partial^2}{\partial \theta^2} = \rho_3 \frac{\partial^2}{\partial \theta^2} \\
\rho_4 \frac{\partial^2}{\partial \theta^2} + \rho_5 \frac{\partial^2}{\partial \theta^2} = \rho_3 \frac{\partial^2}{\partial \theta^2}
\end{align*}
\]

\[\text{Fig. 1 Definitions of Shell-stress Components and Coordinate System in a Circular Cylindrical Shell}\]

(II) Flügge Type Shell Theory \([15]\)

The basic equations of motion derived by Flügge are reduced to a single equation for the radial displacement from a set of the three coupled differential equations given in terms of \((u,v,w)\):

\[
\phi^8 \frac{\partial^2}{\partial t^2} = \rho \frac{\partial^2}{\partial \theta^2} \frac{\partial^2}{\partial \theta^2} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdOTS
\[
L^4_3 = \frac{3v}{2\pi^2} \frac{3^3}{1-v^2} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \\
L_1 = \frac{2v}{\pi^3} \frac{3}{1-v^3} \frac{3^3}{3^3} k^2 \left( \frac{2}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \right) \\
L_2 = \frac{2v}{\pi^3} k^2 \left( \frac{2}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \right)
\]

(III) Donnell Type Shell Theory [16][17]

Donnell's equations including inertia forces in the axial, circumferential and radial directions are obtained in terms of (u,v,w):

\[
\nabla^2 u + \frac{1}{\pi^2} \frac{\partial^2 u}{\partial \alpha^2} = L^a_1 u \quad \text{......... (6)}
\]

and the equations relating the three displacement components become

\[
\begin{align*}
(V^a + L^a)_{u,v} &= \frac{3v}{2\pi^2} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \\
(V^a + L^a)_{u,v} &= \frac{3v}{2\pi^2} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \\
(V^a + L^a)_{u,v} &= \frac{3v}{2\pi^2} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \frac{3}{\pi^3} \frac{3^3}{1-v^3} \\
\end{align*}
\]

(7)

3. Analysis by Means of Mirsky-Herrmann Type Theory

3.1 Solutions under Impulsive Loads

As shown in Fig.2, considering a cantilever cylindrical shell subjected to impulsive line loads in the transverse and tangential directions on its free edge whose magnitudes of \(a_{1}, a_{2} \) vary sinusoidally along the meridian, then the boundary conditions are prescribed as follows

\[
\begin{align*}
(1) \text{ at the clamped edge (x=0)} &: u = \psi = \nu = \omega = 0 \quad \text{................. (8)} \\
(2) \text{ at the free edge (x=\ell)} &: \\
\end{align*}
\]

\[
\begin{align*}
\eta_0 &= q_1 \cos (H(t)) \\
\eta_0 &= q_1 \cos (H(t)) \\
\eta_0 &= q_1 \cos (H(t)) \\
\end{align*}
\]

where \(H(t)\) is Heaviside's unit step function.

The solutions for the shell-displacement components may be sought in a separated form of variables as

\[
\begin{align*}
u/R = u \cos \theta, \quad \nu/R = v \sin \theta, \quad \omega/R = \omega \cos \theta \\
\psi/R = \psi \cos \theta, \quad \psi/R = \psi \sin \theta \\
\end{align*}
\]

(10)

Substitution of Eqs.(10) into the coupled equations of motion (1) and application of Laplace transformation under the initial conditions

\[
\begin{align*}
(f)_{t=0} = (3f/3t)_{t=0} = 0 \\
\end{align*}
\]

lead to a single uncoupled ordinary differential equation of 10th order with constant coefficients. Therefore, the general solutions of the Laplace transformations of the functions of \((u,v,w)\) and \((\nu_\alpha, \nu_\beta)\) can be assumed as

\[
\begin{align*}
\bar{u} = \sum_{j=1}^{10} \tilde{a}_j \bar{e}_j = \sum_{j=1}^{10} \tilde{a}_j \bar{e}_j \\
\bar{v} = \sum_{j=1}^{10} \tilde{a}_j \bar{e}_j = \sum_{j=1}^{10} \tilde{a}_j \bar{e}_j \\
\bar{w} = \sum_{j=1}^{10} \tilde{a}_j \bar{e}_j = \sum_{j=1}^{10} \tilde{a}_j \bar{e}_j
\end{align*}
\]

(11)

where \(\bar{f} = \int f e^{-\eta_0 t} dt\)

Substitution of Eqs.(11) into the Laplace transformations of Eqs.(1) gives

\[
\begin{align*}
\begin{bmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3 \\
\tilde{a}_4 \\
\tilde{a}_5 \\
\tilde{a}_6 \\
\tilde{a}_7 \\
\tilde{a}_8 \\
\tilde{a}_9 \\
\tilde{a}_{10}
\end{bmatrix}
&= A
\end{align*}
\]

(12)

where \([A]\) is a symmetric matrix of 5th order and \(a_{kj}\) denotes the element standing in \(k\)-th row and \(l\)-th column of \([A]\) given as follows

\[
\begin{align*}
a_{11} &= a_{21} + 2v(a_{12} - \lambda^2 \rho_1) \\
a_{12} &= a_{11} - 2(a_{12} - \lambda^2 \rho_1) \\
a_{13} &= \rho_1 a_{12} \\
a_{14} &= \rho_1 a_{12} \\
a_{15} &= \rho_1 a_{12} \\
a_{21} &= a_{11} - 2(a_{12} - \lambda^2 \rho_1) \\
a_{22} &= a_{11} - 2(a_{12} - \lambda^2 \rho_1) \\
a_{23} &= \rho_1 a_{12} \\
a_{24} &= \rho_1 a_{12} \\
a_{25} &= \rho_1 a_{12} \\
a_{31} &= \rho_1 a_{12} \\
a_{32} &= \rho_1 a_{12} \\
a_{33} &= \rho_1 a_{12} \\
a_{34} &= \rho_1 a_{12} \\
a_{35} &= \rho_1 a_{12} \\
a_{41} &= \rho_1 a_{12} \\
a_{42} &= \rho_1 a_{12} \\
a_{43} &= \rho_1 a_{12} \\
a_{44} &= \rho_1 a_{12} \\
a_{45} &= \rho_1 a_{12} \\
a_{51} &= \rho_1 a_{12} \\
a_{52} &= \rho_1 a_{12} \\
a_{53} &= \rho_1 a_{12} \\
a_{54} &= \rho_1 a_{12} \\
a_{55} &= \rho_1 a_{12}
\end{align*}
\]

(13)

Then, the determinant \([A]\) of the matrix \([A]\) leads to the following algebraic equation of 10th degree and \(a_0 (j=1, \ldots, 10)\) in Eqs. (11) are the 10 characteristic roots of the equation:

\[
a_{10} + a_{10} a_{20} + a_{40} a_{20} + a_{20} + a_{0} = 0 \quad \ldots \quad (13)
\]

The coefficients of \((a_{10}, \ldots, a_{0})\) are written as

\[
\begin{align*}
a_{10} &= X(10), \quad a_6 = \sum_{j=1}^{5} X(6) \\
a_6 &= \sum_{j=1}^{5} X(6), \quad a_4 = \sum_{j=1}^{5} X(6) \\
a_2 &= \sum_{j=1}^{5} X(2), \quad a_0 = X(0)
\end{align*}
\]
where \([X^{(10)}], [X^{(8)}], [X^{(6)}], [X^{(4)}], [X^{(2)}], \text{ and } [X^{(0)}]\) are the matrices of 5th order and \(x_{k_1}^{(10)}, x_{k_2}^{(8)}, x_{k_3}^{(6)}, x_{k_4}^{(4)}, x_{k_5}^{(2)}, x_{k_6}^{(0)}\) denote the elements standing in the \(k\)-th row and \(l\)-th column of the respective matrices, given as

\[
(1) \quad x_{k_1}^{(10)} = r_{kl}
\]

(ii) If \(l = j\), \(x_{k_1}^{(8)} = s_{kl}\), and if \(l = j\), \(x_{k_1}^{(8)} = r_{kl}\)

(iii) If \(l = \delta\) or \(l = j\), \(x_{k_2}^{(6)} = s_{kl}\), and if \(l = \delta\) and \(l = j\), \(x_{k_2}^{(6)} = r_{kl}\)

(iv) If \(l = \delta\) or \(l = j\), \(x_{k_3}^{(4)} = s_{kl}\), and if \(l = \delta\) and \(l = j\), \(x_{k_3}^{(4)} = s_{kl}\)

(v) If \(l = j\), \(x_{k_4}^{(2)} = r_{kl}\), and if \(l = \delta\), \(x_{k_4}^{(2)} = s_{kl}\)

(vi) \(x_{k_6}^{(0)} = s_{kl}\)

where the elements \(r_{kl}\) and \(s_{kl}\) are

\[
\begin{align*}
  r_{11} &= -1, & r_{12} &= -k^2, & r_{13} &= r_{14} = r_{15} = 0, & r_{21} = r_{22} = -k^2, \\
  r_{23} &= r_{24} = r_{25} = 0, & r_{31} &= 0, & r_{32} &= 0, & r_{34} &= r_{35} = 0, & r_{42} &= -k^2, \\
  r_{43} &= r_{45} = 0, & r_{51} &= 0, & r_{52} &= 0, & r_{53} &= r_{54} = 0, & r_{55} &= 0, \\
  s_{11} &= (1 + k^2)\delta_1 + \lambda^2, & s_{12} &= -k^2, & s_{13} &= -\delta_2, & s_{14} &= -\delta_3, & s_{15} &= -\lambda^2, \\
  s_{21} &= 0, & s_{22} &= 0, & s_{23} &= 0, & s_{24} &= 0, & s_{25} &= 0, \\
  s_{31} &= 0, & s_{32} &= 0, & s_{33} &= 0, & s_{34} &= 0, & s_{35} &= 0, \\
  s_{41} &= 0, & s_{42} &= 0, & s_{43} &= 0, & s_{44} &= 0, & s_{45} &= 0, \\
  s_{51} &= 0, & s_{52} &= 0, & s_{53} &= 0, & s_{54} &= 0, & s_{55} &= 0,
\end{align*}
\]

The coefficients of \((e_j, f_j, g_j, h_j)\) in Eqs.(11) are obtained by solving the following equation

\[
\begin{pmatrix}
  e_j \\
  f_j \\
  g_j \\
  h_j
\end{pmatrix} = \begin{pmatrix}
  -\omega a_j \\
  k_\lambda^2 a_j \\
  -\rho a_j \\
  \rho a_j
\end{pmatrix}
\]

........................................ (14)

where \([A_{55}]\) is the square submatrix of 4th order obtained by excluding both the 5th row and the 5th column of the 5th-order matrix \([A]\).

The boundary conditions (9) may be expressed in terms of the shell-displacement components by the use of the constitutive relations derived by Miskky and Herrmann. The coefficients of \(a_j (j=1, \ldots, 10)\) in Eqs.(11) are thus determined from the Laplace transformations of Eqs.(8) and (9) as

\[
a_j = \frac{1}{i\lambda} \frac{1}{[B_{i1}] \frac{q_1}{B_{i1}}} \left\{ \frac{q_1}{\lambda} B_{12} \right\} + q_2 \left[ B_{10} \right] \] ............................ (15)

where \([B]\) is the square matrix of 10th order and \(b_{i1}\) denotes the element standing in the \(i\)-th row and \(j\)-th column, given as

\[
\begin{align*}
  b_{11} &= e_j, & b_{12} &= f_j, & b_{13} &= g_j, & b_{14} &= h_j, & b_{15} &= 1, \\
  b_{2j} &= e_j^{(2j)}, & b_{3j} &= e_j^{(3j)} + \xi_j^{(2j)}, & b_{4j} &= e_j^{(4j)}, & b_{5j} &= 1, \\
  b_{6j} &= \xi_j^{(3j)}, & b_{7j} &= e_j^{(3j)} + \xi_j^{(2j)}, & b_{10j} &= \xi_j^{(2j)} + \xi_j^{(3j)}.
\end{align*}
\]

where \(I = I/R\) and \([B_{i1}]\) is the cofactor of the element \(b_{i1}\) and \(\xi_j^{(2j)}\) is given by

\[
\xi_j^{(2j)} = (e_j^{(2j)} a_j^{(1j)}) a_j^{(2j)} + \xi_j^{(2j)} e_j^{(2j)} a_j^{(2j)} + \eta_j^{(2j)} e_j^{(2j)} a_j^{(2j)} + \xi_j^{(2j)} e_j^{(2j)} a_j^{(2j)}.
\]

The solutions of the Laplace transformations of the shell-displacement components are thus obtained.

The Laplace inverse transformations of these solutions are determined as the sum of all the residues of simple poles existing at the origin and on the imaginary axis in the \(p\)-plane. Then, putting the poles as 0 and \(\lambda p_m = (0, \lambda^2, \lambda^3, \ldots)\), the Laplace inverse transformations of the shell-displacement components are given by the equations :

\[
\begin{pmatrix}
  U \\
  V \\
  W
\end{pmatrix} = \begin{pmatrix}
  e_j \\
  f_j \\
  g_j \\
  h_j
\end{pmatrix} \begin{pmatrix}
  \cos \lambda \eta \Gamma \\
  \frac{1}{\lambda^2} |B_{1j}| \sum_{j=1}^{10} \frac{q_1}{B_{1j}} [B_{12}] \right| e_j^{(2j)} \eta_{j}\left[ \frac{q_1}{B_{1j}} |B_{1j}| + q_2 |B_{10j}| |e_j^{(2j)} \eta_{j}| \right] \\
  \frac{1}{\lambda^2} |B_{1j}| \sum_{j=1}^{10} \frac{q_1}{B_{1j}} [B_{12}] \right| e_j^{(2j)} \eta_{j}\left[ \frac{q_1}{B_{1j}} |B_{1j}| + q_2 |B_{10j}| |e_j^{(2j)} \eta_{j}| \right]
\end{pmatrix} \lambda = i\lambda_{m} \] ............................ (16)
where \(|B_{ij}^*|\) is the determinant of 10th order and \(b_{ij}^*\) denotes the element standing in the \(i\)-th row and \(j\)-th column shown as:

\[
(1) \quad \text{If } j=k, \ b_{ij}^* = b_{ij} \\
(11) \quad \text{If } j=k, \ b_{1j}^* = E_j, \ b_{2j}^* = F_j, \ b_{3j}^* = G_j, \ b_{4j}^* = H_j, \ b_{5j}^* = 0, \ b_{6j}^* = (L\beta_j \zeta_j + Z_2 j) e_{i,j}^L, \\
b_{7j}^* = (L\beta_j \zeta_j + Z_2 j) e_{i,j}^L, \ b_{8j}^* = (L\beta_j \zeta_j + Z_2 j) e_{i,j}^L, \ b_{9j}^* = (L\beta_j \zeta_j + Z_2 j) e_{i,j}^L, \\
b_{10j}^* = (L\beta_j \zeta_j + Z_2 j) e_{i,j}^L \\
\]

The coefficients of \((E_j, F_j, G_j, H_j)\) are then obtained by solving the following equation:

\[
\begin{bmatrix}
E_j \\
F_j \\
G_j \\
H_j \\
\end{bmatrix} = D_{55}^{-1} \begin{bmatrix}
\begin{bmatrix} -\omega \beta_j, & 0, & 0, & 0 \end{bmatrix} \\
\begin{bmatrix} K_x, 0, & 0, & 0 \end{bmatrix} \\
0, & 0, & 0, & 0 \\
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix} \alpha_j \end{bmatrix} \\
\begin{bmatrix} \beta_j \end{bmatrix} \\
0, & 0, & 0, & 0 \end{bmatrix} \\
\end{bmatrix} \quad \quad \quad \quad \quad \text{(17)}
\]

where \([D_{55}]\) is the square submatrix of 4th order obtained by excluding both the 5th row and the 5th column of the \(5\)-th order matrix \([D]\) and \(d_{ki}\) denotes the element standing in the \(k\)-th row and \(i\)-th column of the matrix \([D]\) given as:

\[
d_{11} = \gamma_{11} \beta_j^2, \quad d_{12} = \gamma_{12} \beta_j^2 k^2, \quad d_{13} = \gamma_{13} \beta_j^2, \quad d_{14} = 0, \quad d_{15} = \gamma_{15} \beta_j, \quad d_{22} = \gamma_{22} \beta_j^2 + k^2 \\
d_{23} = 0, \quad d_{24} = \gamma_{24} \beta_j, \quad d_{25} = \gamma_{25} \beta_j, \quad d_{33} = \gamma_{33} \beta_j^2 - 1, \quad d_{34} = \gamma_{34} \beta_j^2 - k^2, \quad d_{35} = 0 \\
d_{44} = \gamma_{44} \beta_j^2 k^2, \quad d_{45} = 0, \quad d_{55} = \gamma_{55} \beta_j - 1, \quad d_{kl} = d_{lk} \\
\]

where \(\gamma_{11} = -2 \alpha_j, \quad \gamma_{12} = -2 \alpha_j k, \quad \gamma_{13} = -2 \alpha_j, \quad \gamma_{14} = 0, \quad \gamma_{15} = \nu, \quad \gamma_{22} = -2 \alpha_j k^2, \quad \gamma_{23} = 0, \quad \gamma_{24} = -2 \alpha_j k, \quad \gamma_{25} = 0, \quad \gamma_{33} = -2 \alpha_j, \quad \gamma_{34} = -2 \alpha_j k, \quad \gamma_{35} = 0, \quad \gamma_{44} = -2 \alpha_j k, \quad \gamma_{45} = 0, \quad \gamma_{55} = -2 \alpha_j k, \quad \gamma_{kl} = d_{lk} \]

and the coefficients \((Z_{1j}, Z_{2j}, Z_{3j}, Z_{4j}, Z_{5j})\) are given in the form:

\[
Z_{1j} = (E_j + k^2 F_j) \alpha_j + (E_j + k^2 F_j) \beta_j + (E_j + k^2 F_j) \gamma_j, \\
Z_{2j} = (E_j + k^2 F_j) \alpha_j + (E_j + k^2 F_j) \beta_j + (E_j + k^2 F_j) \gamma_j, \\
Z_{3j} = (E_j + k^2 F_j) \alpha_j + (E_j + k^2 F_j) \beta_j + (E_j + k^2 F_j) \gamma_j, \\
Z_{4j} = (E_j + k^2 F_j) \alpha_j + (E_j + k^2 F_j) \beta_j + (E_j + k^2 F_j) \gamma_j, \\
Z_{5j} = (E_j + k^2 F_j) \alpha_j + (E_j + k^2 F_j) \beta_j + (E_j + k^2 F_j) \gamma_j, \\
\]

and the coefficient \(\beta_j\) is given in the form:

\[
\beta_j = -\sum_{i=1}^{5} |M_i| / \sum_{i=1}^{5} |N_i| \quad \quad \quad \quad \quad \text{(18)}
\]

where \(|M_i|\) and \(|N_i|\) are determinants of the 5th order of the matrices \([M_i]\) and \([N_i]\), and \(m_{ki}\) and \(n_{ki}\) denote the elements standing in the \(k\)-th row and \(i\)-th column, respectively, given as:

\[
(11) \quad \text{If } i = 1, \quad m_{ki} = m_{ki}, \quad n_{ki} = m_{ki} \\
(11) \quad \text{If } i = 1, \quad m_{ki} = m_{ki}, \quad n_{ki} = m_{ki} \\
\]

The values of the poles \(p_m (\lambda = \mp / 2)\) are determined from the roots of the transcendental equation of \(B = 0\) and they coincide with the natural frequencies of a cantilever circular cylindrical shell. The values \(\alpha_j, (j = 1, \ldots, 10)\) are calculated from the characteristic equation (13) by equating \(\lambda\) to \(il\). The first term of Eq.(16) is determined from the residue at the origin and it coincides with the solution of the corresponding static problem. The second term is determined from the residues at the poles on the imaginary axis. Namely, the complete solution is represented as a superposition of the static solution and the solutions for all the modes of the natural vibrations of the shell. The first term of Eq.(16), namely, the static solution, is derived from another analysis as mentioned in the following section.

3.2 Solutions under Static Loads

In the static problem, the uncoupled ordinary differential equation of 10th order reduced by substitution of Eqs.(10) into Eqs.(1) can be written in the form:
\[
\frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 u = 0
\]

(19)

where

\[
a_{10} = |X^{(10)}|, \quad a_{20} = \sum_{j=1}^{5} |X_j^{(0)}|_{\lambda=0}, \quad a_{21} = \sum_{i=1}^{5} \sum_{j=1}^{5} |X_j^{(0)}|_{\lambda=0}, \quad a_{4} = \sum_{i=1}^{5} \sum_{j=1}^{5} |X_j^{(0)}|_{\lambda=0}
\]

Therefore, the general solutions for the shell-displacement components are expressed as:

\[
\begin{bmatrix}
U \\
\psi_x \\
V \\
\psi_\theta \\
W
\end{bmatrix}
= \begin{bmatrix}
e_j \\
f_j \\
g_j \\
h_j
\end{bmatrix}
= \begin{bmatrix}
e_j \\
f_j \\
g_j \\
h_j
\end{bmatrix}
\]

(20)

where \(a_{ij}(j=1, \ldots, 6)\) are the 6 characteristic roots of the equation:

\[
a_{10} e^{a_{10} x} + a_{20} e^{a_{20} x} + a_{40} e^{a_{40} x} + a_4 e^{x} = 0
\]

(21)

The coefficients \(e_j, f_j, g_j, h_j\) in Eq. (20) are given as follows:

(i) If \(j=1, \ldots, 6\), these are obtained by solving Eq. (14) by equating \(\lambda\) to 0.

(ii) If \(j=7, \ldots, 10\),

\[
\begin{align*}
e_{10} = 2 \xi^2 - 2 \eta \xi &
\quad f_{10} = 2 \eta \xi + 2 \eta^2 \\
g_{10} = 2 \eta \xi - 2 \eta^2 &
\quad h_{10} = 2 \eta^2 - 2 \eta \xi
\end{align*}
\]

The coefficients \(a_{ij}(j=1, \ldots, 10)\) in Eq. (20) are determined from the boundary conditions (8) and (9) under the condition that \(H(t)=1\). The static solutions for the shell-displacement components are hereby obtained by the equation

\[
\begin{bmatrix}
0 & 1 & 2 \xi & 3(\eta \xi + \xi^2) \\
0 & 1 & 2 \eta & 3(\eta \xi + \xi^2) \\
0 & 0 & -2 \eta & -6 \eta \xi \\
0 & 1 & \xi & \xi^2
\end{bmatrix}
\begin{bmatrix}
B_{ij} & B_{ij} & B_{ij} \\
B_{ij} & B_{ij} & B_{ij} \\
B_{ij} & B_{ij} & B_{ij}
\end{bmatrix}
= \begin{bmatrix}
0 \xi & 2 \xi & 3 \eta \xi \\
1 \xi & \xi^2 & \xi^3 \\
0 \xi & -2 \eta \xi & -6 \eta \xi \\
0 \xi & \xi^2 & \xi^3
\end{bmatrix}
\begin{bmatrix}
B_{ij} & B_{ij} & B_{ij} \\
B_{ij} & B_{ij} & B_{ij} \\
B_{ij} & B_{ij} & B_{ij}
\end{bmatrix}
\]

(22)

where \([B]\) is the square matrix of 10th order and \(b_{ij}\) denotes the element standing in the \(i\)-th row and \(j\)-th column as shown in Table 1. The matrices \([X^{(10)}], \ldots, [X^{(4)}]\) in Eq. (19) and the coefficients \((\xi_{ij}, \ldots, \xi_{10j})\) coincide with the forms defined in Eqs. (13) and (15), respectively, and the constants \((\eta_1, \ldots, \eta_{10})\) are written as

\[
\begin{align*}
\eta_1 &= 1 + k^2, \quad \eta_2 = 1 + k_0^2, \quad \eta_3 = 2(1 + \eta_1) \\
\eta_4 &= 2(1 + \eta_2) \\
\eta_5 &= 2(1 + \eta_3) \\
\eta_6 &= 2(1 + \eta_4) \\
\eta_7 &= 2(1 + \eta_5) \\
\eta_8 &= 2(1 + \eta_6) \\
\eta_9 &= 2(1 + \eta_7) \\
\eta_{10} &= 2(1 + \eta_8)
\end{align*}
\]

Table 1. Component \(b_{ij}\) of matrix \([B]\)

<table>
<thead>
<tr>
<th>(i)</th>
<th>(b_{i1}, \ldots, b_{i6})</th>
<th>(b_{i7})</th>
<th>(b_{i8})</th>
<th>(b_{i9})</th>
<th>(b_{i10})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(e_j)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3\eta_6</td>
</tr>
<tr>
<td>2</td>
<td>(f_j)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3\eta_7</td>
</tr>
<tr>
<td>3</td>
<td>(g_j)</td>
<td>1</td>
<td>0</td>
<td>-2\eta_8</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>(h_j)</td>
<td>0</td>
<td>0</td>
<td>-2\eta_9</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>(i)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>(\xi_{1j}e_j)</td>
<td>0</td>
<td>0</td>
<td>\eta_10</td>
<td>3\eta_{10}</td>
</tr>
<tr>
<td>7</td>
<td>(\xi_{2j}e_j)</td>
<td>0</td>
<td>0</td>
<td>\eta_{11}</td>
<td>3\eta_{11}</td>
</tr>
<tr>
<td>8</td>
<td>(\xi_{3j}e_j)</td>
<td>0</td>
<td>0</td>
<td>-3\eta_{12}</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>(\xi_{4j}e_j)</td>
<td>0</td>
<td>0</td>
<td>-3\eta_{13}</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>(\xi_{5j}e_j)</td>
<td>0</td>
<td>0</td>
<td>-3\eta_{14}</td>
<td>0</td>
</tr>
</tbody>
</table>
4. Numerical Results

In numerical calculations, the condition that \( q_x = q_y = q = q_0 \) is considered, where the external resultant force orients in a definite direction and is constant in magnitude at any place along the free edge of the shell.

The numerical data of \( \omega = 0.3 \) for Poisson’s ratio and \( \kappa = \kappa_0 = \frac{3}{12} \) for the shear coefficients are used. The results are normalized in terms of the applied loads, \( F_0 = 3\pi R q_0 \), the shell-thickness ratio, \( H = h/R \), the shell-length ratio, \( L = L/R \), and the bending stiffness, \( D = E h^3/12(1-\nu^2) \). First, Tables 2 and 3 show the values of the static deflection, \( w \).

**Table 2** Comparison of the Deflections at the Free Edge \((H=0.1)\)

<table>
<thead>
<tr>
<th>( L )</th>
<th>Misky-Herrmann</th>
<th>Flügge</th>
<th>Donnell</th>
<th>Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.15190</td>
<td>0.14903</td>
<td>0.14933</td>
<td>0.13901</td>
</tr>
<tr>
<td>1</td>
<td>0.20294</td>
<td>0.19983</td>
<td>0.20078</td>
<td>0.08529</td>
</tr>
<tr>
<td>2</td>
<td>0.34228</td>
<td>0.33905</td>
<td>0.34120</td>
<td>0.22873</td>
</tr>
<tr>
<td>4</td>
<td>1.0270</td>
<td>1.0243</td>
<td>1.0180</td>
<td>0.92269</td>
</tr>
<tr>
<td>6</td>
<td>2.6375</td>
<td>2.6366</td>
<td>2.4068</td>
<td>2.5471</td>
</tr>
<tr>
<td>8</td>
<td>5.6390</td>
<td>5.6149</td>
<td>4.3364</td>
<td>5.5671</td>
</tr>
<tr>
<td>10</td>
<td>10.4960</td>
<td>10.506</td>
<td>6.1421</td>
<td>10.448</td>
</tr>
</tbody>
</table>

**Table 3** Comparison of the Deflections at the Free Edge \((H=0.02)\)

<table>
<thead>
<tr>
<th>( L )</th>
<th>Misky-Herrmann</th>
<th>Flügge</th>
<th>Donnell</th>
<th>Beam</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.012177</td>
<td>0.012122</td>
<td>0.012135</td>
<td>0.0015641</td>
</tr>
<tr>
<td>1</td>
<td>0.014010</td>
<td>0.013955</td>
<td>0.013976</td>
<td>0.0034198</td>
</tr>
<tr>
<td>2</td>
<td>0.019718</td>
<td>0.019663</td>
<td>0.019706</td>
<td>0.0091713</td>
</tr>
<tr>
<td>4</td>
<td>0.047389</td>
<td>0.047333</td>
<td>0.047469</td>
<td>0.036996</td>
</tr>
<tr>
<td>6</td>
<td>0.11227</td>
<td>0.11222</td>
<td>0.11216</td>
<td>0.10212</td>
</tr>
<tr>
<td>8</td>
<td>0.23303</td>
<td>0.23298</td>
<td>0.23091</td>
<td>0.22232</td>
</tr>
<tr>
<td>10</td>
<td>0.42831</td>
<td>0.42827</td>
<td>0.41716</td>
<td>0.41893</td>
</tr>
</tbody>
</table>

**Table 4** Comparison of the Lowest Natural Frequency \( \lambda_1 \)

<table>
<thead>
<tr>
<th>( H )</th>
<th>( L )</th>
<th>Misky-Herrmann</th>
<th>Flügge</th>
<th>Donnell</th>
<th>Approx. Flügge</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.5</td>
<td>0.53693</td>
<td>0.52956</td>
<td>0.52969</td>
<td>0.85946</td>
</tr>
<tr>
<td>1</td>
<td>0.55583</td>
<td>0.55575</td>
<td>0.55595</td>
<td>0.64742</td>
<td>0.37045</td>
</tr>
<tr>
<td>2</td>
<td>0.27595</td>
<td>0.27594</td>
<td>0.27602</td>
<td>0.15381</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.10801</td>
<td>0.10800</td>
<td>0.10810</td>
<td>0.033536</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.033536</td>
<td>0.033528</td>
<td>0.03378</td>
<td>0.047865</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.015720</td>
<td>0.015713</td>
<td>0.016238</td>
<td>0.022350</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.090022</td>
<td>0.0900151</td>
<td>0.0098972</td>
<td>0.012795</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.0058297</td>
<td>0.0058226</td>
<td>0.0071101</td>
<td>0.0082544</td>
<td></td>
</tr>
</tbody>
</table>

As compared with the deflections obtained by Misky-Herrmann type theory, Flügge’s theory yields quite accurate results. On the other hand, Donnell’s theory yields good results for short shells, while the beam theory yields good results for long shells. Next, values of the natural frequencies \( \omega_m \), which are determined from the roots of the transcendental equation, \( |B| = 0 \), are numerically found out by the trial and error procedure. The values of the lowest natural frequency are listed in Table 4.

As compared with the results obtained by Misky-Herrmann’s theory, Flügge’s theory yields quite accurate results and Donnell’s theory gives slightly higher results for long shells. Approximated Flügge’s theory neglecting the in-plane inertia effect leads to considerably higher results irrespective of the length of the shell.

![Fig. 3 Displacement Variations in Time](image)

![Fig. 4 In-plane Effective Shearing Force Variations in Time](image)
The numerical calculations for the displacement components and the stress resultsants variations in time are made by the use of the values of these natural frequencies \( \lambda_m \). The convergence of Eq. (16) etc. with respect to \( m \) is verified by equating \( \tau \) to 0.

Figure 3 shows the long time responses of the circumferential and the radial displacement at the free edge obtained by Flügge's theory for a shell of \( h/R = 0.02 \) and \( L/R = 2 \), where chain lines denote the static results. The results given by Mirsky-Herrmann's and Donnell's theories coincide identically with Fig. 3. Similarly, the responses of the in-plane effective shearing force \( S_\phi \) (equivalent to \( N_\phi \)) defined by Mirsky-Herrmann's theory at two different points obtained by Flügge's theory are shown in Fig. 1, where A \( (\tau = L) \) and B \( (\tau = L/2/(1-V^2)) \) denote respectively the time when the elastic longitudinal and shear waves emanating from the free edge due to the loads arrive first at each point and the results given by Mirsky-Herrmann's and Donnell's theories coincide identically with these results. A discontinuous change by \( 1/\pi \) at \( \xi = 0 \) and \( 1/2\pi \) at \( \xi = L/2 \) is observed at the time (Point B) when the shear wave originated at the free edge reaches each point. This phenomenon is repeated periodically due to the reflection of a shear wave at both edges. There is no inconsistency in the responses of the in-plane shearing force obtained by Flügge's theory. On the other hand, no discontinuous change is observed in the result obtained by an approximated Flügge's theory, in which the in-plane inertia effect is neglected. Thus, the propagation of elastic waves in shells is not observed in time histories when the in-plane inertia effect is neglected. Figures 5, 6 and 7 show the initial responses of the bending moment at the clamped edge obtained by Mirsky-Herrmann's and Flügge's theories for a shell of \( h/R = 2 \) and \( h/R = 0.02 \) or 0.1. There exists a distinct difference between the two response curves. In the results obtained by Mirsky-Herrmann's theory, no response is observed before the time (Point A), while in the results obtained by Flügge's theory, an earlier response is predicted. The results given by Donnell's theory agree identically with those given by Flügge's theory.

The long time responses of the same bending moment as shown in Fig. 7 are predicted in Fig. 8. The curve obtained by Flügge's theory is so jagged because of a contribution of high-frequency components, but the periods corresponding to the lowest frequency and the static results are nearly identical to those given by Mirsky-Herrmann's theory.
Figure 9 shows the long time response of the transverse shearing force at the clamped edge obtained by Mirsky-Herrmann's theory for a shell of $h/R=0.1$ and $l/R=8$, where the results given by Flügge's and Donnell's theories can't be shown because of a poor convergence of the infinite series. It is observed that the influences of the rotatory inertia and the transverse shear deformation are important in predicting the bending moment and the transverse shearing force.

Figure 10 shows the long time responses of the membrane force $N_z$ obtained by these three different classical theories. The results are nearly identical for both Flügge's and Donnell's theories during the initial stage but the deviations among these become more and more prominent as time increases. There exist differences in the phase and the period between the two response curves obtained by Flügge's and approximated Flügge's theories, while the result obtained by Mirsky-Herrmann's theory agrees identically with that by Flügge's theory. The initial responses of the bending moment at the clamped edge derived by the four different theories are also shown in Figs. 11 and 12 for a shell of $h/R=0.1$ and $l/R=8$. Similarly, the initial responses of the transverse shearing force $T_x$ : effective shearing force defined by Flügge's theory, $Q_x$ : shearing force defined by Mirsky-Herrmann's theory) at the clamped edge are shown in Fig. 13.

The results obtained by Flügge's and Donnell's theories agree identically during the initial stage but earlier responses are also observed.

5. Conclusions

In this paper, the transient responses of a cantilever circular cylindrical shell subjected to impulse loads on its free edge are analyzed on the basis of an improved shell theory considering the influences of the rotatory inertia and the transverse shear deformation and the following conclusions are obtained:

(A) The results predicted by the classical and the improved shell theories are in good agreement about the displacement components and the in-plane stress resultants, namely, the membrane forces and the tangential shearing forces. Meanwhile, the results appear to be quite different about the out-of-plane stress resultants, namely, the bending moments and the transverse shearing forces.

(B) The in-plane translational inertia force affects considerably the phase and the period in time histories.

(C) The results obtained by Donnell type theory agree identically with those by Flügge's theory during the initial stage.

(D) The lowest frequency and the static displacement calculated by the beam theory considering the influence of the transverse shear deformation are in good agreement with those by shell theories.
Fig. 13 Initial Responses of Transverse Shearing Force

References