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Analysis of Vibration of Bevel Gears*

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Equation of motion for free vibrations of a pair of gears is derived in a general form. A vibration system of two-degrees of freedom which represents the fundamental characteristics of bevel gears is constructed mathematically. It is found from the analysis of the system that the fundamental characteristics of vibrations which distinguish bevel gears from spur and helical gears are caused by a change in the direction of the tooth-normal in tooth-meshing. The influences of parameters such as the contact ratio, stiffness of gear-carrying shafts and damping ratios on the band width of unstable regions of the vibration are also shown.

1. Introduction

The dynamic behaviours of spur and helical gears are known better than those of bevel gears. Especially the vibration of spur gears has been investigated theoretically as well as experimentally. In order to clarify the dynamic behaviours of bevel gears it is useful, therefore, to introduce a vibration model which can represent bevel gears as well as spur gears. In the present paper an equation of motion for a pair of gears is derived in a general form, which can make evident the fundamental difference of vibrations between bevel gears and spur and helical gears. Then it is clear from the equation that the difference between both types of gears can be typically represented by a coupled Hill-type vibration model of two degrees of freedom.

It is shown that the coupled Hill-type equations of the model can be analyzed strictly under some simplifications of the model. The dynamic behaviours of bevel gears are compared with those of spur and helical gears in terms of vibration instability. The model and the method of analysis can be used also to estimate the accuracy of approximate methods of analysis for coupled Hill-type equations, which will be applied for a more complicated model of gear vibrations.

2. Equation of motion

Vibrations of a pair of gears are described by the coordinate systems in Fig. 1. The subscripts I and II show the driving and driven sides respectively. The origin O_j ($j=I, II$) coincides with the centroid of the corresponding gear when the gears are matched without displacements. The Z_j -axis ($j=I, II$)

is taken in the axial direction of each gear. The unit vectors specifying the X_j, Y_j and Z_j directions are designated by i_j, j_j and k_j respectively. Each gear has six degrees of freedom, which are displacements in the direction of X_j, Y_j and Z_j -axes and angular displacements about these axes. The vibrational displacements are represented by a vector which has the components of $(X_j, Y_j, Z_j, \Phi_j, \Psi_j, \Theta_j)$. Suppose that l -pairs of teeth are in contact and the point P_i ($i=1 \sim l$) is the centre of distributed load of the i -th pair of meshing teeth. The vector r_{ji} extends from the origin O_j to the point P_i . The unit vector of the tooth-normal at the point P_i is designated by n_{ji} . Then the deflection of the i -th meshing pair of teeth due to the vibration displacements is expressed as follows:

$$\delta_{ni} = (p_{Ii} n_{Ii}) + (p_{IIi} n_{IIi}) \quad \left. \begin{aligned} \text{where } p_{ji} &= X_j i_j + Y_j j_j + Z_j k_j \\ &+ \Phi_j [i_j r_{ji}] + \Psi_j [j_j r_{ji}] + \Theta_j [k_j r_{ji}] \end{aligned} \right\} \dots (1)$$

The unit vectors of the tangential and binormal directions at the point P_i are designated by t_{ji} and b_{ji} respectively. Then the distortion of the i -th meshing teeth due to the vibration displacements is written as follows:

$$\sigma_{ni} = (b_{Ii} \Omega_I) + (b_{IIi} \Omega_{II}) \quad \left. \begin{aligned} \text{where } \Omega_j &= \Phi_j i_j + \Psi_j j_j + \Theta_j k_j \end{aligned} \right\} \dots (2)$$

Fig. 1 Coordinate systems for a pair of gears

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Let K_{ni} and $J_{\sigma i}$ be the stiffnesses of a pair of teeth against the deflection and the distortion respectively. Then the tooth load W_{ni} and the tooth moment M_{bi} are represented as follows:⁽¹⁾

$$W_{ni} = K_{ni} \delta_{ni}, \quad M_{bi} = J_{\sigma i} \sigma_{bi} \quad \dots\dots\dots (3)$$

In most of bevel gears, the load W_{ni} is distributed over a small region of the contacting surface. The moments about three axes due to M_{bi} are, therefore, negligibly small in comparison with those due to W_{ni} . For this reason the term M_{bi} is neglected in the following discussion.

Equation (1) can be re-written as follows:

$$\delta_{ni} = (q_{1i} X_I) + (q_{2i} X_{II})$$

$$\text{where } q_{ji} = \{(n_{ji} i_j), (n_{ji} j_j), (n_{ji} k_j), (r_{ji} n_{ji} i_j), (r_{ji} n_{ji} j_j), (r_{ji} n_{ji} k_j)\} \quad \dots\dots\dots (4)$$

The vector q_{ji} represents the direction of the tooth stiffness in the six degrees of freedom system. This direction is that of the tooth normal when only the rectilinear displacements are considered.

Supposing q_{ji} and q_{ji} are the column and the row matrices, the stiffness matrix of meshing teeth is expressed as follows:

$$K_{Gjk} = \sum_{i=1}^l K_{ni} q_{ji}^T q_{ki} \quad (j, k = I, II) \quad \dots\dots\dots (5)$$

The damping matrix with respect to the damping force between meshing teeth is represented as follows:

$$C_{Gjk} = \sum_{i=1}^l C_{ni} q_{ji}^T q_{ki} \quad (j, k = I, II) \quad \dots\dots\dots (6)$$

where c_{ni} is the damping coefficient of the damper of the i -th pair of meshing teeth. Using Eqs. (5) and (6), the stiffness and damping matrix with respect to the vector $X = (X_I, X_{II})$ are written as:

$$K_G = \begin{pmatrix} K_{GI I} & K_{GI II} \\ K_{GII I} & K_{GII II} \end{pmatrix}, \quad C_G = \begin{pmatrix} C_{GI I} & C_{GI II} \\ C_{GII I} & C_{GII II} \end{pmatrix}.$$

The equation of motion for a pair of gears is expressed as follows:

$$(m_G + m_S) \ddot{X} + (C_G + C_S) \dot{X} + (K_G + K_S) X + C_G X = 0 \quad \dots\dots\dots (7)$$

$$\text{where } \dot{C}_G = \begin{pmatrix} \dot{C}_{GI I} & \dot{C}_{GI II} \\ \dot{C}_{GII I} & \dot{C}_{GII II} \end{pmatrix}, \quad \dot{C}_{Gjk} = \sum_{i=1}^l C_{ni} q_{ji}^T \dot{q}_{ki}.$$

The terms m_G and m_S in Eq. (7) are mass matrices of gears and shafts respectively and C_S and K_S are damping and stiffness matrices with respect to the gear carrying shafts. Using Eqs. (4), (5) and (6), Eq. (7) can be re-written as:

$$(m_G + m_S) \ddot{X} + C_S \dot{X} + \sum_{i=1}^l C_{ni} \delta_{ni} q_i^T \dot{X} + K_S X + \sum_{i=1}^l K_{ni} \delta_{ni} q_i^T X = 0 \quad \dots\dots\dots (8)$$

$$\text{where } q_i = (q_{1i}, q_{2i}).$$

Neglecting the terms m_S , C_S and K_S , Eq. (8) is written as follows:

$$\left. \begin{aligned} (\ddot{X} q_k) + \sum_{i=1}^l C_{ni} \delta_{ni} / m_{ki} + \sum_{i=1}^l K_{ni} \delta_{ni} / m_{ki} &= 0 \\ (k=1 \sim l) \end{aligned} \right\} \quad \dots\dots\dots (9)$$

$$\text{where } 1/m_{ki} = (q_k m_G^{-1} q_i^T), \quad \delta_{ni} = (X q_i), \quad \dot{\delta}_{ni} = (X \dot{q}_i) + (\dot{X} q_i),$$

$$\ddot{\delta}_{ni} = (X \ddot{q}_i) + 2(\dot{X} \dot{q}_i) + (\ddot{X} q_i) \quad \dots\dots\dots (9)$$

In spur gears the vector q_i is not dependent on time nor on subscript 'i'. Equation (9) is, therefore, reduced to an equation for the variable δ_{ni} . On the other hand, in bevel gears, the vector q_i is a function of time and depends on 'i'. Therefore in bevel gears Eq. (9) represents a coupled vibration of l -degrees of freedom. It is of course necessary to note that in any cases the vibration represented by Eq. (9) is a Hill-type vibration, because the stiffness K_n is a function of time.

3. A Vibration model of two degrees of freedom

Figure 2 shows one of the simplest vibration systems which can represent the above introduced difference between spur and bevel gears. The springs k_{sx} and k_{sy} represent the stiffnesses of gear carrying shafts. The damping force acting through the shafts is represented by the dampers c_{sx} and c_{sy} . The directions of forces acting on these elements are fixed in the space. The springs k_{n1} , k_{n2} and the dampers c_{n1} , c_{n2} represent the meshing pairs of teeth. These elements rotate with an angular velocity ω which corresponds to the term $\dot{q}_i$ in Eq. (9). A pair of teeth continue to contact in the interval of θ_f and rotate by angle θ_z in one meshing period. A double contact occurs in the interval of $(\theta_f - \theta_z)$.

For simplicity, the case of $k_{sx} = k_{sy} = k_s$ and $c_{sx} = c_{sy} = c_s$ is investigated in the following discussion. The essential difference between spur and bevel gears will not be lost by this simplification. The coefficients are nondimensionalized as follows:

$$k_0 = k_s + k_{n1} + k_{n2}(\varepsilon - 1), \quad \omega_0^2 = k_0/m, \quad \tau = \omega_0 t, \\ \kappa_{ni} = k_{ni}/k_0, \quad \kappa_s = k_s/k_0, \quad \tau_z = \omega_0 t_z = 2\pi\omega_0/\omega_s$$

where ε and t_z are the contact ratio and the tooth meshing period respectively. Defining the vector $X = (x, y)$, the vector q_i is given as:

$$\left. \begin{aligned} q_1 &= (\cos \omega t, \sin \omega t), \\ q_2 &= \{\cos(\omega t + \theta_z), \sin(\omega t + \theta_z)\} \end{aligned} \right\} \quad \dots\dots\dots (10)$$

Instead of q_2 , the vector q_3 which is perpendicular to the vector q_1 can be used:

$$q_3 = (-\sin \omega t, \cos \omega t) \quad \dots\dots\dots (11)$$

Fig. 2 A simplified vibration model of bevel gears

Defining the vector $\mathbf{x}$ as

$$\mathbf{x} = (x_1, x_2), \quad x_1 = (\mathbf{q}_1 \mathbf{X}), \quad x_2 = (\mathbf{q}_2 \mathbf{X}). \quad (12)$$

the equation of motion for the system in Fig. 2 is expressed with respect to the coordinates x_1 and x_2 as follows:

$$\mathbf{I} \ddot{\mathbf{x}} + \mathbf{F} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} = 0. \quad (13)$$

where the elements of the matrix $\mathbf{F}$ and $\mathbf{K}$ are

$$\left. \begin{aligned} \gamma_{11} &= 2\zeta_s \sqrt{\kappa_s} + 2\zeta_G (\sqrt{\kappa_{n1}} + \sqrt{\kappa_{n2}} \cos^2 \theta_s), \\ \gamma_{21} &= 2\zeta_G \sqrt{\kappa_{n2}} \sin \theta_s \cos \theta_s + 2\omega, \quad \gamma_{12} = \gamma_{21} - 4\omega, \\ \gamma_{22} &= 2\zeta_s \sqrt{\kappa_s} + 2\zeta_G \sqrt{\kappa_{n2}} \sin^2 \theta_s, \\ \kappa_{11} &= \kappa_s - \omega^2 + \kappa_{n1} + \kappa_{n2} \cos^2 \theta_s, \quad \kappa_{22} = \kappa_s - \omega^2 + \kappa_{n2} \sin^2 \theta_s, \\ \kappa_{21} &= \kappa_{n2} \cos \theta_s \sin \theta_s + 2\zeta_s \sqrt{\kappa_s} \omega, \quad \kappa_{12} = \kappa_{21} - 4\zeta_s \sqrt{\kappa_s} \omega. \end{aligned} \right\} \quad (14)$$

In the single contact region the term κ_{n2} should be eliminated. In the spur and helical gears $\theta_s = \omega = 0$.

Strictly speaking the stiffness κ_{ni} ($i=1, 2$) is dependent on time. But the differences between spur and helical gears and bevel gears are not lost by regarding the term κ_{ni} as a constant. In the following discussion κ_{ni} is regarded as a constant and necessarily κ_{n1} is equal to κ_{n2} , which will be denoted by κ_n .

The normalized form of Eq. (13) can be expressed as:

$$\dot{\mathbf{y}} = \mathbf{A} \mathbf{y}, \quad \mathbf{y} = (x_1, x_2, \dot{x}_1, \dot{x}_2). \quad (15)$$

The elements of matrix $\mathbf{A}$ are constant in the single and double contact regions respectively. The eigen values and the eigen vectors of $\mathbf{A}$ can be, therefore, calculated for each region. The general solutions of Eq. (13) can be expressed by using these eigen values and eigen vectors.⁽²⁾ Denoting the double and the single contact regions by subscripts 'a' and 'b' respectively, and expressing the general solutions by $\mathbf{y}_a(\tau)$ and $\mathbf{y}_b(\tau - \tau_a)$, the continuity conditions of the solution at the time $\tau = \tau_a$ are:

$$\mathbf{y}_a(\tau_a) = \mathbf{y}_b(0), \quad \tau_a = (\varepsilon - 1)\tau_s. \quad (16)$$

The conditions of normal solutions are expressed as follows:

$$\sigma \mathbf{y}_a(0) = \begin{pmatrix} \mathbf{R} & 0 \\ 0 & \mathbf{R} \end{pmatrix} \mathbf{y}_b(\tau_b),$$

$$\mathbf{R} = \begin{pmatrix} \cos \theta_s & -\sin \theta_s \\ \sin \theta_s & \cos \theta_s \end{pmatrix}, \quad \tau_b = \tau_s - \tau_a. \quad (17)$$

From Eqs. (16) and (17) a set of equations for eight unknown constants which are included in the expressions of $\mathbf{y}_a$ and $\mathbf{y}_b$ are obtained. When the coefficients matrix is expressed by $\mathbf{B}$, the condition in which a nontrivial solution of the set of equations is possible is expressed:

$$|\mathbf{B}| = 0. \quad (18)$$

Equation (18) is a fourth order equation in σ . When absolute values of the roots σ_i of Eq. (18) are less than unity, the system will be stable.⁽³⁾ In the following section the results of numerical analysis of Eq. (18) for various conditions of the system will be shown.

The unstable region of the system can be simply estimated by making an approximation

in the damping matrix. Eliminating the terms of damping in Eq. (13), the natural frequencies and modes of the system can be expressed as

$$\left. \begin{aligned} \omega_{a1}^2 &= \kappa_s - \omega^2 + \kappa_n (1 - \cos \theta_s), \quad x_{a1} = (-\sin \theta_s/2, \cos \theta_s/2), \\ \omega_{a2}^2 &= \kappa_s - \omega^2 + \kappa_n (1 + \cos \theta_s), \quad x_{a2} = (\cos \theta_s/2, \sin \theta_s/2), \\ \omega_{b1}^2 &= \kappa_s - \omega^2, \quad x_{b1} = (0, 1), \\ \omega_{b2}^2 &= \kappa_s - \omega^2 + \kappa_n, \quad x_{b2} = (1, 0). \end{aligned} \right\} \quad (19)$$

For simplicity the damping ratio of the vibration in each natural mode is estimated as

$$\left. \begin{aligned} 2\zeta_{a1} &= (2\zeta_s \sqrt{\kappa_s} + 2\zeta_G \sqrt{\kappa_n} (1 - \cos \theta_s)) / \omega_{a1}, \\ 2\zeta_{a2} &= (2\zeta_s \sqrt{\kappa_s} + 2\zeta_G \sqrt{\kappa_n} (1 + \cos \theta_s)) / \omega_{a2}, \\ 2\zeta_{b1} &= 2\zeta_s \sqrt{\kappa_s} / \omega_{b1}, \\ 2\zeta_{b2} &= (2\zeta_s \sqrt{\kappa_s} + 2\zeta_G \sqrt{\kappa_n}) / \omega_{b2}. \end{aligned} \right\} \quad (20)$$

The off-diagonal elements of damping are neglected. The solutions of the system are

$$\left. \begin{aligned} x_a &= \sum_{i=1}^2 e^{-\zeta_{ai} \omega_{ai} \tau} (C_{2i-1} \cos \sqrt{1 - \zeta_{ai}^2} \omega_{ai} \tau \\ &\quad + C_{2i} \sin \sqrt{1 - \zeta_{ai}^2} \omega_{ai} \tau) x_{ai}, \\ x_b &= \sum_{i=1}^2 e^{-\zeta_{bi} \omega_{bi} \tau'} (C_{2i+3} \cos \sqrt{1 - \zeta_{bi}^2} \omega_{bi} \tau' \\ &\quad + C_{2i+4} \sin \sqrt{1 - \zeta_{bi}^2} \omega_{bi} \tau') x_{bi}, \quad \tau' = \tau - \tau_a. \end{aligned} \right\} \quad (21)$$

The expression by Eqs. (16) and (17) can be reduced to Eq. (22) and Eq. (23) respectively:

$$x_a(\tau_a) = x_b(0), \quad \dot{x}_a(\tau_a) = \dot{x}_b(0), \quad (22)$$

$$\sigma x_a(0) = \mathbf{R} x_b(\tau_b), \quad \sigma \dot{x}_a(0) = \mathbf{R} \dot{x}_b(\tau_b). \quad (23)$$

The relationship among the modal vectors is written as follows:

$$\left. \begin{aligned} \tau = \tau_a, \quad (x_{a1} x_{b1}) &= (x_{a2} x_{b2}) = \cos \theta_s/2 = p, \\ (x_{a2} x_{b1}) &= (x_{a1} x_{b2}) = \sin \theta_s/2 = q, \\ \tau = \tau_s, \quad x_{a1} \mathbf{R} x_{b1} &= x_{a2} \mathbf{R} x_{b2} = p, \\ x_{a1} \mathbf{R} x_{b2} &= -x_{a2} \mathbf{R} x_{b1} = q. \end{aligned} \right\} \quad (24)$$

Then the characteristic equation corresponding to Eq. (18) is written:

$$|\mathbf{B}| = f_4(\sigma_p) + f_3(\sigma_p) \alpha^2 + f_2(\sigma_p) \alpha^4 + f_1(\sigma_p) \alpha^6 + f_0(\sigma_p) \alpha^8,$$

$$\alpha = q/p, \quad \sigma_p = \sigma/p^2. \quad (25)$$

where f_i is a polynomial of order i in σ_p .

It is easily shown that the $f_4(\sigma_p)$ is resolved as a multiplication of the two second order factors in σ_p . They are correspondent to independent characteristic equations of Meißner-type vibration of one-degree of freedom.⁽⁴⁾ It can also be shown approximately that the zeros of $f_4(\sigma_p)$ approach maximum value under the following condition:

$$\sqrt{1 - \zeta_{a1}^2} \omega_{a1} \tau_a + \sqrt{1 - \zeta_{b1}^2} \omega_{b1} \tau_b = 2n\pi$$

$$(i=1, 2, n=1/2, 3/2, \dots). \quad (26)$$

When the i -th natural frequency $\bar{\omega}_i$ which is measured over one tooth-mesh period is defined by

$$\bar{\omega}_i = \{\sqrt{1 - \zeta_{a1}^2} \omega_{a1} (\varepsilon - 1) + \sqrt{1 - \zeta_{b1}^2} \omega_{b1} (2 - \varepsilon)\} \omega_0, \quad (27)$$

Eq. (26) can be re-written as:

$$\bar{\omega}_i / \omega_s = n_i \quad (i=1, 2, n_i=1/2, 1, 3/2, \dots). \quad (28)$$

It should be noted that the frequencies $\bar{\omega}_i$ in Eq. (27) are calculated by choosing pairs of modal vectors $\mathbf{x}_{a_i}$ and $\mathbf{x}_{b_i}$ whose direction cosine is largest. In the corresponding pairs of natural modes the ratios of stiffness variation are given as follows:

$$\Delta_1 = \frac{\kappa_n(1 - \cos \theta_z)}{1 - \kappa_n[1 + (\varepsilon - 1) \cos \theta_z]}, \quad \Delta_2 = \frac{\kappa_n(1 + \cos \theta_z)}{1 - \kappa_n(1 - \cos \theta_z)} \quad (29)$$

Since $\alpha \neq 0$, Eq. (25) may have zeros of large absolute value even if the condition given in Eq. (26) is not satisfied.⁽⁵⁾ The unstable regions with respect to these zeros occur at frequencies in the neighbourhood of

$$\left. \begin{aligned} \bar{\omega}_3/\omega_s &= (\bar{\omega}_1 + \bar{\omega}_2)/2\omega_s = n_3, \quad (n_3 = 1/2, 1, 3/2, \dots), \\ \bar{\omega}_4/\omega_s &= |\bar{\omega}_1 - \bar{\omega}_2|/2\omega_s = n_4, \quad (n_4 = 1/2, 1, 3/2, \dots). \end{aligned} \right\} \quad (30)$$

The values n_i given in Eqs. (28) and (30) are called the order of resonance with natural frequency $\bar{\omega}_i$.

4. Results of numerical analysis

The tooth stiffness κ_n , the contact ratio ε , the angle θ_z which corresponds to the rotation angle of the tooth-normal vector in a tooth-mesh period and damping ratios ζ_s , ζ_q are essential parameters of the model under consideration. The parameters κ_n and ε respectively determine the ratio of stiffness variation Δ_i and the spectrum of the stiffness function. Figures 3 ~ 6 show the results of a numerical analysis of Eq. (13) by the procedure of Eq. (15) ~ Eq. (18).

Figure 3 shows the dependence of unstable regions on the angle θ_z with respect to the tooth mesh frequency ω_z/ω_b . The resonant order n_i which is obtained from Eqs. (28) and (30) is indicated by chained lines. The natural frequencies ω_{a_i} , ω_{b_i} and $\bar{\omega}_i$ for Eqs. (28) and (30) are calculated from Eq. (19) by neglecting the term ω . The curve $\omega_{b_i} = 0$ in Fig. 3 shows the condition that the angular velocity ω is equal to ω_{b_i} obtained

in the fixed coordinate system. Unstable regions with respect to ω_4 are not found in the model under consideration. The curves of n_i ($i = 1, 2, 3$) are almost coincident with the centre lines of corresponding unstable regions. This fact means that the three kinds of resonant frequencies which cause instability can be approximately estimated by Eqs. (28) and (30).

The unstable regions due to resonance with $\bar{\omega}_2$, which are typically seen in the regions of $n_2 = 1/2, 1$, tend to be narrower with an increase of θ_z . On the other hand the unstable regions due to resonance with $\bar{\omega}_1$ tend to be wider with θ_z . These tendencies are correspondent to those of Δ_2 and Δ_1 given in Eq. (29). The unstable regions of n_3 due to resonance with $\bar{\omega}_3$ show an intermediate tendency between the others.

An unstable region which is caused by the influence of the angular velocity ω is seen under the condition of $\omega_z/\omega_b > 2$ and $\theta_z > 50^\circ$. No evident influence of ω is found except under this condition. It should be noted that the value θ_z may not exceed 30° and ω_z/ω_b may not exceed 2 in usual bevel gears.⁽⁶⁾ The importance of θ_z is therefore recognized in the fact that the direction of modal vectors and the stiffness variation of the natural modes are dependent on the angle θ_z .

Figure 4 shows the dependence of the unstable regions on the damping ratios ζ_s and ζ_q with respect to ω_z/ω_b . In the case where ζ_q is increased from 0 to 0.15 the value of ζ_s is fixed at zero. In the case where ζ_s is varied the value of ζ_q is fixed at 0.01. It is apparent that the unstable regions of n_1 do not diminish rapidly with an increase of ζ_q . On the other hand the unstable regions of n_2 disappear rapidly as ζ_q increases. The tendencies of the effect of ζ_s on these two kinds of unstable region are contrary to those of that of ζ_q . The dependence of the unstable regions of n_3 on ζ_q and ζ_s shows an intermediate tendency between the others. These influences of ζ_q and ζ_s on instability found in Fig. 4 are

Fig. 3 Influence of the rotation angle of the tooth-normal vector on the unstable regions

correspondent to the effects of ζ_g and ζ_s on the damping ratios of natural modes given in Eq. (20).

It is known that the damping ratio ζ_g is about 0.07 in spur gears.^{(4),(7)} Although the value of ζ_s may change in cases, it may not be expected to be larger than ζ_g . For example the value of ζ_s is estimated to be 0.02 for bearings of shafts.⁽⁸⁾

Figure 5 shows the dependence of the unstable regions on the contact ratio with respect to the tooth-mesh frequency ω_z/ω_b . The dependence of instability on the contact ratio may be explained by the relationship of it with the Fourier coefficients of the stiffness function. The k -th Fourier coefficient of the stiffness function can be expressed by the stiffness variation ratio Δ_i and the contact ratio ε as follows:

$$C_{ik} = (\Delta_i / \pi k) \sin k(\varepsilon - 1)\pi. \quad (31)$$

It is found in Fig. 5 that the band width of unstable regions of $n_2 = 1/2, 1$ and $3/2$ corresponds to the magnitude of the coefficient C_{2k} , $k=1, 2$ and 3 respectively. There is seen

a similar tendency between the band width of the unstable region of n_1 and the corresponding coefficients C_{1k} . Strictly speaking, the value of Δ_i in Eq. (31) increases with the contact ratio, of which the influence can be also found on the unstable regions of n_1 . It should be noted that the possibility of k_n decreasing with an increase of ε is not considered in the calculation of Fig. 5.

For the unstable regions of n_3 as seen typically in the case of $n_3=1/2$ can not be found any simple relationship with the Fourier coefficients C_{1k} . But in most cases the unstable regions of n_1 and n_3 which distinguish the bevel gears from spur and helical gears, tends to be wider with an increase of ε .

Figure 6 shows the dependence of the unstable regions on the magnitude κ_n with respect to the tooth-mesh frequency ω_z/ω_b . The case of $\kappa_n=1/\varepsilon$ is correspondent to the case where the stiffness of shafts is reduced to zero. In the case where $\zeta_g = \zeta_s = 0$, the unstable regions may appear even in the neighbourhood of the axis of $\kappa_n = 0$. The unstable regions seen in small κ_n disappear when a small damping of $\zeta_g = \zeta_s = 0.025$ is

Fig. 4 Influence of damping ratios on unstable regions

Fig. 5 Influence of the contact ratio on unstable regions

Fig. 6 Influence of the stiffness of shafts on unstable regions

taken into consideration. Especially the unstable regions of n_1 become narrower rapidly with a decrease of K_n under existence of small damping ζ_s . This is because the reduction of K_n makes strong the influence of ζ_s on the damping ratio of the natural mode corresponding to $\bar{\omega}$, while it reduces the stiffness variation Δ_1 .

It is also seen in Fig. 6 that the unstable region of n_1 and n_3 shifts to a higher frequency range with reduction of K_n . The increasing of the stiffness of gear carrying shafts is, also for this reason, useful for stabilization of the vibration system.

5. Conclusions

A mathematical model of two degrees of freedom which can represent the essential difference of vibrational characteristics between bevel gears and spur and helical gears has been constructed and investigated. From the analysis of the coupled Hill-type vibration of the model, some fundamental characteristics of the vibrations of bevel gears have been clarified:

- (1) The vibrational characteristics of bevel gears are due to the fact that the tooth-normal varies its direction with rotation of gears. Number of natural modes which are represented by parametric vibrations of single-degree of freedom tends to increase because of this rotation of the tooth-normal direction.
- (2) In the equation of motion the angular velocity of the tooth-normal vector may be regarded as a characteristic of bevel gears. But this term can not have any important influence on the vibration of bevel gears under usual driving conditions.
- (3) The unstable regions due to the higher order natural mode show a similar tendency to those of the spur gears which can be regarded as a single degree of freedom system. The lower order natural mode can not be affected by the damping force due to tooth-meshing. The unstable region caused by this natural mode may become noticeably wide when the damping force acting on gear carrying

shafts is not large.

(4) The variation of contact ratio between 1 ~ 2 may cause a change of the frequency spectra of the stiffness function. The width of unstable regions has close relation to the frequency spectra. But as a whole, the variation of the contact ratio does not have such an important influence in the problem of instability of gear vibrations as the other parameters.

(5) By increasing the stiffness of gear carrying shafts the system can be stabilized through effects of reducing the stiffness variation ratio as well as effect of raising the lower natural frequency of the system.

(6) The coupled Hill-type vibration of bevel gears can be approximately resolved as uncoupled Hill-type vibrations by applying the modal analysis in the single and the double contact regions separately. The uncoupled Hill-type equations can be estimated by applying the results which have been obtained for spur gears of Meißner-type vibration.

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