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Nonstationary Vibration during Acceleration through Two Critical Speeds*

(Without damping)

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The nonstationary vibration of a rotating shaft which passes through two critical speeds successively under conditions of uniform acceleration is theoretically analyzed by both an exact solution and approximate expressions for the vibration, neglecting the damping force. The analytical results show that the nonstationary vibration is characterized by one parameter containing both the critical speed ratio and the acceleration rate, and that the vibration is directly affected by the unbalance distribution in the case of fast acceleration. Furthermore, formulae for evaluating the maximum amplitudes and the rotational speeds of the nonstationary response are derived, and the evaluated values are compared with those obtained from the exact solution.

1. Introduction

Nonstationary vibration during acceleration through a critical speed has been studied by many authors. Most of their works were devoted to clarifying the vibrational characteristics (especially the largest maximum amplitude and the rotational speed) of a linear single-degree-of-freedom system passing through a resonance. The present author also has studied the same problem and derived approximate expressions⁽¹⁾ for the nonstationary vibration.

With respect to a multi-degree-of-freedom system, however, as yet insufficient study has been made, because the vibrational characteristics are complicated and are affected by many factors. McCann and Bennett⁽²⁾ obtained the solutions for the vibration of a two-degree-of-freedom system using an analog computer and pointed out that cancellation or additional effects may occur near the second critical speed when the acceleration is fast. Goloskokow and Filippow⁽³⁾ calculated the nonstationary responses of a torsional vibrating system with three disks using the asymptotic method and showed that there is no interference from other modes when the system passes through one resonance region. Nonami and Miyashita⁽⁴⁾ analyzed the same problem and pointed out that it may be necessary to consider not only a single mode in resonance but also other modes out of resonance when two critical speeds are situated close together. In these works, the nonstationary characteristics of a multi-degree-of-freedom system are partially analyzed, but it is not clearly shown how each factor, for example critical speed ratio, damping ratio, angular acceleration or excitation force, affects them, and how to evaluate the largest maximum amplitude of the response.

In this report, the nonstationary vi-

bration occurring when a two-degree-of-freedom system (a rigid rotor with two unbalance disks supported by springs) passes through the first and the second critical speeds successively at a uniform acceleration rate is theoretically analyzed using both an exact solution and approximate expressions. The calculated results under various acceleration conditions are first shown and the outlines of the nonstationary vibration are made clear. Next, it is shown that the vibration is characterized by one parameter containing both the critical speed ratio and the acceleration rate, and formulae for both the maximum amplitudes and the rotational speeds of the nonstationary response are derived through approximate analysis. The effect of the unbalance distribution on the nonstationary vibration is then discussed.

2. Nomenclature

$$C(u) = \int_0^u \cos\left(\frac{\pi}{2} s^2\right) ds \quad S(u) = \int_0^u \sin\left(\frac{\pi}{2} s^2\right) ds$$

- e_A, e_B : Eccentricities of disks A and B
 $n(=\omega_{n2}/\omega_{n1})$: Critical speed ratio
 R_{max}, R_{0max} : The highest maximum value of a nonstationary response and the maximum value which occurs first after passing through the first critical speed
 Z_A, Z_B : Complex displacements of the centers of gravity of disks
 α : Angular acceleration
 β : Phase angle between e_A and e_B
 ϵ_1, ϵ_2 : The first and second modal eccentricities, see Eq.(10)
 $\tau_1(=\alpha t/\omega_{n1}), \tau_2(=\alpha t/\omega_{n2})$: Nondimensional angular speeds of rotation
 τ_{max}, τ_{0max} : τ_1 values when $|Z_A|$ (or $|Z_B|$) equals to R_{max} and R_{0max} respectively
 ϕ_{jk} : Element of modal matrix, see Eq.(7)
 ω_{n1}, ω_{n2} : The first and second critical speeds
 $\Omega_1(=\omega_{n1}/\sqrt{\alpha}), \Omega_2(=\omega_{n2}/\sqrt{\alpha})$: Acceleration parameter

3. Analysis

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3.1 Critical speeds and natural modes

Figure 1 shows both the rotor-bearing system and the coordinate system used in this study. The rotating shaft with two disks is supported by isotropic springs. The mass of shaft, the gyroscopic moment and the effect of damping force are neglected. The disks, which are attached at the position λl measured from each end of shaft, are equal in weight and have eccentricities e_A and e_B respectively. Under a constant acceleration condition, equations of motion may be written as in Eq. (1) using complex displacement values $Z_A = x_A + iy_A$ and $Z_B = x_B + iy_B$ (see Fig. 2).

$$\left. \begin{aligned} m\ddot{Z}_A + k_A Z_A + k_B Z_B &= k_A e_A \exp(i\psi) + k_B e_B \exp(i(\psi + \beta)) \\ m\ddot{Z}_B + k_B Z_A + k_A Z_B &= k_B e_A \exp(i\psi) + k_A e_B \exp(i(\psi + \beta)) \end{aligned} \right\} \dots (1)$$

$$\dot{\psi} = \alpha$$

$$\text{where } k_A = \frac{k}{2} \left\{ 1 + \frac{1}{(1-2\lambda)^2} \right\} > 0, \quad k_B = \frac{k}{2} \left\{ 1 - \frac{1}{(1-2\lambda)^2} \right\} < 0 \dots (2)$$

In order to obtain the critical speeds and the modal matrix of the system, we put the right side of Eq. (1) equal to zero and substitute $Z_A = A_0 \exp(i\omega_n t)$ and $Z_B = B_0 \exp(i\omega_n t)$. We obtain

$$\begin{bmatrix} k_A - m\omega_n^2 & k_B \\ k_B & k_A - m\omega_n^2 \end{bmatrix} \begin{Bmatrix} A_0 \\ B_0 \end{Bmatrix} = 0 \dots (3)$$

Two critical speeds are obtained as

$$\omega_{n1} = \sqrt{(k_A + k_B)/m}, \quad \omega_{n2} = \sqrt{(k_A - k_B)/m} \dots (4)$$

and the critical speed ratio n can be expressed as

$$n = \frac{\omega_{n2}}{\omega_{n1}} = \sqrt{\frac{k_A - k_B}{k_A + k_B}} = \frac{1}{1-2\lambda} \dots (5)$$

Two natural modes of vibration become

$$\left(\frac{B_0}{A_0} \right)_{\omega_n = \omega_{n1}} = 1, \quad \left(\frac{B_0}{A_0} \right)_{\omega_n = \omega_{n2}} = -1 \dots (6)$$

The normalized modal matrix ϕ_{jk} can then be represented as

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \dots (7)$$

3.2 Exact solutions for the steady state and the nonstationary vibrations

$$\begin{Bmatrix} Z_A \\ Z_B \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} \dots (8)$$

Using Eq. (8), Eq. (1) may be rewritten as in Eq. (9).

$$\begin{aligned} \ddot{w}_1 + \omega_{n1}^2 w_1 &= \omega_{n1}^2 \varepsilon_1 \exp(i\psi), \\ \ddot{w}_2 + \omega_{n2}^2 w_2 &= \omega_{n2}^2 \varepsilon_2 \exp(i\psi), \end{aligned} \quad \dot{\psi} = \alpha \dots (9)$$

$$\text{where } \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} = \frac{1}{\phi_{11}\phi_{22} - \phi_{12}\phi_{21}} \begin{Bmatrix} \phi_{22}e_A - \phi_{12}e_B \exp(i\beta) \\ -\phi_{21}e_A + \phi_{11}e_B \exp(i\beta) \end{Bmatrix} \dots (10)$$

The solutions for the steady state vibration in Eq. (9) can be easily obtained if $\psi = \omega t$ is assumed. Then the steady state solutions of Eq. (1) may be represented as in Eq. (11).

Fig. 1 Rotating rigid shaft with two disks supported by isotropic springs

$$\left. \begin{aligned} Z_A &= \left\{ \frac{\phi_{11}\varepsilon_1}{1-(\omega/\omega_{n1})^2} + \frac{\phi_{12}\varepsilon_2}{1-(\omega/\omega_{n2})^2} \right\} \exp(i\omega t) \\ Z_B &= \left\{ \frac{\phi_{21}\varepsilon_1}{1-(\omega/\omega_{n1})^2} + \frac{\phi_{22}\varepsilon_2}{1-(\omega/\omega_{n2})^2} \right\} \exp(i\omega t) \end{aligned} \right\} \dots (11)$$

The solutions for the nonstationary vibration in Eq. (9) can be written as follows according to Eq. (37) in Appendix if the rest state is selected as the initial condition. (The conditions $Z_A = e_A$, $\dot{Z}_A = 0$, $Z_B = e_B \exp(i\beta)$, $\dot{Z}_B = 0$ at $t=0$ correspond to $w_1 = \varepsilon_1$, $\dot{w}_1 = 0$, $w_2 = \varepsilon_2$, $\dot{w}_2 = 0$ at $t=0$.)

$$\begin{aligned} w_j(\tau_j) &= \frac{\sqrt{\pi}}{2} \varepsilon_j Q_j \left\{ \exp\left[i Q_j^2 \left(\tau_j - \frac{1}{2}\right)\right] \times \left[C\left(\frac{Q_j}{\sqrt{\pi}}(\tau_j - 1)\right) \right. \right. \\ &\quad \left. \left. + iS\left(\frac{Q_j}{\sqrt{\pi}}(\tau_j - 1)\right) + C\left(\frac{Q_j}{\sqrt{\pi}}\right) + iS\left(\frac{Q_j}{\sqrt{\pi}}\right) \right] - \exp\left[-i Q_j^2 \left(\tau_j + \frac{1}{2}\right)\right] \right\} \\ &\quad \times \left[C\left(\frac{Q_j}{\sqrt{\pi}}(\tau_j + 1)\right) + iS\left(\frac{Q_j}{\sqrt{\pi}}(\tau_j + 1)\right) - C\left(\frac{Q_j}{\sqrt{\pi}}\right) - iS\left(\frac{Q_j}{\sqrt{\pi}}\right) \right] \\ &\quad + \varepsilon_j \cos Q_j^2 \tau_j \quad (j=1, 2) \dots (12) \end{aligned}$$

where

$$Q_1 = \omega_{n1}/\sqrt{\alpha}, \quad Q_2 = \omega_{n2}/\sqrt{\alpha}, \quad \tau_1 = \alpha t/\omega_{n1}, \quad \tau_2 = \alpha t/\omega_{n2} \dots (13)$$

The exact solutions for the nonstationary vibration of Eq. (1) can then be expressed as in Eq. (14) by substituting Eq. (12) into Eq. (8).

$$Z_A = \phi_{11}w_1 + \phi_{12}w_2, \quad Z_B = \phi_{21}w_1 + \phi_{22}w_2 \dots (14)$$

3.3 Approximate expressions for the nonstationary vibration

Using Eq. (38) in Appendix, the approximations of w_1 and w_2 in Eq. (9) may be written as follows:

$$\left. \begin{aligned} \text{when } \tau_j < 1 - (\sqrt{2}/Q_j), \\ w_{j1}(\tau_j) &= (\varepsilon_j/(1-\tau_j^2)) \exp\left\{i \frac{1}{2} Q_j^2 \tau_j^2\right\}, \\ \text{when } 1 - (\sqrt{2}/Q_j) < \tau_j < 1 + (\sqrt{2}/Q_j), \\ w_{j2}(\tau_j) &= \frac{\sqrt{\pi}}{2} \varepsilon_j Q_j \left\{ \left[\frac{1}{2} + \frac{Q_j}{\sqrt{\pi}}(\tau_j - 1) \right] \exp\left\{i \left(Q_j^2 \tau_j - \frac{Q_j^2}{2} - \frac{\pi}{2} \right) \right\} \right. \\ &\quad \left. + \left[\frac{1}{2} + \frac{Q_j^3}{6\sqrt{\pi}}(\tau_j - 1)^3 \right] \exp\left\{i \left(Q_j^2 \tau_j - \frac{Q_j^2}{2} \right) \right\} \right. \\ &\quad \left. + \frac{1}{\sqrt{\pi} Q_j(\tau_j + 1)} \exp\left\{i \frac{1}{2} Q_j^2 \tau_j^2\right\} \right\}, \\ \text{and when } 1 + (\sqrt{2}/Q_j) < \tau_j, \\ w_{j3}(\tau_j) &= \sqrt{\frac{\pi}{2}} \varepsilon_j Q_j \exp\left\{i \left(Q_j^2 \tau_j - \frac{Q_j^2}{2} - \frac{\pi}{4} \right) \right\} \\ &\quad - \frac{\varepsilon_j}{\tau_j^2 - 1} \exp\left\{i \frac{1}{2} Q_j^2 \tau_j^2\right\} \quad (j=1, 2). \end{aligned} \right\} \dots (15)$$

Approximate expressions for the nonstationary vibration of Eq. (1) will be obtained by substituting Eq. (15) into Eq. (8) considering their application ranges carefully. We shall limit ourselves to discussing the vibration after passing through the first critical speed, when we assume τ_1 is larger than $1 + \sqrt{2}/Q_1$. Under this condition, w_1 may be approximated by w_{13} . With regard to w_2 , the suffix j should be replaced by 2 in these equations and inequalities and corresponding to the three ranges of τ_2 , w_2 can be approximated by

Fig. 2 Relationship between the center of gravity and the volume center

w_{21} , w_{22} and w_{23} respectively. The application ranges represented by τ_1 and Ω_1 , remembering $\tau_2 = \tau_1/n$ and $\Omega_2 = n\Omega_1$.

Figure 3 shows schematically the application ranges of the various w_{jk} . This figure tells us that there exist two cases (I) and (II), depending on the characteristic value $(n-1)\Omega_1$. On the basis of this figure, Z_A may be approximated by the following equations by using Eq.(8), and the approximation for Z_B will be obtained by substituting ϕ_{21} and ϕ_{22} into ϕ_{11} and ϕ_{12} respectively.

(I) In case of $2\sqrt{2} < (n-1)\Omega_1$
 when $1 + (\sqrt{2}/\Omega_1) < \tau_1 < n - (\sqrt{2}/\Omega_1)$,

$$Z_A = \phi_{11} \sqrt{\frac{\pi}{2}} \varepsilon_1 \Omega_1 \exp \left\{ i \left(\Omega_1^2 \tau_1 - \frac{\Omega_1^2}{2} - \frac{\pi}{4} \right) \right\} + \left(-\frac{\phi_{11}\varepsilon_1}{\tau_1^2 - 1} + \frac{\phi_{12}\varepsilon_2}{1 - (\tau_1/n)^2} \right) \exp \left\{ i \frac{1}{2} \Omega_1^2 \tau_1^2 \right\}, \dots \dots \dots (16)$$

when $n - (\sqrt{2}/\Omega_1) < \tau_1 < n + (\sqrt{2}/\Omega_1)$,

$$Z_A = \phi_{11} \sqrt{\frac{\pi}{2}} \varepsilon_1 \Omega_1 \exp \left\{ i \left(\Omega_1^2 \tau_1 - \frac{\Omega_1^2}{2} - \frac{\pi}{4} \right) \right\} - \frac{\phi_{11}\varepsilon_1}{\tau_1^2 - 1} \exp \left\{ i \frac{1}{2} \Omega_1^2 \tau_1^2 \right\} + \phi_{12} \sqrt{\frac{\pi}{2}} \varepsilon_2 n \Omega_1 \left[\left\{ \frac{1}{2} + \frac{n\Omega_1}{\sqrt{\pi}} \left(\frac{\tau_1}{n} - 1 \right) \right\} \exp \left\{ i \left(n\Omega_1^2 \tau_1 - \frac{n^2\Omega_1^2}{2} - \frac{\pi}{2} \right) \right\} + \left\{ \frac{1}{2} + \frac{n^3\Omega_1^3}{6\sqrt{\pi}} \left(\frac{\tau_1}{n} - 1 \right)^3 \right\} \exp \left\{ i \left(n\Omega_1^2 \tau_1 - \frac{n^2\Omega_1^2}{2} \right) \right\} + \frac{1}{\sqrt{\pi} n \Omega_1 (\tau_1/n + 1)} \exp \left\{ i \frac{1}{2} \Omega_1^2 \tau_1^2 \right\} \right], \dots \dots \dots (17)$$

and when $n + (\sqrt{2}/\Omega_1) < \tau_1$,

$$Z_A = \phi_{11} \sqrt{\frac{\pi}{2}} \varepsilon_1 \Omega_1 \exp \left\{ i \left(\Omega_1^2 \tau_1 - \frac{\Omega_1^2}{2} - \frac{\pi}{4} \right) \right\} + \phi_{12} \sqrt{\frac{\pi}{2}} \varepsilon_2 n \Omega_1 \exp \left\{ i \left(n\Omega_1^2 \tau_1 - \frac{n^2\Omega_1^2}{2} - \frac{\pi}{4} \right) \right\} + \left(-\frac{\phi_{11}\varepsilon_1}{\tau_1^2 - 1} - \frac{\phi_{12}\varepsilon_2}{(\tau_1/n)^2 - 1} \right) \exp \left\{ i \frac{1}{2} \Omega_1^2 \tau_1^2 \right\}, \dots \dots \dots (18)$$

(II) In case of $(n-1)\Omega_1 < 2\sqrt{2}$
 when $1 + (\sqrt{2}/\Omega_1) < \tau_1 < n + (\sqrt{2}/\Omega_1)$, Z_A is approximated by Eq.(17), and when $n + (\sqrt{2}/\Omega_1) < \tau_1$, Z_A is approximated by Eq.(18).

The first case (I) corresponds to the case where the two critical speeds are far apart and the acceleration is slow and the second (II) does the opposite case. Considering $\Omega_1^2 \tau_1 = \omega_{n1} \tau_1$, $n\Omega_1^2 \tau_1 = \omega_{n2} \tau_1$ and $\Omega_1^2 \tau_1^2 / 2 = \alpha \tau^2 / 2$, it is found that Eq.(16) consists of two types of vibration. One is a free vibration with the first natural frequency and the other a forced vibration synchronous with the shaft rotation. Equations (17) and (18) consist of free vibration with the first and second natural frequencies and a forced vibration.

Fig.3 Application ranges of w_{jk}

4. Calculated results

The system parameters are selected as $n=1.1$ and $\varepsilon_1 = -\varepsilon_2 = \sqrt{2}$, and the acceleration parameter is selected to be either 100, 50, 25 or 12.5. The solution for the steady state vibration Eq.(11), and the exact and approximate solutions for the nonstationary vibration Eqs.(14) and (16)~(18) are calculated and the envelope curve of each response of the disk A is plotted in Fig.4.

Fig.4 Comparison between the exact solution and the approximate expressions
 ($n=1.1$, $\varepsilon_1 = -\varepsilon_2 = \sqrt{2}$)

Figure 4 shows that the two nonstationary curves (the exact solution and approximate expressions) are in good agreement over a wide speed range. Furthermore, the nonstationary characteristics of the vibration occurring when a two-degree-of-freedom system without damping passes through two critical speeds successively are stated as follows:

(1) When the two critical speeds are situated far apart and the acceleration is slow [(a) and (b)], the nonstationary response before passing through the second critical speed is a little different from that of a single-degree-of-freedom system. For example the beat, which consists of a free vibration at the first natural frequency and a forced vibration, disappears momentarily in some cases or the maximum amplitude which occurs first happens to be smaller than that which comes second. The response after passing through the second critical speed shows the other type of beat which mainly consists of free vibrations at the first and second natural frequencies and the effect of the forced vibration is small.

(2) When the two critical speeds are situated close together and the acceleration is fast, the beat is mainly composed of the first and second free vibrations. A maximum amplitude of the beat does not occur until after passing through the second critical speed as if a rotor passed through a single critical speed. In most of these cases, the maximum amplitude which occurs first becomes the largest maximum amplitude. But in some cases this does not hold good as shown in (d).†

† (3) In case (I), when the response of $|Z_A|$ reaches the maximum, the response of $|Z_B|$ does the minimum and vice versa except at the initial stage of beats.

(4) Calculated results in cases of $n=1.25$ and $n=1.5$ (these are not shown) indicate that the nonstationary vibration exhibits the same characteristics as described in the statement (1) as n becomes large even when the acceleration is same.

In order to examine the nonstationary characteristics quantitatively, the two typical points on the response curves are selected. One is the largest maximum amplitude point ($\tau_{\max}$, $R_{\max}$) and the other is the maximum amplitude point which occurs first after passage through the first critical speed ($\tau_{0\max}$, $R_{0\max}$). These values are summarized in Table 1. Table 1 tells us the following.

When the acceleration is slow ($\Omega_1=100$ and 50), $\tau_{0\max}$ hardly changes regardless of the critical speed ratio n and the difference between $|Z_A|$ and $|Z_B|$, and it is nearly equal to $1+\sqrt{3\pi/2}(1/\Omega_1)$ where the largest maximum amplitude of the nonstationary vibration occurs in a single-degree-of-freedom system. $\tau_{0\max}$ may be approximated by Eq.(19) which can be easily obtained by using the preceding result.

$$\tau_{\max} = n + \sqrt{\frac{3\pi}{2}} \frac{1}{\Omega_1} \quad \dots\dots\dots(19)$$

When the acceleration is fast ($\Omega_1=12.5$), both $\tau_{0\max}$ and $\tau_{\max}$ become extremely large as n becomes small and these values of $|Z_A|$ are very different from that of $|Z_B|$.

5. Evaluation of maximum amplitudes and rotational speeds

5.1 Derivation of formulae

Considering the results of Fig.4 and Table 1, formulae for both maximum amplitudes and rotational speeds will be derived by using the approximate expressions for the nonstationary vibration. Especially, for the system which is used in calculating the results shown in Fig.4 (in the following, it is called the model system), simple formulae for $\tau_{0\max}$, $\tau_{\max}$, $R_{0\max}$ and $R_{\max}$ will be shown.

5.1.1 In case of $2\sqrt{2} < (n-1)\Omega_1$

(i) When $1+(\sqrt{2}/\Omega_1) < \tau_1 < n-(\sqrt{2}/\Omega_1)$, the nonstationary vibration may be approximated by Eq.(16). If both amplitudes of the free vibration and the forced vibration in Eq.(16), generally ϵ_1 and ϵ_2 are complex numbers as shown in Eq.(10), may be written as in Eq.(20), Eq.(16) becomes

$$\phi_{11} \sqrt{\frac{\pi}{2}} \epsilon_1 \Omega_1 = F_1 \exp(ig_1), \quad -\frac{\phi_{11}\epsilon_1}{\tau_1^2-1} + \frac{\phi_{12}\epsilon_2}{1-(\tau_1/n)^2} = G_1 \exp(ig_1) \quad F_1, G_1 > 0 \quad \dots\dots\dots(20)$$

$$Z_A = F_1 \exp \left\{ i \left(\Omega_1^2 \tau_1 - \frac{\Omega_1^2}{2} - \frac{\pi}{4} + f_1 \right) \right\} + G_1 \exp \left\{ i \left(\frac{1}{2} \Omega_1^2 \tau_1^2 + g_1 \right) \right\} \quad \dots\dots\dots(21)$$

From the above equations, it is found that maximum amplitudes will occur when

$$\frac{1}{2} \Omega_1^2 \tau_1^2 + g_1 - \left(\Omega_1^2 \tau_1 - \frac{\Omega_1^2}{2} - \frac{\pi}{4} + f_1 \right) = 2N\pi \quad \text{is nearly satisfied. Then}$$

$$\tau_1 = 1 + \frac{\sqrt{2}}{\Omega_1} \sqrt{2N\pi - \frac{\pi}{4} + f_1 - g_1} \quad (N: \text{integer}, 1+(\sqrt{2}/\Omega_1) < \tau_1 < n-(\sqrt{2}/\Omega_1)) \quad \dots\dots\dots(22)$$

In obtaining τ_1 , recurrent calculations will be needed if g_1 in Eq.(22) varies with τ_1 . The maximum amplitudes of Z_A will be calculated by the following equation.

$$(Z_A)_{\max} = F_1 + G_1(\tau_1) = \left| \phi_{11} \sqrt{\frac{\pi}{2}} \epsilon_1 \Omega_1 \right| + \left| -\frac{\phi_{11}\epsilon_1}{\tau_1^2-1} + \frac{\phi_{12}\epsilon_2}{1-(\tau_1/n)^2} \right| \quad \dots\dots\dots(23)$$

$\tau_{0\max}$ of the model system can be obtained as the minimum value which satisfies Eq.(22) and $R_{0\max}$ can be obtained by substituting $\tau_{0\max}$ into Eq.(23). The approximate formulae for these values may be expressed as follows:

$$\left. \begin{aligned} \text{Disk A} \quad \tau_{0\max} &= 1 + \sqrt{\frac{2\pi}{2}} \frac{1}{\Omega_1}, \quad R_{0\max} = \sqrt{\frac{\pi}{2}} \Omega_1 + \left| -\frac{1}{\tau_{0\max}^2-1} - \frac{1}{1-(\tau_{0\max}/n)^2} \right| \\ &\quad \text{if } \tau_{0\max} > \sqrt{2n^2/(1+n^2)}, \text{ then } \tau_{0\max} = 1 + (\sqrt{2}/\Omega_1) \\ \text{Disk B} \quad \tau_{0\max} &= 1 + \sqrt{\frac{3\pi}{2}} \frac{1}{\Omega_1}, \quad R_{0\max} = \sqrt{\frac{\pi}{2}} \Omega_1 + \left| -\frac{1}{\tau_{0\max}^2-1} + \frac{1}{1-(\tau_{0\max}/n)^2} \right| \end{aligned} \right\} \quad \dots\dots\dots(24)$$

It should be noted here that as for the response of $|Z_A|$, $\tau_{0\max}$ is approximated by the lower limit of the application range of Eq.(22), that is, $1+(\sqrt{2}/Q_1)$, when the amplitude of the forced vibration due to the second modal eccentricity is larger than that of the first in Eq.(20).

(ii) When $n+(\sqrt{2}/Q_1) < \tau_1$, the nonstationary vibration may be approximated by Eq.(18). But it is very difficult to obtain $\tau_{\max}$ analytically. So Eq.(19) is used here. $R_{\max}$ may be evaluated as in Eq.(26) using the relationships Eq.(25).

$$F_1 = \left| \phi_{11} \sqrt{\frac{\pi}{2}} \varepsilon_1 Q_1 \right|, \quad F_2 = \left| \phi_{12} \sqrt{\frac{\pi}{2}} \varepsilon_2 n Q_1 \right|, \quad G_1 = \left| -\frac{\phi_{11} \varepsilon_1}{\tau_{\max}^2 - 1} - \frac{\phi_{12} \varepsilon_2}{(\tau_{\max}/n)^2 - 1} \right| \dots (25)$$

$$F_1 + F_2 < R_{\max} < F_1 + F_2 + G_1 \dots (26)$$

The reason why the inequality is used is to avoid miss-evaluation of $R_{\max}$ due to the approximation of $\tau_{\max}$. Formulae for $\tau_{\max}$ and $R_{\max}$ of the model system are as follows:

$$\tau_{\max} = n + \sqrt{\frac{3\pi}{2}} \frac{1}{Q_1}, \quad (1+n)\sqrt{\frac{\pi}{2}} Q_1 < R_{\max} < (1+n)\sqrt{\frac{\pi}{2}} Q_1 + \left| \frac{-1}{\tau_{\max}^2 - 1} \pm \frac{1}{(\tau_{\max}/n)^2 - 1} \right| \dots (27)$$

+: Disk A, -: Disk B

5.1.2 In case of $(n-1)Q_1 < 2\sqrt{2}$

(iii) When $1+(\sqrt{2}/Q_1) < \tau_1 < n+(\sqrt{2}/Q_1)$, the nonstationary vibration may be approximated by Eq.(17). This condition includes a case as shown in Fig.4(c), where $R_{0\max}$ occurs in the neighborhood of the second critical speed. Under this condition the approximation $(1/2)Q_1^2 \tau_1 \approx nQ_1^2 \tau_1 - (n^2 Q_1^2/2)$ is satisfied and if we use notations F_1 , F_3 , f_1 and f_3 in Eq.(28), Eq.(17) can be rewritten as Eq.(29) neglecting the term $1/(\sqrt{\pi} n Q_1 (\tau_1/n+1))$.

$$\left. \begin{aligned} \phi_{11} \varepsilon_1 \sqrt{\frac{\pi}{2}} Q_1 &= F_1 \exp(if_1) \\ -\frac{\phi_{11} \varepsilon_1}{\tau_1^2 - 1} + \phi_{12} \varepsilon_2 \sqrt{\frac{\pi}{2}} n Q_1 \left[\left\{ \frac{1}{2} + \frac{n Q_1}{\sqrt{\pi}} \left(\frac{\tau_1}{n} - 1 \right) \right\} (-i) + \left\{ \frac{1}{2} + \frac{n^2 Q_1^2}{6\sqrt{\pi}} \left(\frac{\tau_1}{n} - 1 \right)^2 \right\} \right] &= F_3 \exp(if_3) \end{aligned} \right\} \dots (28)$$

$$Z_A = F_1 \exp \left\{ i \left(Q_1^2 \tau_1 - \frac{Q_1^2}{2} - \frac{\pi}{4} + f_1 \right) \right\} + F_3 \exp \left\{ i \left(n Q_1^2 \tau_1 - \frac{n^2 Q_1^2}{2} + f_3 \right) \right\} \dots (29)$$

The rotational speeds where the maximum amplitudes occur can then be obtained as

$$\tau_1 = \frac{n+1}{2} + \frac{2N\pi - (\pi/4) + f_1 - f_3}{(n-1)Q_1^2} \quad (N: \text{integer}, 1+(\sqrt{2}/Q_1) < \tau_1 < n+(\sqrt{2}/Q_1) \dots (30)$$

The maximum amplitude then becomes

$$(Z_A)_{\max} = F_1 + F_3(\tau_1) \dots (31)$$

Formulae for $\tau_{0\max}$ and $R_{0\max}$ of the model system may be represented as in Eq.(32).

$$\left. \begin{aligned} \text{Disk A} \quad \tau_{0\max} &= 1+(\sqrt{2}/Q_1) \\ \text{Disk B} \quad \tau_{0\max} &= \frac{n+1}{2} + \frac{(3\pi/4) + \gamma}{(n-1)Q_1^2}, \quad \tan \gamma = \frac{(\sqrt{\pi}/4)n(n^2-1)Q_1}{1+(\sqrt{\pi}/4)n(n^2-1)Q_1} \\ R_{0\max} &= \sqrt{\frac{\pi}{2}} Q_1 + \left| -\frac{1}{\tau_{0\max}^2 - 1} \pm \frac{\sqrt{\pi}}{2} n Q_1 \left[\frac{1}{2} + \frac{Q_1^2}{6\sqrt{\pi}} (\tau_{0\max} - n)^2 - i \left\{ \frac{1}{2} + \frac{Q_1}{\sqrt{\pi}} (\tau_{0\max} - n) \right\} \right] \right| \end{aligned} \right\} \dots (32)$$

+: Disk A, -: Disk B

According to $|Z_A|$, $\tau_{0\max}$ is approximated by the lower boundary of the application range since τ_1 cannot be found in this range. As for $|Z_B|$, τ_1 in Eq.(28) is approximated by n for simplicity.

(iv) When $n+(\sqrt{2}/Q_1) < \tau_1$, the nonstationary vibration may be approximated by Eq.(18). In order to obtain the rotational speeds where the maximum amplitudes occur, we neglect the forced vibration terms in Eq.(18) since the beats in this region are mainly composed of free vibrations at the first and second natural frequencies. Using the notations f_1 and f_2 in Eq.(33)

$$\phi_{11} \varepsilon_1 \sqrt{\frac{\pi}{2}} Q_1 = F_1 \exp(if_1), \quad \phi_{12} \varepsilon_2 \sqrt{\frac{\pi}{2}} n Q_1 = F_2 \exp(if_2) \quad F_1, F_2 > 0 \dots (33)$$

the rotational speeds can be expressed as

$$\tau_1 = \frac{n+1}{2} + \frac{2N\pi + f_1 - f_2}{(n-1)Q_1^2} \quad (N: \text{integer}, n+(\sqrt{2}/Q_1) < \tau_1) \dots (34)$$

The maximum amplitudes can be obtained by substituting τ_1 into Eq.(26).

Formulae for $\tau_{0\max}$ and $R_{0\max}$ of the model system may be represented as Eq.(35).

$$\left. \begin{aligned} \tau_{\max} &= \frac{n+1}{2} + \frac{2N\pi}{(n-1)Q_1^2} \\ (1+n)\sqrt{\frac{\pi}{2}} Q_1 &< R_{\max} < (1+n)\sqrt{\frac{\pi}{2}} Q_1 + \left| -\frac{1}{\tau_{\max}^2 - 1} + \frac{1}{(\tau_{\max}/n)^2 - 1} \right| \\ \tau_{\max} &= \frac{n+1}{2} + \frac{(2N-1)\pi}{(n-1)Q_1^2} \\ (1+n)\sqrt{\frac{\pi}{2}} Q_1 &< R_{\max} < (1+n)\sqrt{\frac{\pi}{2}} Q_1 + \left| -\frac{1}{\tau_{\max}^2 - 1} - \frac{1}{(\tau_{\max}/n)^2 - 1} \right| \end{aligned} \right\} \dots (35)$$

Where N is chosen as the minimum integer which satisfies $n+(\sqrt{2}/Q_1) < \tau_{\max}$.

5.2 Comparison between exact values and evaluated ones as for $\tau_{0\max}$, $\tau_{\max}$, $R_{0\max}$ and $R_{\max}$

In order to ascertain the validity of the formulae for the maximum amplitudes and the rotational speeds of the nonstationary responses, the evaluated values of $\tau_{0\max}$, $\tau_{\max}$, $R_{0\max}$ and $R_{\max}$ calculated by the formulae are compared with the results obtained

from the responses which are calculated by using the exact solution (14).

As for $\tau_{0\max}$ and $\tau_{\max}$, the compared results are shown in Figs.5 and 6 schematically. In Fig.5, $\tau_{0\max}$ and $\tau_{\max}-Q_1$ curves with various n , which are calculated by the formulae Eqs.(24) and (27), are drawn and the exact values of $|Z_B|$ (only those which are included in case Sec.5.1.1) are plotted. As a whole,

both results are in good agreement. In case of $n=1.1$, the evaluation is more complex, because the characteristic value $(n-1)\Omega_1$ is one in case Sec.5.1.1 or 5.1.2 depending on Ω_1 . The details are shown in Fig.6. The thick lines denote the evaluated values calculated from various formulae and the thin solid lines the application boundary of formulae.

The exact values shown in Table 1 including the results of $\Omega_1=15, 20, 30$ and 40 are plotted in the same figure. This figure tells us that the evaluation of $\tau_{0\max}$ and $\tau_{\max}$ are in good agreement with the exact results quantitatively, except in the vicinity of the application boundary of the formulae.

As for $R_{0\max}$ and $R_{\max}$, the comparison is shown in Table 2. In the parentheses the evaluation errors are shown in the form of %. The formulae used to calculate the estimated results are different according to the values of n and Ω_1 , and the details are shown in the foot note of Table 2. When a formula is expressed as an inequality, the upper value is adopted. (Within the extent of Table 2 the upper value is about 10% larger than the lower one.) The two cases ($n=1.1$, $\Omega_1=25$) and ($n=1.25$, $\Omega_1=12.5$) in Table 2 have not a small evaluation error since these cases are close to the application boundary $(n-1)\Omega_1=2\sqrt{2}$.

The largest error is 9% in amplitude and 3.6% in rotational speed. Including the above two cases the evaluation seems to have a fairly good accuracy.

5.3 Effect of unbalance distribution

Table 3 shows the variation of $\tau_{0\max}$, $\tau_{\max}$, $R_{0\max}$ and $R_{\max}$ with the phase angle between the first and second modal unbalances. This table shows that $\tau_{0\max}$ and $\tau_{\max}$ change violently in case of the fast acceleration. Other calculations (the results are not shown) indicate that a change in the amount of the modal eccentricity has a great influence on $R_{0\max}$ and $R_{\max}$ but hardly affects $\tau_{0\max}$ and $\tau_{\max}$.

Fig.6 Comparison between the evaluated value and the exact one as for $\tau_{0\max}$ and $\tau_{\max}$ (the model system, $n=1.1$)

Fig.5 Comparison between the evaluated value and the exact one as for $\tau_{0\max}$ and $\tau_{\max}$ (the model system, disk B)

Table 1 Variation of $\tau_{0\max}$, $\tau_{\max}$, $R_{0\max}$ and $R_{\max}$ with n and Ω_1 (Exact solution)

n	Ω_1				
		12.5	25	50	100
1.1	$\tau_{0\max}$	A 1.144	1.060	1.038	1.022
		B —	1.103	1.044	1.022
	$R_{0\max}$	A 22.64	32.27	64.17	139.2
		B —	52.64	82.86	153.7
	$\tau_{\max}$	A 1.438	1.258	1.150	1.120
		B 1.260	1.199	1.139	1.123
1.25	$R_{\max}$	A 35.01	68.18	142.6	285.4
		B 33.89	68.28	138.5	280.2
	$\tau_{0\max}$	A 1.118	1.081	1.043	1.022
		B 1.194	1.088	1.044	1.022
	$R_{\max}$	A 15.41	32.49	69.82	143.5
		B 24.58	40.51	76.41	149.5
1.5	$\tau_{\max}$	A 1.442	1.327	1.296	1.271
		B 1.375	1.346	1.291	1.272
	$R_{\max}$	A 38.73	77.43	154.8	299.7
		B 35.60	75.01	152.3	306.5
	$\tau_{0\max}$	A 1.154	1.086	1.043	1.022
		B 1.177	1.087	1.043	1.022
1.5	$R_{\max}$	A 15.68	34.34	71.18	144.6
		B 20.68	38.54	75.05	148.3
	$\tau_{\max}$	A 1.654	1.592	1.542	1.520
		B 1.692	1.582	1.544	1.521
	$R_{\max}$	A 43.04	86.32	167.5	344.6
		B 41.97	85.27	171.5	343.8

$$e_1 = \sqrt{2}, e_2 = -\sqrt{2}$$

Note: $\tau_{0\max}$ and $R_{0\max}$ values of — part do not exist and they coincide with $\tau_{\max}$ and $R_{\max}$ respectively.

Table 2 Comparison between evaluated values and exact ones
as for $\tau_{0\max}$, $\tau_{\max}$, $R_{0\max}$ and $R_{\max}$

Ω_1		12.5	25	50	100
π	$\tau_{0\max}$	A 1.113* (-2.7) B —	1.057* (-0.3) 1.088* (-0.5)	1.043 (+0.5) (-0.1)	1.022 (0) (0)
	$R_{0\max}$	A 23.1* (+2.0) B —	34.0* (+5.4) 52.2* (-0.8)	64.1 (-0.1) 83.9 (+1.3)	141 (+1.3) 155 (+0.8)
1.1	$\tau_{\max}$	A 1.452**(+1.0) B 1.251**(-0.7)	1.251**(-0.6) 1.201**(+0.2)	1.143 (-0.6) (+0.4)	1.122 (+0.2) (-0.1)
	$R_{\max}$	A 34.2**(-2.3) B 36.3**(+7.1)	69.2**(+1.5) 68.1**(-0.3)	145 (+1.7) (+4.7)	288 (+0.9) (+2.8)
1.25	$\tau_{0\max}$	A 1.113***(-0.4) B 1.174 (-1.7)	1.087 (+0.6) (-0.1)	1.043 (0) (-0.1)	1.022 (0) (0)
	$R_{0\max}$	A 16.3***(+5.8) B 26.8 (+9.0)	32.7 (+0.6) 40.9 (+1.0)	70.7 (+1.3) 77.3 (+1.2)	145 (+1.0) 151 (+1.0)
1.5	$\tau_{\max}$	A 1.424 (-1.2) B (+3.6)	1.337 (+0.8) (-0.7)	1.293 (-0.2) (+0.2)	1.272 (+0.1) (0)
	$R_{\max}$	A 38.6 (-0.3) B (+8.4)	77.4 (0) (+3.2)	155 (+0.1) (+1.8)	310 (+3.4) (+1.1)
1.5	$\tau_{0\max}$	A 1.174 (+1.7) B (-0.3)	1.087 (+0.1) (0)	1.043 (0) (0)	1.022 (0) (0)
	$R_{0\max}$	A 15.7 (+0.1) B 20.9 (+1.1)	34.7 (+1.0) 38.9 (+0.9)	72.1 (+1.3) 76.0 (+1.3)	146 (+1.0) 150 (+1.1)
1.5	$\tau_{\max}$	A 1.674 (+1.2) B (-1.1)	1.587 (-0.3) (+0.3)	1.543 (+0.1) (-0.1)	1.522 (+0.1) (+0.1)
	$R_{\max}$	A 43.3 (+0.6) B (+3.2)	86.7 (+0.4) (+1.7)	174 (+3.9) (+1.5)	347 (+0.7) (+1.0)

$$\varepsilon_1 = -\varepsilon_2 = \sqrt{2}$$

Note: The formulae used in evaluation are as follows:

* Eq.(32), ** Eq.(35), *** Eq.(24) in case of $\tau_{0\max} > \sqrt{\frac{2n^2}{n^2+1}}$
and no mark Eqs.(24) and (32).

Table 3 Variation of $\tau_{0\max}$, $\tau_{\max}$, $R_{0\max}$ and $R_{\max}$ with the phase angle between the modal eccentricities

Ω_1		12.5	25	50	100	Note
$\frac{2}{3}\pi$	$\tau_{0\max}$	A 1.180 B —	1.073 1.124	1.037 1.049	1.020 1.023	$e_A = \sqrt{3}$ $e_B = 1$ $\beta = \pi/2$
	$R_{0\max}$	A 30.64 B —	42.49 52.87	72.42 80.17	143.9 150.8	
	$\tau_{\max}$	A 1.528 B 1.306	1.171 1.213	1.131 1.143	1.120 1.124	
	$R_{\max}$	A 34.38 B 34.88	72.25 67.76	138.7 141.1	286.3 282.5	
$\frac{\pi}{3}$	$\tau_{0\max}$	A — B 1.100	1.087 1.146	1.040 1.063	1.021 1.023	$e_A = 1$ $e_B = \sqrt{3}$ $\beta = \pi/2$
	$R_{0\max}$	A — B 12.95	49.86 58.69	79.98 72.97	150.8 143.7	
	$\tau_{\max}$	A 1.212 B 1.422	1.183 1.433	1.135 1.146	1.121 1.125	
	$R_{\max}$	A 34.05 B 33.18	70.60 67.55	139.5 144.7	282.8 284.0	

6. Conclusions

The conclusions obtained in this study are summarized as follows:

(1) When the condition $2\sqrt{2} < (n-1)\Omega_1$ is satisfied, the nonstationary vibration exhibits

two types of beat. One, which is composed of both a forced vibration synchronous with shaft rotation and a free vibration at the first natural frequency, occurs between the first and second critical speeds. The other, which is composed of a free vibration at the second natural frequency in addition to the above beat, occurs after passing through the second critical speed. The largest maximum amplitude occurs in the latter beat and is nearly equal to the sum of the three vibration amplitudes, while the rotational speed is nearly equal to $n + \sqrt{3}\pi/2 \times 1/\Omega_1$ regardless of the rotor unbalance distribution.

(2) When the condition $(n-1)\Omega_1 < 2\sqrt{2}$ is satisfied, the nonstationary vibration exhibits a response similar to that obtained when the rotor passes through a single critical speed, and the maximum amplitude does not occur until after passing through the second critical speed. In this case, the beat is mainly composed of free vibration at the first and second natural frequencies. The largest maximum amplitude is a little greater than the sum of the amplitudes of two free vibrations. The rotational speed at which the largest maximum amplitude occurs is greatly affected by the rotor unbalance distribution.

(3) Formulae for evaluating the maximum (including the largest maximum) amplitudes and the rotational speeds have been derived. The evaluated results obtained from these formulae are in good agreement with the results from the exact solutions. The maximum error was 9% in amplitude and 3.6% in rotational speed.

Appendix The exact solution and the approximate expressions for
the nonstationary vibration of a single-degree-of-freedom system

When a rotating shaft which has a disk at the center of the span is accelerated at a constant angular acceleration(α), the equations of motion of the center of gravity of the disk (x, y) may be written as Eq.(36) neglecting the damping force.

$$m\ddot{w} + kw = k\epsilon \exp(i\psi), \quad \ddot{\psi} = \alpha, \quad w = x + iy \quad \dots\dots\dots(36)$$

Where m , k , ϵ and ψ denote the mass of disk, the spring constant of shaft, the eccentricity of disk and the rotational angle of shaft respectively. When the rest state ($w = \epsilon$, $\dot{w} = 0$ at $t = 0$) is selected, ψ can be expressed as $\alpha t^2/2$. Under this condition, the exact solution of Eq.(36) may be represented as Eq.(37) using Fresnel integrals C and S (see Nomenclature), $\tau (= \alpha t / \omega_n)$ and $\Omega (= \omega_n / \sqrt{\alpha})$.

$$\frac{w(\tau)}{\epsilon} = \frac{\sqrt{\pi}}{2i} \Omega \left\{ \begin{array}{l} \exp\left\{i\Omega^2\left(\tau - \frac{1}{2}\right)\right\} \times \left[C\left\{\frac{\Omega}{\sqrt{\pi}}(\tau - 1)\right\} + iS\left\{\frac{\Omega}{\sqrt{\pi}}(\tau - 1)\right\} + C\left(\frac{\Omega}{\sqrt{\pi}}\right) + iS\left(\frac{\Omega}{\sqrt{\pi}}\right) \right] \\ - \exp\left\{i\Omega^2\left(\tau + \frac{1}{2}\right)\right\} \times \left[C\left\{\frac{\Omega}{\sqrt{\pi}}(\tau + 1)\right\} + iS\left\{\frac{\Omega}{\sqrt{\pi}}(\tau + 1)\right\} - C\left(\frac{\Omega}{\sqrt{\pi}}\right) - iS\left(\frac{\Omega}{\sqrt{\pi}}\right) \right] \end{array} \right\} + \cos \Omega^2 \tau \quad \dots\dots(37)$$

According to the reference (1), Eq.(37) may be approximated as follows:

$$\left. \begin{array}{l} \text{when } \tau < 1 - (\sqrt{2}/\Omega), \quad w(\tau)/\epsilon = (1/(1 - \tau^2)) \exp\left\{i \frac{1}{2} \Omega^2 \tau^2\right\}, \\ \text{when } 1 - (\sqrt{2}/\Omega) < \tau < 1 + (\sqrt{2}/\Omega), \\ \quad w(\tau)/\epsilon = \frac{\sqrt{\pi}}{2} \Omega \left[\left\{ \frac{1}{2} + \frac{\Omega}{\sqrt{\pi}}(\tau - 1) \right\} \exp\left\{i \left(\Omega^2 \tau - \frac{\Omega^2}{2} - \frac{\pi}{2} \right)\right\} \right. \\ \quad \left. + \left\{ \frac{1}{2} + \frac{\Omega^3}{6\sqrt{\pi}}(\tau - 1)^3 \right\} \exp\left\{i \left(\Omega^2 \tau - \frac{\Omega^2}{2} \right)\right\} + \frac{1}{\sqrt{\pi} \Omega(\tau + 1)} \exp\left\{i \frac{1}{2} \Omega^2 \tau^2\right\} \right], \\ \text{and when } 1 + (\sqrt{2}/\Omega) < \tau, \quad w(\tau)/\epsilon = \sqrt{\frac{\pi}{2}} \Omega \exp\left\{i \left(\Omega^2 \tau - \frac{\Omega^2}{2} - \frac{\pi}{4} \right)\right\} - \frac{1}{\tau^2 - 1} \exp\left\{i \frac{1}{2} \Omega^2 \tau^2\right\}. \end{array} \right\} \quad \dots\dots(38)$$

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